COMPARING RELATIVISTIC AND NEWTONIAN DYNAMICS IN FIRST-ORDER LOGIC

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1. Introduction

In this paper we introduce and compare Newtonian and relativistic dynamics as two theories of first-order logic (FOL). To illustrate the similarities between Newtonian and relativistic dynamics, we axiomatize them such that they differ in one axiom only. This one axiom difference, however, leads to radical differences in the predictions of the two theories. One of their major differences manifests itself in the relation between relativistic and rest masses, see Thms. 4.2 and 4.3.

The statement that the center-lines of a system of point masses viewed from two different reference frames are related exactly by the coordinate transformation between them seems to be a natural and harmless assumption; and it is natural and harmless in Newtonian dynamics, see Cor.4.8. However, in relativistic dynamics it leads to a contradiction, see Thm.4.1. Showing this surprising fact, which also illustrates the great difference between the two theories, is the main result of this paper.

Our work is directly related to Hilbert's 6th problem on axiomatization of physics. Moreover, it goes beyond this program since our general aim is not only to axiomatize physical theories but to investigate the relationship between the basic assumptions (axioms) and

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the predictions (theorems) of the theories and to compare the axiom systems of related theories. Our another general aim is to provide a foundation of physics similar to that of mathematics.

For good reasons, the foundation of mathematics was performed strictly within FOL. One of these reasons is that staying within FOL helps to avoid tacit assumptions. Another reason is that FOL has a complete inference system while second-order logic (and thus any higher-order logic) cannot have one, see, e.g., [11, §IX. 1.6]. For further reasons for staying within FOL, see, e.g., [5, §Why FOL?], [7], [18, §11], [19], [20].

There are many FOL axiomatizations of relativistic kinematics both special and general, see, e.g., [7], [8], [9], [12], [17]. However, as far as we know, our co-authored paper [6] is the only one which deals with the FOL axiomatization of relativistic dynamics, too. Newtonian and relativistic kinematics are compared in the level of axioms in [5, §4.1]. The main aim of this paper is to compare the key axioms and theorems of Newtonian and relativistic dynamics, too.

2. A first-order logic frame for dynamics

Our choice of vocabulary (basic concepts) is explained as follows. We represent motion as the changing of spatial location of *bodies* in time. To do so, we have reference-frames for coordinatizing events (sets of bodies) and, for simplicity, we associate reference-frames with *observers*. There are special kind of bodies which we call *photons*. For coordinatizing events, we use an *ordered field* in place of the field of real numbers.[1] Thus the elements of this field are the *quantities* which we use for marking time and space. In our axioms of dynamics we use *relativistic masses* of bodies as a basic concept.

Motivated by the above, we now turn to fixing the FOL language of our axiom systems. First we fix a natural number $d \geq 2$ for the dimension of spacetime. Our language contains the following non-logical symbols:

- unary relation symbols IOb (inertial **observers**), B (**bodies**), Ph (**photons**) and Q (**quantities**),
- binary function symbols $+$, $\cdot$ and a binary relation symbol $<$ (the field operations and the ordering on Q),
- a $2+d$-ary relation symbol W (**world-view relation**), and
- a binary function symbol M (**mass function**).

We translate IOb(x), B(x), Ph(x) and Q(x) into natural language as "x is an (inertial) observer," "x is a body," "x is a photon," and "x is a quantity." (A more careful wording would be "x is a possible observer," "x is a possible body," etc.) The bodies play the role of the "main characters" of our spacetime models and they are "observed" (coordinatized using the quantities) by the observers. This observation is coded by the world-view relation by translating $\mathrm{W}(x, y, z_1, \ldots, z_d)$ as "observer x coordinatizes body y at spacetime location $\langle z_1, \ldots, z_d \rangle$," (i.e., at space location $\langle z_2, \ldots, z_d \rangle$ at instant z_1). Finally we use the mass function to speak about the relativistic masses of bodies according to observers, i.e., "$\mathrm{M}(x, y)$ is the relativistic mass of body y according to observer x."

$\mathrm{IOb}(x), \mathrm{B}(x)$, $\mathrm{Ph}(x)$, $\mathrm{Q}(x)$, $\mathrm{W}(x, y, z_1, \ldots, z_d)$, $x = y$ and $x < y$ are the atomic formulas, where x, y, z_1, ..., z_d can be arbitrary variables or terms built up from variables by using the field-operations and the mass function M. The **formulas** are built up from these atomic formulas by using the logical connectives *not* ($\neg$), *and* ($\wedge$), *or* ($\vee$), *implies* ($\longrightarrow$), *if-and-only-if* ($\longleftrightarrow$) and the quantifiers *exists* x ($\exists x$) and *for all* x ($\forall x$) for every variable x.

The **models** of this language are of the form

$$\langle U; \mathrm{IOb}, \mathrm{B}, \mathrm{Ph}, \mathrm{Q}, +, \cdot, <, \mathrm{W}, \mathrm{M}\rangle,$$

where U is a nonempty set and IOb, B, Ph and Q are unary relations on U, etc. For simplicity we write $k \in \mathrm{IOb}$ in place of $\mathrm{IOb}(k)$, etc.

We use the notation $\mathrm{Q}^n := \mathrm{Q} \times \ldots \times \mathrm{Q}$ (n-times) for the set of all n-tuples of elements of Q. If $p \in \mathrm{Q}^n$, then we assume that $p = \langle p_1, \ldots, p_n\rangle$, i.e., $p_i \in \mathrm{Q}$ denotes the i-th component of the n-tuple p. We write $\mathrm{W}(k, b, p)$ in place of $\mathrm{W}(k, b, p_1, \ldots, p_d)$, and we write $\forall p$ in place of $\forall p_1, \ldots, p_d$, etc.

We present each axiom at two levels. First we give an intuitive formulation, then we give a precise formalization using our logical notation (which can easily be translated into FOL formulas by inserting the definitions into the formalizations). We seek to formulate easily understandable axioms in FOL.

Our first axiom expresses very basic assumptions, such as: photons are bodies, etc.

> AxFrame **:** $\mathrm{Ph} \subseteq \mathrm{B}$, the **quantity part** $\langle \mathrm{Q}; +, \cdot, <\rangle$ is a Euclidean[2] ordered field, and the masses are positive elements of the quantity part, i.e., $\mathrm{Q}\big(\mathrm{M}(k, b)\big) \wedge \mathrm{M}(k, b) > 0$.

For the FOL definition of linearly ordered field, see, e.g., [10]. We use the usual field operations $0, 1, -, /, \sqrt{}$ definable within FOL. We also use the vector-space structure of Q^n, i.e., if $p, q \in \mathrm{Q}^n$ and $\lambda \in \mathrm{Q}$, then $p + q, -p, \lambda p \in \mathrm{Q}^n$. The **Euclidean length** of $p \in \mathrm{Q}^n$ is defined as $|p| := \sqrt{p_1^2 + \ldots + p_n^2}$, for any $n \geq 1$. The **Euclidean distance** of $p, q \in \mathrm{Q}^n$ is defined as $|pq| := |p - q|$. As usual, ℓ is called a **line** iff there are $p, q \in \mathrm{Q}^d$ such that $q \neq \langle 0, \ldots, 0\rangle$ and $\ell = \{p + \lambda q : \lambda \in \mathrm{Q}\}$. And $\mathrm{Q}^+ := \{\, \lambda \in \mathrm{Q} : 0 < \lambda\}$ denotes the set of **positive elements** of Q. Set Q^d is called **coordinate system** and its elements are referred to as **coordinate points**. We use the notations

$$p_\sigma := \langle p_2, \dots, p_d \rangle \text{ and } p_\tau := p_1$$

for the **space component** and the **time component** of $p \in \mathrm{Q}^d$, respectively. The **event** $ev_k(p)$ is the set of bodies observed by observer k at coordinate point p, i.e.,

$$ev_k(p) := \{\, b \in \mathrm{B} \ : \ \mathrm{W}(k, b, p) \,\}.$$

The **world-line** of body b according to observer k is defined as the set of coordinate points where b was observed by k, i.e.,

$$wl_k(b) := \left\{\, p \in \mathrm{Q}^d \ : \ W(k, b, p) \,\right\}.$$

3. Kinematics

In this section we formulate our axioms on kinematics. Our first axiom on observers states that they see the same events.

AxEv: All observers coordinatize the very same events:

$$\forall k, h \in \mathrm{IOb} \ \ \forall p \in \mathrm{Q}^d \ \exists q \in \mathrm{Q}^d \quad ev_k(p) = ev_h(q).$$

To introduce our next axiom, we need a concept of inertial bodies. A body is called **inertial** iff its world-line is a line for every observer. The set of inertial bodies is denoted by IB, i.e.,

$$\mathrm{IB} := \{ b \in \mathrm{B} \ : \forall k \in \mathrm{IOb} \ wl_k(b) \text{ is a line} \}.$$

AxThEx below states that each observer can make thought experiments in which it assumes the existence of "slowly moving" inertial bodies (see, e.g., [4, p.622]):

AxThEx: For each observer there is a positive speed limit such that in each spacetime location, in each direction, with any speed less than this limit it is possible to "send out" an inertial body:

$$\forall k \in \mathrm{IOb}\ \exists \lambda \in \mathrm{Q}^+\ \forall p, q \in \mathrm{Q}^d\ \exists b \in \mathrm{IB}\ \big(|(p-q)_\sigma| < \lambda (p-q)_\tau\ \rightarrow\ p, q \in wl_k(b)\big).$$

The following axiom system will be the common core of our axiom systems for relativistic and Newtonian kinematics:

$$\mathsf{Kin} := \{\mathsf{AxEv}, \mathsf{AxThEx}, \mathsf{AxFrame}\}.$$

The **world-view transformation** between the coordinate systems of observers k and h is the set of pairs of coordinate points $\langle p, q\rangle$ such that k and h observe the same event in p and q, respectively, i.e.,

$$w_{kh} := \{\langle p, q\rangle \in \mathrm{Q}^d \times \mathrm{Q}^d\ :\ ev_k(p) = ev_h(q)\}.$$

If R is a binary relation and X is a set, $R[X]$ denotes the R-**image** of X, i.e., $R[X] := \{b : \exists a \in X\ \langle a, b\rangle \in R\}$.

Proposition 3.1. Assuming Kin, the world-view transformations are bijections and take lines to lines, i.e., $w_{kh}[\ell]$ is a line for every line ℓ and $k, h \in \mathrm{IOb}$.

A *proof* can be obtained from that of Thm.3.1.1 in [5, pp.160-170].

We extend Kin to an axiom system for special relativity by assuming that the speed of light is 1 according to any observer.

AxPh**:** The world-lines of photons are of slope 1, and for every observer, there is a photon through two coordinate points if their slope is 1:

$$\forall k \in \mathrm{IOb}\ \forall p, q \in \mathrm{Q}^d\quad \big(|p_\sigma - q_\sigma| = |p_\tau - q_\tau| \leftrightarrow \exists ph \in \mathrm{Ph}\quad p, q \in wl_k(ph)\big).$$

We axiomatize special relativistic kinematics as follows:

$$\mathsf{SpecRelKin} := \mathsf{Kin} \cup \{\mathsf{AxPh}\}.$$

CONVENTION 3.2. Whenever we write "$w_{kh}(p)$," we mean that there is a unique $q \in \mathrm{Q}^d$ such that $\langle p, q\rangle \in w_{kh}$, and $w_{kh}(p)$ denotes this q.

CONVENTION 3.3. We use the equation sign "$=$" in the sense of existential equality (of partial algebra theory [1]), i.e., $\alpha = \beta$ abbreviates that both α and β are defined and they are equal. See [13, Conv.2.3.10, p.31] and [5, Conv.2.3.10, p.61]. Similar convention applies for the binary relations "$<$" and "$\neq$."

To get an axiom system for Newtonian kinematics, we extend Kin by an axiom saying that the simultaneity of events is independent from observers.

AxAbsSim: Simultaneity is absolute, i.e.,

$$\forall k, h \in \mathrm{IOb}\ \forall p, q \in \mathrm{Q}^d\ \big(p_\tau = q_\tau\ \rightarrow\ w_{kh}(p)_\tau = w_{kh}(q)_\tau\big).$$

We axiomatize Newtonian kinematics as follows:

$$\mathsf{NewtKin} := \mathsf{Kin} \cup \{\mathsf{AxAbsSim}\}.$$

Let us note that SpecRelKin and NewtKin differ in one axiom only. But we will see in Prop.3.4 below that these two axiom systems are very different, e.g., they are inconsistent together if we assume that there are observers moving relative to each other. To formulate this statement we need the following definition.

The **speed** $v_k(b)$ of body b according to observer k is defined as:

$$v_k(b) := \frac{|p_\sigma - q_\sigma|}{|p_\tau - q_\tau|},\ \text{ for } p, q \in wl_k(b) \text{ with } p_\tau \neq q_\tau$$

if $wl_k(b)$ is a subset of a line and contains coordinate points p and q with $p_\tau \neq q_\tau$, otherwise $v_k(b)$ is undefined.

Ax∃IOb: There are observers moving relative to each other.

$$\exists k, h \in \mathrm{IOb}\ \exists b \in \mathrm{IB} \quad v_h(b) \neq v_k(b) = 0.$$

Proposition 3.4. SpecRelKin ∪ NewtKin ∪ {Ax∃IOb} is inconsistent.

This proposition is a corollary of Thm.3.6 below.

While in Newtonian kinematics there is no speed limit for observers SpecRelKin implies that no observer can move faster than light if $d \geq 3$ by the following theorem.

Theorem 3.5. Assume $d \geq 3$ and SpecRelKin. Then there are no faster than light observers, i.e.,

$$\forall k, h \in \mathrm{IOb}\ \forall b \in \mathrm{B}\ \big(v_k(b) = 0\ \rightarrow\ v_h(b) < 1\big).$$

Moreover, $\forall k, h \in \mathrm{IOb}\ \forall b \in \mathrm{B}\ \big(v_k(b) < 1\ \rightarrow\ v_h(b) < 1\big).$

In the first formula of the theorem, the speeds of observers are captured by speaking about resting bodies. For *proof*, see, e.g., [2, Prop.1, Thm.3], [13, 2.3.5, 2.8.25, 3.2.13], [16, Thm.3, Thm.5]. We note that the theorem remains true if we omit AxThEx from SpecRelKin.

Lines ℓ and ℓ' are said to be **orthogonal** in the Euclidean sense iff there are $p, p' \in \ell$ and $q, q' \in \ell'$ such that $p \neq p'$, $q \neq q'$, and

$$(p_1 - p_1')(q_1 - q_1') + (p_2 - p_2')(q_2 - q_2') + \ldots + (p_d - p_d')(q_d - q_d') = 0.$$

If $p, q \in \mathrm{Q}^d$ and $p \neq q$, then pq denotes the line passing through coordinate points p and q.

By Thm.3.6, two clocks separated in direction not orthogonal to the direction of movement get out of synchronism.

Theorem 3.6. Assume SpecRelKin. Then two clocks remain in synchronism iff they are separated in direction orthogonal to the direction of movement. Formally: Let $k, h \in \mathrm{IOb}$, $b \in \mathrm{IB}$ and $p, q \in \mathrm{Q}^d$ be such that $v_k(b) = 0$, $p \neq q$ and $p_\tau = q_\tau$. Then $w_{hk}(p)_\tau = w_{hk}(q)_\tau$ iff pq is orthogonal to $wl_h(b)$ in the Euclidean sense.

For *proof*, see, e.g., [4, Thm.11.4, p.626].

To formulate one more theorem on SpecRelKin, we need the following definitions: Let $p, q, r, s \in \mathrm{Q}^d$. The **Minkowski length** of p is

$$\mu(p) := \begin{cases} \sqrt{|p_\tau^2 - |p_\sigma|^2|} & \text{if } p_\tau^2 - |p_\sigma|^2 \geq 0 \\ -\sqrt{|p_\tau^2 - |p_\sigma|^2|} & \text{otherwise} \end{cases}$$

and the **Minkowski distance** of p and q is $\mu(p, q) := \mu(p - q)$. Segments $[pq]$ and $[rs]$ are called **Minkowski equidistant** iff $\mu(p, q) = \mu(r, s)$.

Theorem 3.7. Assume SpecRelKin. Then the world-view transformations preserve the Minkowski equidistance, i.e.,

$$\forall k, h \in \mathrm{IOb}\ \forall p, q, r, s \in \mathrm{Q}^d$$
$$\big(\mu(p, q) = \mu(r, s) \rightarrow \mu(w_{kh}(p), w_{kh}(q)) = \mu(w_{kh}(r), w_{kh}(s))\big).$$

Idea of *proof* of Thm.3.7 is in §5.

4. Dynamics

In this section we formulate our axioms on dynamics. For convenience we use the notation $m_k(b) := M(k, b)$ for the relativistic mass of body b according to observer k.

The **spacetime location** $loc_k(b, t)$ of body b at time instance $t \in \mathrm{Q}$ according to observer k is defined to be the coordinate point p for which

$p \in wl_k(b)$ and $p_\tau = t$ if there is such a unique p, otherwise $loc_k(b,t)$ is undefined, see Fig.1.

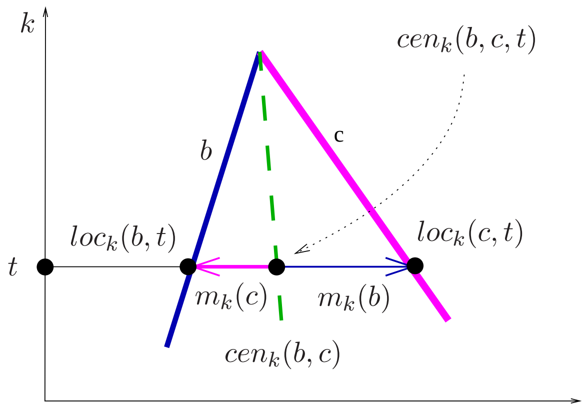

FIGURE 1. Illustration of $loc_k(b,t)$, $cen_k(b,c,t)$ and $cen_k(b,c)$.

The **center of the masses** $cen_k(b,c,t)$ of bodies b and c at time instance t according to observer k is defined to be the coordinate point q such that $q_\tau = t$ and q is the point on the line-segment between $loc_k(b,t)$ and $loc_k(c,t)$ whose distances from these two end-points have the same proportion as that of the relativistic masses of b and c; and it is closer to the "more massive" body, i.e.:

$$m_k(b)\big(loc_k(b,t) - cen_k(b,c,t)\big) = m_k(c)\big(cen_k(b,c,t) - loc_k(c,t)\big)$$

if $loc_k(b,t)$ and $loc_k(c,t)$ are defined, and otherwise $cen_k(b,c,t)$ is undefined, see Fig.1. We note that an explicit definition for $cen_k(b,c,t)$ is the following:

$$cen_k(b,c,t) = \frac{m_k(b)}{m_k(b)+m_k(c)} loc_k(b,t) + \frac{m_k(c)}{m_k(b)+m_k(c)} loc_k(c,t),$$

if $loc_k(b,t)$ and $loc_k(c,t)$ are defined, otherwise $cen_k(b,c,t)$ is undefined. The **center-line of the masses** of bodies b and c according to observer

k is defined as

$$cen_k(b,c) := \{cen_k(b,c,t) \ : \ t \in \mathrm{Q} \text{ and } cen_k(b,c,t) \text{ is defined}\}.$$

In Newtonian dynamics two bodies can be substituted by one body living on the center-line of the two bodies and having mass equal to the sum of the masses of the two bodies. The crucial point in this statement is that different observers agree as for the center-line of inertial bodies (up to world-view transformations), which can be formalized as follows.

AxCen: The world-view transformations take the center-line of two inertial bodies to the center-line of the two bodies.

$$\forall k,h \in \mathrm{IOb} \ \ \forall b,c \in \mathrm{IB} \quad w_{kh}[cen_k(b,c)] = cen_h(b,c).$$

However intuitive and natural AxCen is, it does not hold in the "standard model" of special relativity. Moreover, it is inconsistent with SpecRelKin if we assume that there are observers moving relative to each other.

Theorem 4.1. Assume $d \geq 3$. Then SpecRelKin $\cup$ {Ax$\exists$IOb, AxCen} is inconsistent.

The *proof* of Thm.4.1 is in section 5.

Thus, by Thm.4.1, two bodies cannot be replaced by one in relativistic dynamics. Therefore, if we want to build a consistent relativistic dynamics based on this assumption, we have to weaken AxCen. The solution is to assume it only for meeting bodies.

AxCen$^-$: The world-view transformations take the center-line of two meeting inertial bodies to the center-line of the two bodies.

$$\forall k,h \in \mathrm{IOb} \ \ \forall b,c \in \mathrm{IB}$$
$$\big(wl_k(b) \cap wl_k(c) \neq \emptyset \rightarrow w_{kh}[cen_k(b,c)] = cen_h(b,c)\big).$$

The main axiom of dynamics is AxCen^-. The remaining axioms of our axiom system are only simplifying axioms.

The **rest mass** $m_0(b)$ of body b is defined as $m_0(b) = \lambda$ if (1) there is an observer according to which b is at rest and the relativistic mass of b is λ, and (2) for every observer according to which b is at rest the relativistic mass of b is λ, i.e., $m_0(b) = \lambda$ iff

$$\exists k \in \mathrm{IOb}\, (v_k(b) = 0 \wedge m_k(b) = \lambda) \wedge \forall k \in \mathrm{IOb}\, (v_k(b) = 0 \rightarrow m_k(b) = \lambda).$$

AxCen^- (together with $\mathsf{SpecRelKin}$) implies that the mass of a body necessarily depends on the observer. The reason for this fact is that the simultaneities of observers moving relative to each other in $\mathsf{SpecRelKin}$ differ from each other, and this implies that the proportions involved in AxCen^- change, too. See Prop.4.1 and Fig.3 in [6]. The next axiom states that the relativistic mass of a body depends at most on its rest mass and its speed.

$\mathsf{AxSpeed}$: The relativistic masses of two inertial bodies are the same if both of their rest masses and speeds are equal:

$$\forall k \in \mathrm{IOb}\ \forall b, c \in \mathrm{IB}$$
$$\big((m_0(b) = m_0(c)\ \wedge\ v_k(b) = v_k(c)\big)\ \rightarrow\ m_k(b) = m_k(c)\big).$$

Our next axiom on dynamics states that each observer can make experiments by putting stationary inertial bodies with arbitrary rest mass to any coordinate point.

AxRest: In the coordinate system of any observer there is a resting inertial body with arbitrary rest mass at any coordinate point.

$$\forall k \in \mathrm{IOb}\ \ \forall \lambda \in \mathrm{Q}^+\ \ \forall p \in \mathrm{Q}^d\ \ \exists b \in \mathrm{IB}\ \big(m_0(b) = \lambda\ \wedge\ p \in wl_k(b)\big).$$

Let IB_0 denote the set of inertial bodies having rest mass.

AxMedian: For every two inertial bodies having rest mass, there is an observer for which they have the same speed:

$$\forall b, c \in \mathrm{IB}_0 \;\; \exists k \in \mathrm{IOb} \quad v_k(b) = v_k(c).$$

Let us collect the axioms for dynamics together.

$$\mathsf{Dyn} := \{\mathsf{AxCen}^-, \mathsf{AxSpeed}, \mathsf{AxRest}, \mathsf{AxMedian}\}.$$

By adding Dyn to our kinematical axiom systems we get the respective dynamical ones.

$$\mathsf{NewtDyn} := \mathsf{Dyn} \cup \mathsf{Kin} \cup \{\mathsf{AxAbsTime}\} = \mathsf{Dyn} \cup \mathsf{NewtKin},$$
$$\mathsf{SpecRelDyn} := \mathsf{Dyn} \cup \mathsf{Kin} \cup \{\mathsf{AxPh}\} = \mathsf{Dyn} \cup \mathsf{SpecRelKin}.$$

Let us note that Dyn is the common dynamical core of the two axiom systems, which also differ in one axiom only.

Thms. 4.2 and 4.3 below give the connection between the rest mass and the relativistic mass of an inertial body. Their conclusions are well known results of relativistic and Newtonian dynamics. However, in the usual literature, the assumptions are stronger and not stated explicitly.

Theorem 4.2. Assume SpecRelDyn. Let k be an observer and b be an inertial body having rest mass. Then $v_k(b) < 1$ and

$$m_0(b) = m_k(b)\sqrt{1 - v_k(b)^2}.$$

Theorem 4.3. Assume NewtDyn. Let k be an observer and b be an inertial body having rest mass. Then

$$m_0(b) = m_k(b).$$

The *proofs* of Thms. 4.2 and 4.3 are in section 5.

Let us note that, by Thms. 4.2 and 4.3, axiom systems SpecRelDyn and NewtDyn differing in one axiom have radically different consequences.

Coordinate points p, q and r are called **collinear** iff there is a line ℓ such that $p, q, r \in \ell$. A map $f : \mathrm{Q}^d \longrightarrow \mathrm{Q}^d$ is called an **affine transformation**, if it preserves collinearity and the ratios of distances, i.e., for every distinct and collinear coordinate points p, q and r coordinate points $f(p)$, $f(q)$ and $f(r)$ are collinear and $|pq| \cdot |f(q)f(r)| = |f(p)f(q)| \cdot |qr|$.

By Prop.3.1, it follows that the world-view transformations are affine transformations composed by filed-automorphism induced mappings in models of Kin. It can be proved that, in models of SpecRelKin and NewtKin, the world-view transformations are not necessarily affine transformations, i.e., there are models in which field-automorphism induced mappings occur in the world-view transformations. By Thm.4.4, this is not the case in models of NewtDyn and SpecRelDyn.

Theorem 4.4. The world-view transformations are bijective affine transformations in models of NewtDyn and SpecRelDyn.

The *proof* of Thm.4.4 is in section 5.

By Thm.4.2, SpecRelDyn implies that for every inertial body b the quantity $m_k(b)\sqrt{1 - v_k(b)^2}$ is independent of observer k if b has rest mass. By Thm.4.5, the same holds for every inertial body b moving slower than light.

Theorem 4.5. Assume SpecRelDyn. Let b be an inertial body such that $v_k(b) < 1$ for an observer k. Then $v_k(b) < 1$ for every observer k and

$$\forall k, h \in \mathrm{IOb} \quad m_k(b)\sqrt{1 - v_k(b)^2} = m_h(b)\sqrt{1 - v_h(b)^2}.$$

The *proof* of Thm.4.5 is in section 5.

By our definition, the rest mass of an inertial body can be undefined even if there is an observer according to which the body is at rest. By the following immediate corollary of Thm.4.5, the rest mass of a body is defined whenever there is an observer according to which the body is at rest if we assume SpecRelDyn.

Corollary 4.6. Assume SpecRelDyn. Let b be an inertial body such that $v_k(b) = 0$ for some $k \in \mathrm{IOb}$. Then b has a rest mass.

By Thm.4.3, axiom system NewtDyn implies that relativistic mass of an inertial body is observer independent if the body has rest mass. By Thm.4.7, the same holds for inertial bodies with "finite" speeds.

Theorem 4.7. Assume NewtDyn. Let b be an inertial body such that $v_k(b)$ is defined for some $k \in \mathrm{IOb}$. Then

$$\forall k, h \in \mathrm{IOb} \quad m_k(b) = m_h(b).$$

The *proof* of Thm.4.7 is in section 5.

By the following corollary, theory NewtDyn implies AxCen, the axiom which is inconsistent with SpecRelDyn, see Thm.4.1. This fact also shows great difference between the two theories of dynamics.

Corollary 4.8. NewtDyn $\models$ AxCen.

The *proof* of Cor.4.8 is in section 5.

5. Proofs

Idea of proof of Thm.3.7. Assume first that $d > 2$. One can prove, by Alexandrov-Zeeman theorem, that every world-view transformation is a composition of a Poincaré transformation, a dilation and a field-automorphism-induced mapping, cf. [3, Thm.1.2]. All of these mappings preserve the Minkowski equidistance. Thus the world-view transformations also preserve the Minkowski equidistance.

We note that a similar proof can be obtained for $d = 2$, cf. [3, Thm.1.4].

Another proof can be obtained as follows. Coordinate points p and q are called *lightlike separated*, in symbols $p\lambda q$ iff $|p_\tau - q_\tau| = |p_\sigma - q_\sigma|$. Furthermore, p and q are *timelike separated* iff $|p_\tau - q_\tau| > |p_\sigma - q_\sigma|$. Assume first that $d = 2$. Let p, q and q' be distinct coordinate points. It can be seen that

$$\mu(p,q) = \mu(p,q') \leftrightarrow \exists s, s'\big(s \neq s' \wedge q\lambda s \wedge q'\lambda s \wedge q\lambda s' \wedge q'\lambda s' \wedge coll(p,s,s')\big). \quad (1)$$

Then timelike separatedness is FOL definable from lightlike separatedness by Alexandrov-Zeeman theorem. Let p, q and q' be distinct points such that p and q are timelike separated and the same holds for p and q'. Then one can see that (1) above holds for p, q and q'.

By the above one can prove that Minkowski equidistance for timelike separated pairs of points is FOL definable from lightlike separatedness and collinearity. World-view transformations preserve lightlike separatedness and collinearity by AxPh and Prop.3.1. Thus they preserve Minkowski equidistance for timelike separated pairs of points. The general case can be reduced to the timelike and lightlike cases. □

Proof of Thm.4.1. The proof goes by contradiction. Let $\mathfrak{M}$ be a model of SpecRelKin $\cup$ {Ax∃IOb, AxCen}. Let $k, h \in \mathrm{IOb}$ and $b \in \mathrm{IB}$ be such that $v_h(b) \neq v_k(b) = 0$, see Fig.2. Let $c \in \mathrm{IB}$ be such that $v_h(c) = 0$ and $wl_h(b)$ and $wl_h(c)$ do not meet, i.e., $wl_h(b)$ and $wl_h(c)$ are skew lines. Such c exists by AxThEx. Furthermore, $v_h(b) < 1$ and $v_k(c) < 1$ by Thm.3.5. Thus $wl_h(b)$ and $wl_k(c)$ are not "horizontal" lines, i.e., for every $t \in \mathrm{Q}$ there are $p \in wl_h(b)$ and $q \in wl_k(c)$ such that $t = p_\tau = q_\tau$. Let $p \in wl_h(c)$ and $q \in wl_h(b)$ be such that $p_\tau = q_\tau$ and pq is not orthogonal to $wl_h(b)$ in the Euclidean sense. It is easy to see that there are such p and q. Then by Thm.3.6, $w_{hk}(p)_\tau \neq w_{hk}(q)_\tau$. Let $r \in wl_h(c)$ and $s \in wl_h(b)$ be such that $w_{hk}(q)_\tau = w_{hk}(r)_\tau$ and $r_\tau = s_\tau$. Then pq and rs are skew lines since the world-lines of b and c are skew lines. Center-line $cen_h(b,c)$ intersects lines pq and rs and it does not go through points p, q, r and s. Thus it does not intersect line qr since the world-lines of b and c are skew lines. On the other hand center-line $cen_k(b,c)$ intersects line $w_{hk}(q)w_{hk}(r)$. That is a contradiction since the world-view transformations are bijections taking lines to lines and center-lines to center-lines by Thm.3.1 and AxCen. □

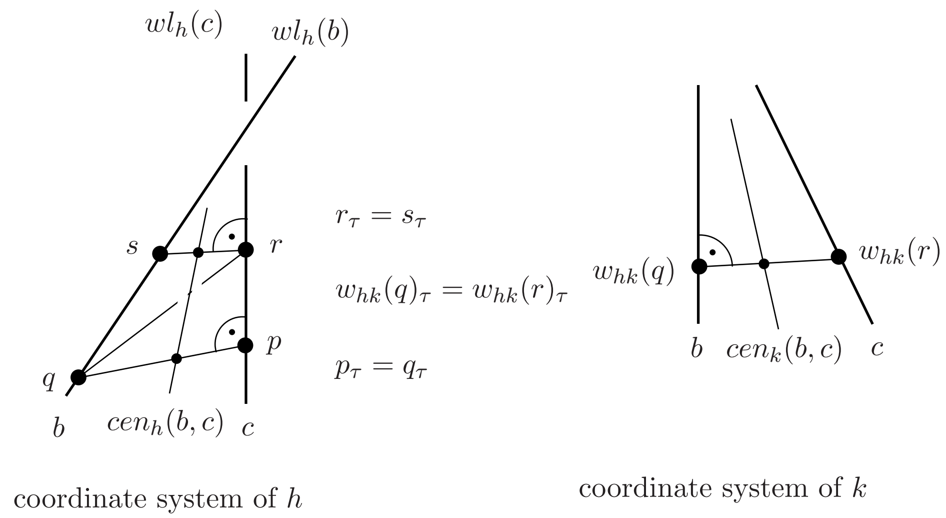

FIGURE 2. Illustration for the proof of Thm.4.1.

Proof of Thm.4.2. Let k be an observer and b be an inertial body having rest mass. Let $v := v_k(b)$.

First we prove from SpecRelDyn that $v < 1$. For $d \geq 3$, $v < 1$ already follows from SpecRelKin by Thm.3.5. To see that $v < 1$ for $d = 2$, let

b' be a body having rest mass such that $v_k(b') = 0$. Such a b' exists by AxRest. By AxMedian there is an observer according to which b and b' have the same speeds. But then, by SpecRelKin, it can be proved that $v < 1$, cf. [5, Thm.2.7.2, p.110].

If $v = 0$, the conclusion of the theorem holds. Thus we can assume that $v \neq 0$. Let c be an inertial body such that $v_k(c) = 0$, $m_0(c) = m_0(b)$ and b and c meet. See Fig.3. Such a c exists by AxRest. Let A be

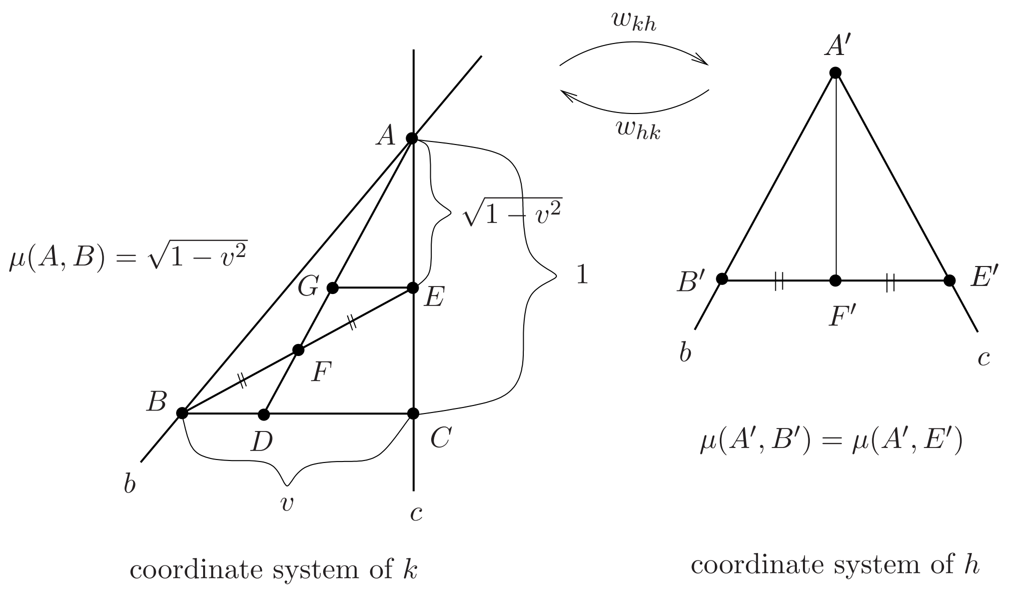

FIGURE 3. Illustration for the proof of Thm.4.2.

the coordinate point where the world-lines of b and c meet. Let B and C be the coordinate points on the world-lines of b and c, respectively, such that $A_\tau - C_\tau = 1$ and $B_\tau = C_\tau$. Then $|AC| = 1$, $|BC| = v$ and $\mu(A, B) = \sqrt{1 - v^2}$. Let D be the center of masses of b and c at $B_\tau = C_\tau$, i.e., $D := cen_k(b, c, B_\tau)$. By definition of center of masses,

$m_k(b)|BD| = m_0(b)|DC|$. Thus

$$m_0(b) = \frac{|BD|}{|DC|} m_k(b). \tag{2}$$

Let h be an observer according to which the speeds of b and c coincide. Such an h exists by AxMedian. Let us turn our attention to the coordinate system of observer h illustrated by the right hand side of Fig.3. Let A' and B' be the w_{kh}-images of A and B, respectively. Let E' be a coordinate point on the world-line of c such that $E'_\tau = B'_\tau$. Let F' be the center of masses of b and c at $E'_\tau = B'_\tau$, i.e., $F' := cen_h(b, c, E'_\tau)$. Since the rest masses and the speeds of b and c coincide, their relativistic masses coincide by AxSpeed. Thus $|B'F'| = |F'E'|$ by the definition of center of masses. Consequently, $\mu(A', B') = \mu(A', E')$, i.e., segments $[A'B']$ and $[A'E']$ are Minkowski equidistant.

Now we turn our attention to the coordinate system of observer k illustrated by the left hand side of Fig.3. Let F and E be the w_{hk} images of F' and E'. Then, by AxCen$^-$, $F \in AD = cen_k(b, c)$ since $F' \in cen_h(b, c)$. Furthermore, $E \in BF$ since w_{hk} takes lines to lines by Prop.3.1. The world-view transformation w_{hk} preserve the Minkowski equidistance by Thm.3.7. Consequently, segments $[AB]$ and $[AE]$ as well as $[BF]$ and $[FE]$ are Minkowski equidistant. Thus $|AE| = \mu(A, E) = \mu(A, B) = \sqrt{1 - v^2}$ and $|BF| = |FE|$.

Triangles BDF and EGF are congruent and triangles AGE and ADC are similar. Thus

$$\frac{|BD|}{|DC|} = \frac{|GE|}{|DC|} = \frac{|AE|}{|AC|} = \sqrt{1 - v^2}.$$

By that and (2), we get $m_0(b) = m_k(b)\sqrt{1 - v^2}$. That completes the proof. □

Proof of Thm.4.3. Let k be an observer and b be an inertial body having rest mass. Then there is an observer k' such that $v_{k'}(b) = 0$. Thus, by AxAbsSim, $v_k(b)$ is not "infinite," i.e., $v_k(b)$ is defined. If $v_k(b) = 0$, the conclusion of the theorem holds. Assume that $v_k(b) \neq 0$. Let c be an inertial body such that $v_k(c) = 0$, $m_0(c) = m_0(b)$ and b and c meet, see the left-hand side of Fig.4. Such a c exists by AxRest. Let h

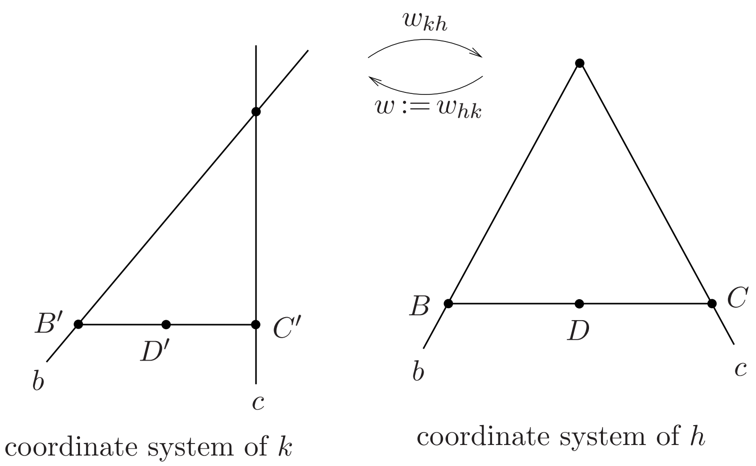

FIGURE 4. Illustration for the proof of Thm.4.3.

be an observer according to which the speeds of b and c coincide. Such an h exists by AxMedian. Let us turn our attention to the coordinate system of observer h illustrated by the right-hand side of Fig.4. Let $B \in wl_h(b)$ and $C \in wl_h(c)$ be distinct points such that $B_\tau = C_\tau$. Let D be the center of masses of b and c at time instance $B_\tau = C_\tau$ according to observer h, i.e., $D := cen_h(b, c, B_\tau)$. Since the speeds and the rest masses of b and c coincide, their relativistic masses coincide by AxSpeed. But then, by definition of center of masses, D is the midpoint of segment $[BC]$, i.e., $|BD| = |DC|$.

$w := w_{hk}$ is a bijection taking lines to lines by Prop.3.1. Furthermore, it takes the midpoint of a segment to the midpoint of the w-image of the segment, i.e., $w\big((p+q)/2\big) = \big(w(p)+w(q)\big)/2$ for every $p,q \in \mathrm{Q}^d$. The proof of this statement is illustrated in Fig.5.

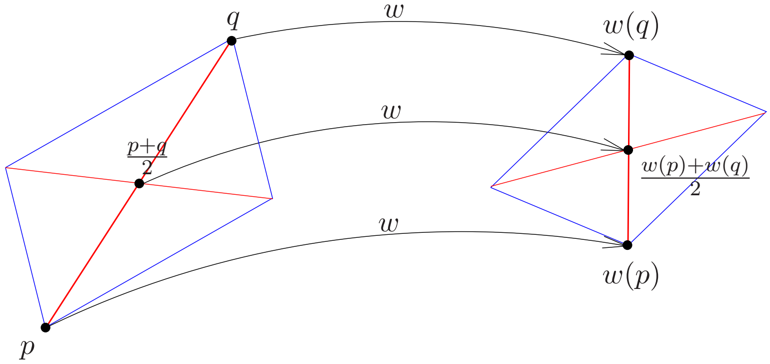

FIGURE 5. w takes midpoint of a segment to the midpoint of the w-image of the segment. The proof of that is based on the fact that w takes parallelograms to parallelograms and the diagonals of parallelograms bisect each other.

Now let us turn our attention to the coordinate system of observer k illustrated by the left-hand side of Fig.4. Let B', C' and D' be the w-images of B, C and D. Then $B'_\tau = C'_\tau = D'_\tau$ by AxAbsSim. Furthermore, D' is the midpoint of segment $[B'C']$, i.e., $|B'D'| = |D'C'|$ since D is the midpoint of segment $[BC]$ and w takes midpoints to midpoints. Furthermore, $D' \in cen_k(b,c)$, by AxCen$^-$, since $D \in cen_h(b,c)$. Thus D' is the center of masses of b and c at time instance $B'_\tau = C'_\tau = D'_\tau$ according to observer k, i.e., $D' = cen_k(b,c,B'_\tau)$. But then, by definition of center of masses, $m_k(b) = m_k(c) = m_0(c) = m_0(b)$ since $|B'D'| = |D'C'|$. That completes the proof. $\square$

Proof of Thm.4.4. Assume first Kin $\cup$ {AxRest}. Let $k,h \in \mathrm{IOb}$. Let $w := w_{kh}$. Then w is a bijection taking lines to lines by Prop.3.1. Thus it takes parallel lines to parallel ones. Furthermore, it takes the midpoint of a segment to the midpoint of the w-image of the segment, i.e., $w\big((p+q)/2\big) = \big(w(p)+w(q)\big)/2$ for every $p, q \in \mathrm{Q}^d$. The proof of this statement is illustrated in Fig.5.

To prove that the world-view transformations are affine transformations in models of NewtDyn and SpecRelDyn, it is enough to prove that $|AC|/|CB| = |w(A)w(C)|/|w(C)w(B)|$ for every distinct $A, B, C \in \mathrm{Q}^d$ with $A_\tau = B_\tau = C_\tau$. To prove that, let A, B and C be distinct coordinate points such that $A_\tau = B_\tau = C_\tau$. We can assume that C is between A and B. See the left-hand side of Fig.6. Let $a \in \mathrm{IB}$

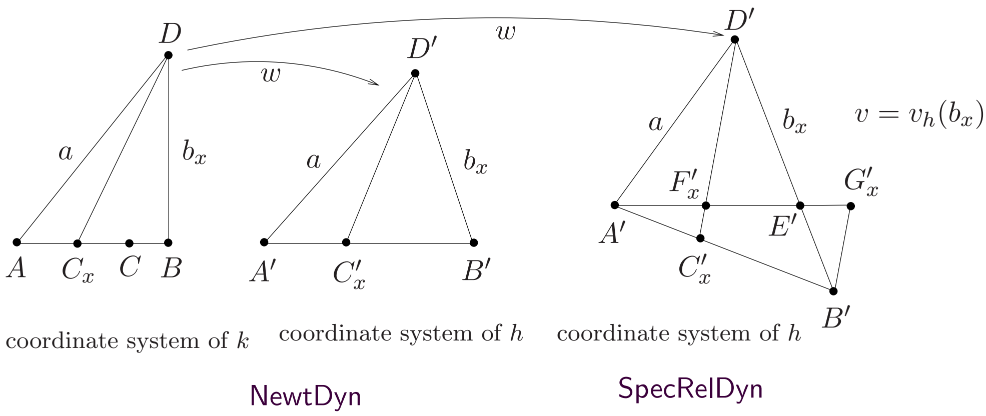

FIGURE 6. Illustration for the proof of Thm.4.4.

be such that $A \in wl_k(a)$ and $v_k(a) \neq 0$. Let $D \in wl_k(a)$ be such that $D_\sigma = B_\sigma$. For every $x \in \mathrm{Q}^+$, fix an inertial body b_x such that $wl_k(b_x) = BD$ and the rest mass of b is x, i.e., $m_0(b_x) = x$. Such bodies exist by AxRest. For every $x \in \mathrm{Q}^+$, let C_x be the center of masses of a and b_x at time-instance $A_\tau = B_\tau$ according to k, i.e., $C_x := cen_k(a, b_x, A_\tau)$. By definition of the center of masses,

$|AC_x|/|C_xB| = m_k(b_x)/m_k(a) = m_0(b_x)/m_k(a) = x/m_k(a)$. Let

$$\lambda := \frac{1}{m_k(a)}. \tag{3}$$

Then

$$\forall x \in \mathrm{Q}^+ \quad \frac{|AC_x|}{|C_xB|} = \lambda x. \tag{4}$$

Let A', B', D' and C'_x be the w-images of A, B, D and C_x, respectively.

Now assume NewtDyn. Then $m_h(b_x) = m_0(b_x) = x$ by Thm.4.3. Furthermore, $A'_\tau = B'_\tau = (C'_x)_\tau$ by AxAbsSim, see the middle of Fig.6. Then $D'C'_x = cen_h(a, b_x)$ by AxCen$^-$. Thus C'_x is the center of masses of a and b_x at time instance A'_τ according to h, i.e., $C'_x = cen_h(a, b_x, A'_\tau)$. Thus $|A'C'_x|/|C'_xB'| = m_h(b_x)/m_h(a) = x/m_h(a)$. Let

$$\lambda' := \frac{1}{m_h(a)}. \tag{5}$$

Then

$$\forall x \in \mathrm{Q}^+ \quad \frac{|A'C'_x|}{|C'_xB'|} = \lambda' x. \tag{6}$$

Let us now consider the case $x = m_k(a)$. Then $C_{m_k(a)}$ is the midpoint of the segment $[AB]$, i.e., $|AC_{m_k(a)}| = |C_{m_k(a)}B|$ by (3) and (4). Then $C'_{m_k(a)}$ is the midpoint of the segment $[A'B']$, since w takes the midpoint of a segment to the midpoint of the w-image of the segment. But then, by (4) and (6), $\lambda m_k(a) = \lambda' m_k(a) = 1$. Hence

$$\lambda = \lambda'. \tag{7}$$

Thus, by (4) and (6), $\forall x \in \mathrm{Q}^+ \ |AC_x|/|C_xB| = |A'C'_x|/|C'_xB'|$. Clearly, there is an x such that $C_x = C$. Then $|AC|/|CB| = |A'C'|/|C'B'|$, which completes the proof for the case of NewtDyn.

Now assume SpecRelDyn. See the right-hand side of Fig.6. Let us note, that the speed $v_h(b_x)$ is independent of the choice of x. Let v denote this speed. Then $m_h(b_x) = m_0(b_x)/\sqrt{1-v^2} = x/\sqrt{1-v^2}$ by Thm.4.2. Clearly, $D'C'_x = cen_h(a, b_x)$ by AxCen$^-$. Thus $A'_\tau \neq D'_\tau \neq$

B'_τ. Let $E' \in D'B'$ be such that $E'_\tau = A'_\tau$. Let F'_x be the intersection of $D'C'_x$ and $A'E'$. Clearly, $F'_x = cen_h(a, b_x, A'_\tau)$. Thus

$$\frac{|A'F'_x|}{|F'_xE'|} = \frac{m_h(b_x)}{m_h(a)} = \frac{x}{m_h(a)\sqrt{1-v^2}}.$$

Let $G'_x \in A'E'$ be such that $B'G'_x$ is parallel to $D'C'_x$. Now,

$$\frac{|A'C'_x|}{|C'_xB'|} = \frac{|A'F'_x|}{|F'_xG'_x|} = \frac{|A'F'_x|}{|F'_xE'|}\frac{|F'_xE'|}{|F'_xG'_x|} = \frac{x}{m_h(a)\sqrt{1-v^2}}\frac{|D'E'|}{|D'B'|}.$$

Let

$$\lambda'' := \frac{|D'E'|}{m_h(a)\sqrt{1-v^2}|D'B'|}. \tag{8}$$

Then

$$\forall x \in \mathrm{Q}^+ \quad \frac{|A'C'_x|}{|C'_xB'|} = \lambda'' x. \tag{9}$$

Now,

$$\lambda = \lambda'' \tag{10}$$

can be proved by (3), (4) and (9) exactly the same way as $\lambda = \lambda'$ was proved for the case of NewtDyn. The rest of the proof is analogous to the proof for the case of NewtDyn. □

Proof of Thm.4.5. Let a be an inertial body and $k, h \in \mathrm{IOb}$ be such that $v_k(a) < 1$. Then, by Thm.3.5, $v_h(a) < 1$ if $d \geq 3$. To prove that $v_h(a) < 1$ for arbitrary d, let b_k and b_h be inertial bodies having rest masses such that $v_k(b_k) = 0$ and $v_h(b_h) = 0$. Such bodies exist by AxRest. By AxMedian, there is an observer according to which b_k and b_h have the same speeds. But then it can be proved that $v_h(b_k) < 1$, cf. [5, Thm.2.7.2, p.110]. By that, it is easy to prove that $v_h(a) < 1$, too.

Assume first, that $v_k(a) \neq 0$. We will use the proof of Thm.4.4 for the case of SpecRelDyn. So we can assume that k, h and a are as in the second paragraph of that proof, and let A, B, b_x etc. be as in that proof, see the left-hand and right-hand sides of Fig.6. It can be proved that

$$\frac{\mu(D',A')}{\mu(D',B')} = \frac{\mu(D,A)}{\mu(D,B)} = \sqrt{1-v_k(a)^2}$$

by Thm.3.7 and Thm.4.4. Then

$$\frac{|D'E'|}{|D'B'|} = \frac{\mu(D',E')}{\mu(D',B')} = \frac{\mu(D',E')}{\mu(D',A')}\frac{\mu(D',A')}{\mu(D',B')} = \frac{\sqrt{1-v^2}\sqrt{1-v_k(a)^2}}{\sqrt{1-v_h(a)^2}}.$$

By this equation, (3), (8) and (10), we conclude that
$m_k(a)\sqrt{1-v_k(a)^2} = m_h(a)\sqrt{1-v_h(a)^2}$.

Now assume that $v_k(a) = 0$. Let a^* be an inertial body such that $wl_k(a^*) = wl_k(a)$ and $m_0(a^*) = m_k(a)$. Such an a^* exists by AxRest. Then

$$m_k(a) = m_0(a^*) = m_h(a^*)\sqrt{1-v_h(a)^2} \tag{11}$$

by Thm.4.2. Let b be an inertial body such that $v_k(b) \neq 0$ and $wl_k(a) \cap wl_k(b) \neq \emptyset$. Clearly, $cen_k(a,b) = cen_k(a^*,b)$. Then $cen_h(a,b) = cen_h(a^*,b)$ by AxCen$^-$. Thus $m_h(a) = m_h(a^*)$. This equation together with (11) completes the proof. □

Proof of Thm.4.7. The proof is analogous to that of Thm.4.5. Let a be an inertial body and $k,h \in \mathrm{IOb}$ be such that $v_k(a)$ is defined.

Assume first, that $v_k(a) \neq 0$. We will use the proof of Thm.4.4 for the case of NewtDyn. We can assume that k, h and a are as in that proof. By (3), (5) and (7) we get that $m_k(a) = m_h(a)$.

Now assume that $v_k(a) = 0$. Let a^* be an inertial body such that $wl_k(a^*) = wl_k(a)$ and $m_0(a^*) = m_k(a)$. Such an a^* exists by AxRest. Then

$$m_k(a) = m_0(a^*) = m_h(a^*) \tag{12}$$

by Thm.4.3. Let b be an inertial body such that $v_k(b) \neq 0$ and $wl_k(a) \cap wl_k(b) \neq \emptyset$. Clearly, $cen_k(a,b) = cen_k(a^*,b)$. Then $cen_h(a,b) =$

$cen_h(a^*, b)$ by AxCen$^-$. Thus $m_h(a) = m_h(a^*)$. This equation together with (12) completes the proof. □

Proof of Cor.4.8. Assume NewtDyn. Let $k, h \in \mathrm{IOb}$ and $a, b \in \mathrm{IB}$. We would like to prove that

$$w_{kh}[cen_k(a, b)] = cen_h(a, b). \tag{13}$$

First assume that $v_k(a)$ is "infinite," i.e., undefined. Then $v_h(a)$ is also "infinite," by AxAbsSim. Hence, $cen_k(a, b) = cen_h(a, b) = \emptyset$. Then (13) holds. The same holds if $v_k(b)$ is "infinite".

If $wl_k(a) \cap wl_k(b) \neq \emptyset$, (13) holds by AxCen$^-$. Thus we can assume that $wl_k(a) \cap wl_k(b) = \emptyset$. Now assume that $v_k(a)$ and $v_k(b)$ are not "infinite," i.e., both are defined. Then $v_h(a)$ and $v_h(b)$ are also defined by AxAbsSim. First we prove that $w_{kh}[cen_k(a, b)] \subseteq cen_h(a, b)$. To prove this statement, let $C \in cen_k(a, b)$. Let $A \in wl_k(a)$ and $B \in wl_k(b)$ be such that $A_\tau = B_\tau = C_\tau$. Then, by definition of center of masses, $|AC|/|CB| = m_k(b)/m_k(a)$. Let A', B' and C' be the w_{kh}-images of A, B and C. Clearly, $A'_\tau = B'_\tau = C'_\tau$ by AxAbsSim. By Thm.4.7, $m_k(b) = m_h(b)$ and $m_k(a) = m_h(a)$. By Thm.4.4, w_{kh} is a bijective affine transformation. Consequently,

$$|A'C'|/|C'B'| = |AC|/|CB| = m_k(b)/m_k(a) = m_h(b)/m_h(a).$$

Hence $C' \in cen_h(a, b)$. Thus $w_{kh}[cen_k(a, b)] \subseteq cen_h(a, b)$. Analogously, $w_{hk}[cen_h(a, b)] \subseteq cen_k(a, b)$ holds. Since w_{kh} and w_{hk} are bijections and inverses of each other, we conclude that (13) holds. □

6. Concluding remarks

We have shown that Newtonian and relativistic dynamics can be axiomatized (within FOL) such that they differ in one axiom only. However, in the level of consequences, they have radical differences.

The most surprising difference is that AxCen, an apparently harmless consequence of Newtonian dynamics, is inconsistent with relativistic dynamics.

Notes

[1]Replacing the field of real numbers by an ordered field not just increases the flexibility of our theories but makes it possible to keep them within FOL, which is crucial in axiomatic foundations, see, e.g., [5, Appendix: Why first-order logic?].

[2]That is, a linearly ordered field in which positive elements have square roots.

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