Reverse Thinking and Axiomatic Method in Foundations of Physics

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Joint work with: H. Andréka, J. X. Madarász, I. Németi

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AxFrame :

Photons and inertial bodies are also bodies. The ***quantity part*** $\langle \mathsf{Q}; +, \cdot, \leq \rangle$ *is an ordered field. And o is an observer, b is a body and x, y, z and t are quantities if relation $\mathsf{W}(o, b, x, y, z, t)$ holds.*

AxSelf :

Any inertial observer coordinatizes itself at a coordinate point if and only if its space component is the origin.

$$\forall o, x, y, z, t \;\; \mathsf{Ob}(o) \land \mathsf{IB}(o) \land \mathsf{Q}(x) \land \mathsf{Q}(y) \land \mathsf{Q}(z) \land \mathsf{Q}(t) \rightarrow [\, \mathsf{W}(o, o, x, y, z, t) \leftrightarrow x = y = z = 0 \,].$$

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$$\forall o,p,x,x',y,y',z,z',t,t' \;\; \mathsf{Ob}(o)\wedge \mathsf{IB}(o)\wedge \mathsf{Ph}(p)\wedge \mathsf{Q}(x)\wedge \mathsf{Q}(x') \wedge \mathsf{Q}(y)\wedge \mathsf{Q}(y')\wedge \mathsf{Q}(z)\wedge \mathsf{Q}(z')\wedge \mathsf{Q}(t)\wedge \mathsf{Q}(t') \rightarrow \Big[\mathsf{W}(o,p,x,y,z,t)\wedge \mathsf{W}(o,p,x',y',z',t') \leftrightarrow (x'-x)^2+(y'-y)^2+(z'-z)^2=(t'-t)^2\Big].$$

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Every inertial observer coordinatizes the very same set of events (i.e., meetings of bodies).

$$\forall o, o', x, y, z, t \;\; \mathrm{Ob}(o) \wedge \mathrm{IB}(o) \wedge \mathrm{Ob}(o') \wedge \mathrm{IB}(o) \wedge \mathrm{Q}(x) \wedge \mathrm{Q}(y) \wedge \mathrm{Q}(z) \wedge \mathrm{Q}(t) \rightarrow [\exists x', y', z', t' \;\; \forall b \;\; \mathrm{W}(o, b, x, y, z, t) \leftrightarrow \mathrm{W}(o', b, x', y', z', t')].$$

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$$\mathsf{SpecRel} := \{\mathsf{AxSelf}, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxSymDist}\}$$

The **world-view transformation** $\mathrm{w}^{o}_{o'}$ between observers o and o' relates the coordinate points where o and o' coordinatize the same events, i.e,:

$$\mathrm{w}^{o}_{o'}(x, y, z, t : x', y', z', t') \stackrel{def}{\Longleftrightarrow} \forall b \ \ \mathrm{W}(o, b, x, y, z, t) \leftrightarrow \mathrm{W}(o', b, x', y', z', t').$$

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$$w^{o}_{o'}(x,y,z,t : x',y',z',t') \overset{def}{\Longleftrightarrow} \forall b \;\; W(o,b,x,y,z,t) \leftrightarrow W(o',b,x',y',z',t').$$

Theorem

$$\mathrm{SpecRel} \models \forall o,o' \;\; \mathrm{Ob}(o) \wedge \mathrm{Ob}(o') \wedge \mathrm{IB}(o) \wedge \mathrm{IB}(o') \rightarrow w^{o}_{o'} \text{ “is a Poincaré transformation.”}$$

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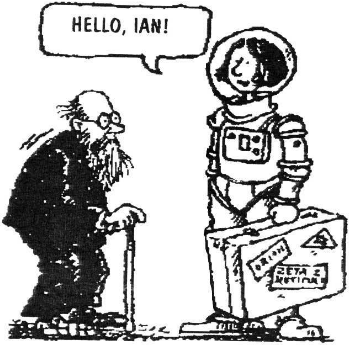
Theorem

The world-view transformation between two observers is differentiable at the points where the two observers meet, and its derivative is a Lorentz transformation if AccRel is assumed.



Twin Paradox (TwP) concerns two twin siblings whom we shall call Ann and Ian. Ann travels in a spaceship to some distant star while Ian remains at home. TwP states that when Ann returns home she will be *younger* than her *twin brother* Ian.

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$$\text{AccRel} \not\models \text{Twin Paradox}$$

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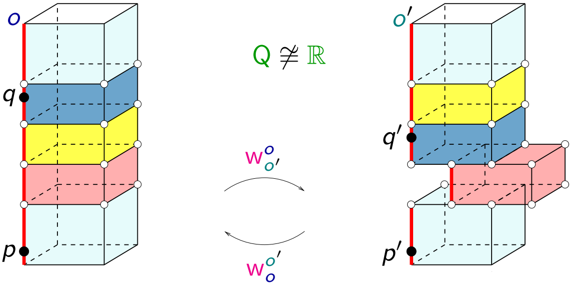

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$$\text{AccRel} \cup Th(\mathbb{R}) \not\models \text{Twin Paradox}$$

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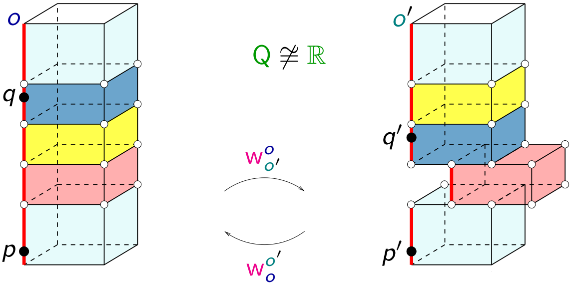

What shall we do now?

Can we stay within first-order logic and assume something which is stronger than $Th(\mathbb{R})$?

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CONT speaks not only about the quantity part, but about its relation to the other parts of the models (e.g., to the observers).

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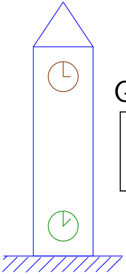

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"The clocks in the bottom of a tower run slower than at its top."

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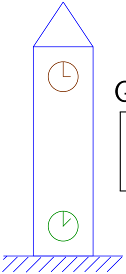

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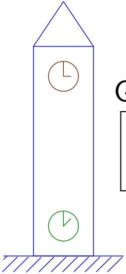

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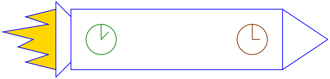
"The clocks in the back of an accelerated spaceship run slower than in its front."



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Meeting observers approximately agree as for the spatial distance of a neighbouring event if this event and the event of meeting are approximately simultaneous enough.

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Theorem (Completeness)

$\mathsf{GenRel_n}$ *is complete with respect to the n-times differentiable Lorentzian manifolds over ordered fields.*

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$$\mathsf{GenRel_\infty} := \bigcup_{n\geq 1} \mathsf{GenRel_n}$$

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$\mathsf{GenRel_\infty}$ *is complete with respect to the smooth Lorentzian manifolds over ordered fields.*

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- Etc.

Background materials are available from:

www.renyi.hu/~turms

Thank you for your attention!