A logical investigation of inertial and accelerated observers in flat space-time (logic and relativity theory)*

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Extended abstract

Abstract

We study relativity theory as a theory in the sense of mathematical logic. We use first-order logic (FOL) as a framework for this. We aim at an "analysis of the logical structure of relativity theories". First we build up (the kinematics of) special relativity, then analyze it, and then we experiment with generalizations in the direction of general relativity. The present note gives samples from an ongoing broader research project (which in turn is part of a research direction going back to Reichenbach and others in the 1920's).

In sections 2,3 we recall a complete FOL-axiomatization Specrel of special relativity from [2],[13]. In section 4 we answer questions from papers by Ax and Mundy concerning the logical status of faster than light motion (FTL) in relativity. We claim that already very small/weak fragments of Specrel prove "*No FTL* ". In section 5 we give a sketchy outlook for the possibility of generalizing Specrel to theories permitting accelerated observers (gravity).

1 Introduction

The interplay between logic and relativity theory goes back for 80 years by now and has been playing a non-negligible role in works of researchers like Reichenbach, Carnap, Suppes, Ax, Szekeres, Malament, and many other contemporaries.

The present paper intends to give samples from the area called *analysis of the logical structure of relativity theories*. The first step in this analysis is building up relativity theory as a theory in the sense of first-order logic (FOL).[1] Axiomatizations of special relativity have been extensively studied in the literature, cf. our list of references. These works usually stop with a kind of completeness theorem for their axiomatizations. As a contrast, what we call the analysis of logical structure of relativity theory begins with proving such a completeness theorem but the real work comes afterwards during which one often concludes that we have to change the axioms.[2] In the present work we try to illustrate what we understand by this analysis of logical structure.

*Research supported by the Hungarian National Foundation for scientific research grants No's T30314, T43242, T35192.

[1]The reason why we chose FOL and not e.g. second-order logic is presented in detail in [2, App.1] as well as in Ax [5], but the reasons in Väänänen [30] also apply.

[2]Very roughly, one could phrase this as "we start off where the others stopped (namely, at completeness)".

To this end we have to start with a list of axioms and a completeness theorem but the emphasis is on what comes beyond these. To illustrate what we mean by the logical analysis of relativity, we choose the much debated topic of *faster than light motion*. We refer to this as the *No FTL* conjecture where "*No FTL* " abbreviates the conjecture that faster than light motion is impossible (in relativity). A large part of the literature claims that *No FTL* is an axiom (of relativity) while another substantial part wants to get rid of *No FTL* and they maintain that *No FTL* is *not* true (i.e. that *FTL is* possible). This controversy in the literature[3] makes *No FTL* an ideal testing-ground for logic in relativity. E.g. we can study which potential axioms of (special) relativity imply *No FTL* , what is implied by *No FTL* , how to obtain the most "conservative" modification of the axiom system such that it will not imply *No FTL* etc.

Mundy [20] gives an axiomatization of special relativity, and he claims that his axioms do not imply *No FTL* . From this Mundy concludes that *No FTL* is not a theorem of special relativity. We will return to quoting Mundy verbatim above Theorem 4.3.[4] At the same time, in the area of connecting relativity with quantum mechanics, the *No FTL* conjecture plays an essential role, e.g. the so called Einstein-Podolsky-Rosen paradox is based on assuming *No FTL* . Many authors think that *No FTL* is the essence of relativity as is suggested by the subtitle "Physics according to Newton – A world with no speed limit" in Kogut's book on relativity. As a further contrast, Ax [5] adds *No FTL* to his list of axioms for special relativity and he raises the question whether his axiom system is redundant.

Here we will give an answer to Ax's question which at the same time will refute Mundy's claim for space-times of more than 2 dimensions. Namely, we will prove that *No FTL* is a logical consequence of Ax's remaining axioms (Theorem 4.2 herein), and will prove that Mundy's axioms do entail *No FTL* (Theorem 4.3 herein). Briefly, we conclude that *No FTL* is not an axiom but a theorem of (special) relativity. In [13, pp.89-90] we also show that *No FTL* does not follow from Einstein's Special Principle of Relativity (SPR) going back to Galileo, in 2-dimensional space-time, contrary to what Einstein claims in [7, pp.126-127]. In [2],[13] we also study how to refine the axioms of special relativity in order to make *No FTL* independent from the rest of the axioms. I.e. there we elaborated variants of special relativity in which *FTL is* possible (i.e. where *No FTL* is no more a theorem). This way we intend to contribute to those works, too, which want to experiment with permitting *FTL* motion in relativity.[5]

We hope the above example (of *No FTL*) illustrates how analyzing the logical structure works in the *theory of inertial observers* (special relativity proper). In the last section of this work we give a brief outlook in the direction of general relativity, as follows. The *theory of accelerated observers* is a proper generalization of the theory of inertial observers and, as explained in the classic textbook [18, pp.163-165], the study of accelerated observers can be regarded as a natural first step towards general relativity. (It is true that accelerated observers "live" in flat space-time, but still this is a natural first step.) In our opinion, the logic based axiomatic approach outlined above and quoted from the literature (e.g. Ax, Mundy) can be extended to the theory of accelerated observers. Further, we think that such an extension would be most fruitful since it provides a first step towards analyzing general relativity. E.g. in the theory of accelerated observers (i) the *Twin Paradox* admits a more natural formulation than

[3]The above-mentioned controversy is documented in [13, §2.7]. Cf. also Lewis [12, pp.67-80] and Novikov [21].

[4]Mundy's work "comes" from the seminars of Suppes on the topic of logic and relativity. Suppes (1974) credits his motivation in this line to interactions with Tarski.

[5]For a thorough discussion of the status and literature of FTL cf. e.g. [13, §2.7, pp.70-73 (especially footnote 163)], and Lewis [12, pp.67-80, 212-213].

in the theory of inertial observers, hence more refined analysis is possible and (ii) the effect of gravity on clocks can be modeled and studied. The last section will be more sketchy than the previous ones for two reasons: (1) While the literature of the axiomatic approach to inertial observers is extensive as we quote below, the same is not true for accelerated observers. Hence we lack a background in the literature from which we could "destill" the notion of e.g. a standard model of the theory (i.e. intended model). So in this direction more work is needed. (2) We intended to give only a hint indicating that the "analysis of logical structure" can be naturally extended from inertial observers to accelerated ones.

To place the present work in context: The interplay between logic and relativity has been extensively investigated in the literature. E.g. axiomatizations of special relativity have been studied e.g. in Robb 1914 [23], Reichenbach 1924 [22], Carathéodory 1924 [6], Suppes 1959-1972 [26],[27],[28], Szekeres 1968 [29], Ax 1978 [5], Friedman 1983 [8], Mundy 1986 [20], Goldblatt 1987 [11], Schutz 1997 [25]. (This is only a small sample.) Although most of these works use the framework of logic, only Ax [5] and Goldblatt [11] are purely in first-order logic. Ax thoroughly presents the intuitive, physical motivation for his approach, his choice of framework, his axioms etc. which are satisfactory even on the level of abstraction of philosophy of physics (and philosophy of science). The motivation for the present work coincides with the motivation given by Ax (and is very close to the one in Suppes [27]), therefore for further motivation, introduction, background etc. we refer the reader to Ax [5] as well as to the related paper Madarász-Tőke [15] submitted to the present workshop.[6]

László Kalmár was interested in broadening the scope of interactions between mathematical logic on one side and the sciences on the other. His pioneering activities in computer science and in artificial intelligence are examples for this interest. The subject matter of the present paper connecting logic with the philosophy of physics/space-time is a further example in which Kalmár expressed interest in personal communication with two of the present authors. The current trends connected to the present work are represented e.g. by the upcoming Conference on Philosophy of Space-time (Oxford, May 2004), or by e.g. Muis [19].

2 The vocabulary of the first-order language of our investigations

In this paper we will deal with kinematics of relativity only, i.e. we will deal with motion of *bodies* (or *test-particles*). The motivation for our choice of vocabulary (for special relativity) is summarized as follows. We will represent motion as changing spatial location in time. For this, we will have reference-frames for co-ordinatizing events and, for simplicity, we will represent reference-frames as special bodies which we will call *observers*. We visualize an observer-as-a-body as "sitting" in the origin of its reference frame, or equivalently, "living" on the time-axis of the reference frame. There will be another special kind of bodies which we will call *photons*, and we will distinguish *inertial bodies* from the non-inertial ones. We will use the word "*accelerated*" as a synonym for "non-inertial". For co-ordinatizing events we will use an arbitrary *ordered field* in place of the field of the real numbers. Thus the elements of this field will be the "*quantities*"

[6] More generally, the interplay between logic and relativity has remained the main "testing-ground" for logical positivism throughout its history, cf. e.g. Friedman [9]. Gödel, a part-time participant of the Vienna Circle, did some revolutionary work in relativity, too (besides logic), cf. his collected works (volumes II, III). Further substantial motivation is elaborated e.g. in [2], [28], [27].

which we will use for measuring time and space.

n will be a natural number, the number of space-time dimensions.

Motivated by the above, our first-order language contains the following symbols:

unary relation symbols $\mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \mathsf{Ib}, \mathsf{F}$
(for bodies, observers, photons, inertial bodies, and quantities, i.e. elements of the field, respectively)

binary function symbols $+, \cdot$, constants $0, 1$ and a binary relation symbol $<$
(for the field-operations and ordering on F)

a $2+n$-ary relation symbol W
(for co-ordinatizing events, i.e. for the world-view relation).

We will read "B(x)" as "x is a body" etc., and we will read "$\mathsf{W}(m, b, x_1, x_2, \ldots, x_n)$" as "observer m sees (or observes) the body b at time x_1 at location $(x_2, \ldots x_n)$". This "seeing" or "observing" has nothing to do with seeing via photons or observing via experiments, it simply means that b is present in the event which has co-ordinates $(x_1, \ldots, x_n)$ in m's co-ordinatization.

We assume, tacitly, throughout this paper that $\mathsf{Ob}, \mathsf{Ph}, \mathsf{Ib} \subseteq \mathsf{B}$. These statemens are all expressible by first-order formulas, e.g. $\mathsf{Ob} \subseteq \mathsf{B}$ is expressed by $\forall x[\mathsf{Ob}(x) \rightarrow \mathsf{B}(x)]$.

3 A (complete) first-order axiom system for inertial observers

We will formulate each axiom on three levels. First we give a very intuitive formulation, then we give a precise formalization using notions that will be useful later, too (like life-line), and finally, for completeness, we give a concrete first-order formula without using the introduced notions.

First we introduce some notions (but one could omit these and read the first-order formalizations of the axioms right away).

We will use the vector-space structure of ${}^n\mathsf{F}$. I.e. if $p, q \in {}^n\mathsf{F}$ and $\lambda \in \mathsf{F}$, then $p+q, \lambda p \in {}^n\mathsf{F}$, and $\overline{0} = (0, \ldots, 0)$ is the *origin*. When $p \in {}^n\mathsf{F}$ we assume that $p = (p_1, \ldots, p_n)$. Let $q, v \in {}^n\mathsf{F}, v \neq \overline{0}$. The (straight) *line* going through q, with *squared speed*[7] (or *slope*) $(v_2^2 + \cdots + v_n^2)/v_1^2$ and in the spatial direction $(v_2, \ldots, v_n)$ is

$\{q + \lambda v : \lambda \in \mathsf{F}\}$.

The set of straight lines is then

$\mathsf{Lines} = \{\{q + \lambda v : \lambda \in \mathsf{F}\} : q, v \in {}^n\mathsf{F}, v \neq \overline{0}\}$.

The *life-line*, or *trace* of a body b in observer m's world-view, or as seen by m, is the set of co-ordinate points at which m sees b:

$\mathsf{tr}_m(b) = \{p \in {}^n\mathsf{F} : \mathsf{W}(m, b, p)\}$.

The *slope* of a vector $p = (p_1, \ldots, p_n) \in {}^n\mathsf{F}$ is defined as $\mathsf{slope}(p) = (p_2^2 + \cdots + p_n^2)/p_1^2$ if $p_1 \neq 0$, and $\mathsf{slope}(p)$ is infinite othewise. If $\ell \in \mathsf{Lines}$, then

$\mathsf{slope}(\ell) = \mathsf{slope}(p - q)$ for some (and then for all) $p, q \in \ell, p \neq q$.

We are ready now for formulating the axioms on all three levels.

[7]For technical reasons, we use the square of the speed instead of speed throughout.

Ax1 Photons are inertial bodies, i.e.

$\mathsf{Ph} \subseteq \mathsf{Ib}$. The first-order formula expressing this is

$\forall x(\mathsf{Ph}(x) \to \mathsf{Ib}(x))$.

Ax2 Inertial observers "see" inertial bodies move at a steady speed along straight lines, i.e. the life-lines of inertial bodies are straight lines in an inertial observer's world-view,

$\mathsf{tr}_m(b) \in \mathsf{Lines}$ if $m, b \in \mathsf{Ib}$, $m \in \mathsf{Ob}$. A first-order formula expressing this is

$$\mathsf{Ob}(m) \wedge \mathsf{Ib}(m) \wedge \mathsf{Ib}(b) \to (\exists q, v \in {}^n\mathsf{F})(v \neq \overline{0} \wedge \\ (\forall p \in {}^n\mathsf{F})[\mathsf{W}(m, b, p) \leftrightarrow (\exists \lambda \in \mathsf{F}) p = q + \lambda v]) \ .$$

Ax3 Each inertial observer sees himself standing still at the origin, i.e. the life-line of m in m's world-view is the time-axis $\bar{t} = \{(t, 0, \dots, 0) : t \in \mathsf{F}\}$,

$\mathsf{tr}_m(m) = \bar{t}$, if $m \in \mathsf{Ob} \cap \mathsf{Ib}$. A first-order formula expressing this is

$$\mathsf{Ob}(m) \wedge \mathsf{Ib}(m) \wedge p \in {}^n\mathsf{F} \to [\mathsf{W}(m, m, p) \leftrightarrow p_2 = \dots = p_n = 0].$$

Ax4 It is possible to move with any given speed less than 1, and it is possible to send out a photon with speed 1, at each point and in each direction, i.e. for all straight lines $\ell \in \mathsf{Lines}$ and $m \in \mathsf{Ob} \cap \mathsf{Ib}$

$\mathsf{slope}(\ell) < 1 \Rightarrow (\exists k \in \mathsf{Ob} \cap \mathsf{Ib})\ell = \mathsf{tr}_m(k)$ and
$\mathsf{slope}(\ell) = 1 \Rightarrow (\exists \mathsf{ph} \in \mathsf{Ph})\ell = \mathsf{tr}_m(\mathsf{ph})$.
A first-order formula expressing this without abbreviations is

$$(\forall q, v \in {}^n\mathsf{F})(\forall m)[\mathsf{Ob}(m) \wedge \mathsf{Ib}(m) \Rightarrow \\ (v_0 \neq 0 \wedge v_2^2 + \dots + v_n^2 < v_1^2 \Rightarrow \\ (\exists k)(\mathsf{Ob}(k) \wedge \mathsf{Ib}(k) \wedge (\forall p \in {}^n\mathsf{F})[\mathsf{W}(m, k, p) \leftrightarrow (\exists \lambda \in \mathsf{F}) p = q + \lambda v]) \wedge \\ (v \neq \overline{0} \wedge v_2^2 + \dots + v_n^2 = v_1^2 \Rightarrow \\ (\exists \mathsf{ph})(\mathsf{Ph}(\mathsf{ph}) \wedge (\forall p \in {}^n\mathsf{F})[\mathsf{W}(m, k, p) \leftrightarrow (\exists \lambda \in \mathsf{F}) p = q + \lambda v]\ )\ ].$$

Ax5 All inertial observers see the same events,

$\mathsf{Events}_m = \mathsf{Events}_k$ for $m, k \in \mathsf{Ob} \cap \mathsf{Ib}$, where
$\mathsf{ev}(m, p) = \{b : \mathsf{W}(m, b, p)\}$, the *event* seen by m at p, and
$\mathsf{Events}_m = \{\mathsf{ev}(m, p) : p \in {}^n\mathsf{F}\}$, the set of events seen by m.
By a first-order formula without abbreviations this is

$$\mathsf{Ob}(m) \wedge \mathsf{Ib}(m) \wedge \mathsf{Ob}(k) \wedge \mathsf{Ib}(k) \to (\forall p \in {}^n\mathsf{F})(\exists q \in {}^n\mathsf{F}) \\ (\forall b)[\mathsf{W}(m, b, p) \leftrightarrow \mathsf{W}(k, b, q)].$$

AxE All photons move with speed 1 in an inertial observer's world-view,

$\mathsf{slope}(\mathsf{tr}_m(\mathsf{ph})) = 1$ when $m \in \mathsf{Ob} \cap \mathsf{Ib}$ and $\mathsf{ph} \in \mathsf{Ph}$. The first-order formula is

$$\mathsf{Ob}(m) \wedge \mathsf{Ib}(m) \wedge \mathsf{Ph}(\mathsf{ph}) \wedge \mathsf{W}(m, \mathsf{ph}, p) \wedge \mathsf{W}(m, \mathsf{ph}, q) \rightarrow$$
$$(p_1 - q_1)^2 = (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2 .$$

The *world-view transformation* f_{mk} between two observers m, k is defined as

$$\mathsf{f}_{mk} = \{(p, q) : \mathsf{ev}(m, p) = \mathsf{ev}(k, q) \neq \emptyset\} \ .$$

From our previous axioms it follows that f_{mk} is a transformation of ${}^n\mathsf{F}$ (and not only an arbitrary binary relation) if m, k are inertial observers. Therefore we wil use f_{mk} as a function. Then $\mathsf{f}_{mk}(p)$ is the place where k sees the same event that m sees at p, i.e.

$$\mathsf{ev}(m, p) = \mathsf{ev}(k, \mathsf{f}_{mk}(p)) \ .$$

Let $p, q \in {}^n\mathsf{F}$. Then $p_1 - q_1$ is the time passed between the events $\mathsf{ev}(m, p)$ and $\mathsf{ev}(m, q)$ as seen by m and $\mathsf{f}_{mk}(p)_1 - \mathsf{f}_{mk}(q)_1$ is the time passed between the same two events as seen by k. Hence $(\mathsf{f}_{mk}(p)_1 - \mathsf{f}_{mk}(q)_1)/(p_1 - q_1)$ is the rate with which k's clock is slowed down as seen by m.

Ax(sym) All inertial observers see each other's clocks show the wrong time to the same extent,

$\mathsf{f}_{mk}(p)_1 - \mathsf{f}_{mk}(q)_1 = \mathsf{f}_{km}(p)_1 - \mathsf{f}_{km}(q)_1$, when $m, k \in \mathsf{Ob} \cap \mathsf{Ib}$ and $p, q \in \bar{t}$. The first-order formula is

$$(\forall b)[\mathsf{W}(m, b, p) \leftrightarrow \mathsf{W}(k, b, p')] \wedge (\forall b)[\mathsf{W}(m, b, q) \leftrightarrow \mathsf{W}(k, b, q')] \wedge$$
$$(\forall b)[\mathsf{W}(m, b, p'') \leftrightarrow \mathsf{W}(k, b, p)] \wedge (\forall b)[\mathsf{W}(m, b, q'') \leftrightarrow \mathsf{W}(k, b, q')]) \rightarrow$$
$$p'_1 - q'_1 = p''_1 - q''_1 \ .$$

Ax(field) The usual first-order axioms saying that $(\mathsf{F}, +, \cdot, 0, 1, <)$ is an ordered field.

$$\mathsf{Specrel} = \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax5}, \mathbf{AxE}, \mathbf{Ax(sym)}, \mathbf{Ax(field)}\} \ .$$

$\mathsf{Specrel}$ is a first-order axiomatization of special relativity theory. It is complete in the sense that the models of $\mathsf{Specrel}$ are basically the *standard* models for special relativity, except that the ordered field $(\mathsf{F}, +, \cdot, 0, 1, <)$ is not necessarily the ordered field of the real numbers. This completeness statement is carefully formulated and proved in [2, Thm.3.8.14, p.301] and [1, Thm.4, p.15]. As we said in the introduction, there are several axiomatizations of the kinematics of special relativity (usually under the name "space-time") in the literature, e.g. Carathéodory [6], Szekeres [29], Schutz [25], Ax [5], Goldblatt [11]. Of these, Ax [5] is very close in spirit to ours. The first truly first-order logic axiomatizations of the kinematics of special relativity (or of space-time) are the ones in Ax [5], Goldblatt [11] and in works of the present team, e.g. [2], [1], [13], present work. Actually, the structures axiomatized by Ax and Goldblatt do not contain the (relativistic) information carried by the so-called Minkowski-metric[8] while those of the present team do. Hence in some sense the presently reported work seems to provide the first full, complete first-order axiomatization of the kinematics of special relativity or of the space-time of special relativity (together with its special metric).[9]

[8]The square of the Minkowski-distance of $p, q \in {}^n\mathsf{F}$ is $(p_1 - q_1)^2 - \sum_{i>1}(p_i - q_i)^2$.

[9]We note that the relativistic metric will play an important role in the theory of accelerated observers, cf. e.g. [2] or [13, §4.7].

4 A piece of conceptual analysis: faster than light motion

Our main aim is more ambitious than providing a complete axiomatization for the kinematics of special relativity. Our complete axiomatization is only a byproduct. Our aim is to provide an analysis of the logical structure of special relativity (or in other words, giving a conceptual analysis in a precise, mathematical logical framework). Below we illustrate what we mean by this analysis on the example of faster than light movement in relativity.

"*No FTL* " abbreviates the formula saying that no observer m can move *faster than light* relative to any other observer k, formally it abbreviates

$\mathsf{slope}(\mathsf{tr}_m(k)) \leq 1$ when $m, k \in \mathsf{Ob} \cap \mathsf{Ib}$.

The issue whether faster than light (FTL) observers can, in principle, exist is being seriously debated even today, cf. what we said in the introduction or e.g. [10], Matolcsi-Rodrigues [17]. In this connection we proved in [2, Thm.3.4.1, p.203.] the following. Let $\mathsf{Specrel}^- = \mathsf{Specrel} \setminus \{\mathbf{Ax(sym)}\}$.

Theorem 4.1. $\mathsf{Specrel}^- \models$ *No FTL* , if $n > 2$.

The above theorem implies that if we do not want to have *No FTL* as a theorem in special relativity, then we have to give up or weaken at least one of the axioms in $\mathsf{Specrel}^-$. However, even those authors who debate the status of "*No FTL* " accept all axioms of $\mathsf{Specrel}^-$. Theorem 4.1 above can be considerably improved. Namely, if we derive "*No FTL* " from a weaker subsystem S_0 of $\mathsf{Specrel}^-$, we get a stronger, more interesting theorem. In particular, we get information about what other usually accepted axioms we also have to give up if we want to permit "*FTL*" travel in special relativity. Indeed, $\mathsf{Specrel}^-$ can be replaced by a much weaker subsystem S_0 in Thm.4.1 above, as was proved in Madarász-Tőke [15, Thm.4],[16], [13, Thm.3.2.13, p.118.], [2, Thm.4.3.24, p.497]. It was also studied in [2] how to weaken $\mathsf{Specrel}^-$ so that "*No FTL* " be no more implied by it (i.e. *FTL* be permitted by it), cf. also Madarász-Németi [14] and [2, section 3.4.2, Thm.3.4.22, Thm.4.3.25.].

Now we turn to applying Thm.4.1 to answering questions in Ax [5], Mundy [20] as was promised in the introduction. Ax [5] contains a finite first-order axiomatization Σ of special relativity. In this axiom system, one of the axioms, namely **AxC4** states explicitly that all observers move slower than light.

Theorem 4.2. In the axiom-system Σ in Ax [5], the axiom **AxC4** stating that all observers move slower than light is superfluous, i.e.

$\Sigma \setminus \{\mathbf{AxC4}\} \models \mathbf{AxC4}$.

Proof-outline: Let $\mathsf{U} = (U, P, S, T, R)$ be a model of $\Sigma \setminus \{\mathbf{AxC4}\}$. Here, P denotes the set of "particles" and S denotes the set of "light-signals". These correspond, roughly, to our Ob and Ph. Using Tarski's first-order axiomatization of Euclidean geometry, on pp. 531-532 Ax constructs a Euclidean ordered field F and to any $a \in P$ a bijection $\sigma_a : {}^4\mathsf{F} \longrightarrow \mathsf{Events}$, where Events corresponds roughly to our set of events. In this construction, Ax uses **AxC4** three times. With some ingenuity, one can replace the first and last uses of **AxC4** by the use of the axioms **T1** and **AxC1** of Σ respectively. The second use of **AxC4** is not needed for the construction itself, only to ensure a specific property. Now, using the above σ_a, one can construct a model $\mathsf{M} = (\mathsf{F}, \mathsf{B}, \mathsf{Ob}, \dots, \mathsf{W})$ of our language and check that $\mathsf{M} \models \mathsf{Specrel}^-$. By Theorem4.1 then $\mathsf{M} \models$ *No FTL* . Using the definition of M, this means that $\mathsf{U} \models$ **AxC4**. $\square$

On the other hand, the axiomatization in Mundy [20] does not contain an axiom explicitly stating "*No FTL* ". But then, Mundy claims that there are models of his axioms in which there exist FTL observers. On p.43 he writes: "Therefore the only line-type information left open by the theories $\mathcal{T}$ and $\mathcal{T}'$ is which if any of the lines on or outside of the light cone are of type T, i.e. are possible paths of inertial motion. In physical terms this amounts to *asking whether inertial motion can proceed at a speed equal to or greater than that of light*[10]. My contention is that nothing in either the classical or the special relativistic space-time theories provides any answer to this question. The evidence for this is that the theories $\mathcal{T}$ and $\mathcal{T}'$ seem to formalise adequately the physical content of those space-time theories, and yet do not fix an answer to this question." (He then goes on and gives more explanation.)

Theorem 4.3. Let $\mathcal{T}$ be the axiom system in Mundy [20]. In any model of $\mathcal{T}$ all the lines in T (i.e. the so-called time-like lines) are within the light-cones. When $c \neq 0$, they are strictly within the light-cones. Hence $\mathcal{T} \models$"*No FTL* ".

Proof-outline: The proof-idea is very similar to the previous one. Let M be a model of $\mathcal{T}$. (Assume that $c \neq 0$. Mundy does not seem to be aware of the fact that $\mathcal{T}$ allows $c = 0$. But if $c = 0$ in a model of $\mathcal{T}$, then the time-like and light-like lines coincide, hence we are done.) In Section 5, on pp. 40-42, Mundy constructs co-ordinate systems for each time-like line in T, based on which one can construct a model M' of our language. One can check then that $\mathsf{M}' \models \mathsf{Specrel}^-$, and so $\mathsf{M}' \models$ *No FTL* by our Theorem 4.1. This means that there are no time-like lines outside the light-cones. That there are no time-like lines on the light-cone follows from [1, Prop.1]. $\square$

5 Axioms for accelerated observers

In Specrel we restricted attention to inertial observers. It is a natural idea to generalize the theory to including accelerated ovservers, too. We will refer to such a generalized theory as theory of accelerated observers. As explained in the classic textbook [18, pp.163-165], the study of accelerated observers can be regarded as a natural first step (from special relativity) towards general relativity.

While first-order logic axiomatizations for the theory of inertial observers can be found in the literature (Ax, Goldblatt), for accelerated observers it is in the works of the present team where the first-order logic axiomatizations and analysis seem to appear first (as far as we know), cf. [2, chapter 8], [4], [3].

A general overview for a first-order logic approach to accelerated observers was outlined in [2, chapter 8], where not only axiomatization but some conceptual analysis was also provided, e.g. "gravity slows down time". Here we concentrate on spelling out the axioms because our experience e.g. with [2] suggests that this is an important aspect of the theory. E.g. at first sight one might guess that the formalism of mathematical analysis will dominate the axioms, but it turns out that they can be presented in a geometry oriented intuitive style. From [2] one can conclude that the key axiom for

[10] Italics by the present authors.

accelerated observers is what we denote by **Axg** below. Hence below we concentrate on giving this axiom a geometric and intuitively convincing form.

The most important axiom for accelerated observers will state that at each moment of his life-time, the accelerated observer "sees" the world near him and for a short time like an inertial observer does, co-moving with him. We now begin to formalize this.

We will use the following **convention**: If we write Tp where T is an arbitrary relation, we will mean that there is a unique element b which is in relation with p according to T (i.e. $(p,b) \in T$) and Tp denotes this unique element b. We will need this convention because for accelerated observers, in general, f_{mk} is not everywhere defined even when it is a function.

We recall the notion of (total) differential from e.g. Rudin [24, items 9.10-9.15], but we do so in a generalized situation when we replace the ordered field of reals with an arbitrary ordered field F and we talk of the differential of an arbitrary relation T, not only of a function T. The usual (Euclidean) length of $p \in {}^n\mathsf{F}$ is $|p| = \sqrt{p_1^2 + \ldots p_n^2}$, and for $\varepsilon \in \mathsf{F}$, the *sphere* with center p and radius ε is $S(p,\varepsilon) = \{q \in {}^n\mathsf{F} : |q-p| \leq \varepsilon\}$. F^+ denotes the set of strictly positive members of F. From now on we always assume that the ordered field F is Euclidean, i.e. $\forall x > 0 \exists y[x = y \cdot y]$ is valid in F.

Definition 5.1. (*differential of a relation*) Let n, m be natural numbers, $T \subseteq {}^n\mathsf{F} \times {}^m\mathsf{F}$ a relation and $q \in {}^n\mathsf{F}$. A linear function $H : {}^n\mathsf{F} \to {}^m\mathsf{F}$ is called the *differential of T in q* iff

$$\forall \varepsilon \in \mathsf{F}^+ \quad \exists \delta \in \mathsf{F}^+ \quad \forall p \in S(q,\delta) \quad |Tp - Tq - H(p-q)| \leq \varepsilon \, |p-q| \, .$$

If there is such a linear function H, then it is unique, and we denote it by $(dT)_q$.

Definition 5.2. (*local-injection at a point*) A relation $T \subseteq {}^n\mathsf{F} \times {}^m\mathsf{F}$ is called a *local-injection at $q \in {}^n\mathsf{F}$* iff there is $\delta \in \mathsf{F}^+$ such that T is an injective function on $S(q,\delta)$, i.e. if $\forall p, r \in S(q,\delta)[\exists! b((p,b) \in T) \wedge (r \neq p \Rightarrow Tp \neq Tr)]$. Here $\exists!$ denotes that "there exists a unique".

Definition 5.3. ($m \sim_q k$) Let $q \in {}^n\mathsf{F}$ and m, k be observers. We say that *m and k see the event at q the same way*, in symbol $m \sim_q k$, iff f_{mk} is a local transformation at q and

$$(d\mathsf{f}_{mk})_q = \mathsf{Id} \quad \wedge \quad \mathsf{f}_{mk}(q) = q.$$

Then $\sim_q$ is a binary relation on Ob for every $q \in {}^n\mathsf{F}$. In the above, Id is the identity function, which is linear.

If we write out the above definition without using the defined terms, we get the following.

Claim 5.4. $m \sim_q k$ iff f_{mk} is an injective function on $S(q,\delta)$ for some $\delta \in \mathsf{F}^+$ and

$$\forall \varepsilon \in \mathsf{F}^+ \quad \exists \delta \in \mathsf{F}^+ \quad \forall p \in S(q,\delta) \quad |p - \mathsf{f}_{mk}(p)| \leq \varepsilon \, |p-q| \, .$$

Remark 1. We could have defined the binary relation $\sim_q$ without requiring that the world-view transformation f_{mk} be an injective function in a neighbourhood of q. I.e., we could have defined $m \sim_q k$ as

$$\forall \varepsilon \in \mathsf{F}^+ \quad \exists \delta \in \mathsf{F}^+ \quad \forall p \in S(q,\delta) \quad \forall r[(p,r) \in \mathsf{f}_{mk} \Rightarrow |p-r| \leq \varepsilon \, |p-q|].$$

We did not do so because the above definition allows that m sees every event at most once in all neighbourhoods of q while, say, k sees every event that it sees at least twice in all neighbourhoods of q. This does not match our intuition when m and k see the world the same way at q.

$\sim_q$ is an equivalence relation, for all $q \in {}^n\mathsf{F}$. We note that $k \sim_q m$ implies that $\mathsf{ev}(m, q) = \mathsf{ev}(k, q)$ (since $k \sim_q m$ implies $\mathsf{f}_{mk}(q) = q$).

Definition 5.5. (*co-moving observers*) If $m \sim_q k$ and $m, k \in \mathsf{ev}(m, q) = \mathsf{ev}(k, q)$, we say that *$m$ and k are co-moving at q*, and $\mathsf{comove}(m, k, q)$ will denote this.

The following lemma gives the intuition behind the above definition. We will need the notion of life-curve of an observer. The difference between life-line and life-curve is that the life-line is a subset of ${}^n\mathsf{F}$, while the life-curve is a function mapping F into ${}^n\mathsf{F}$, as curves do: the life-curve of an observer k is the life-line of k parametrized by the proper time of k.

Definition 5.6. (*life-curve*) We define the *life-curve* $\mathsf{Tr}_m(k)$ *of the observer k as seen by m* to be $i \circ \mathsf{f}_{km}$ where $i : \mathsf{F} \to {}^n\mathsf{F}$ is the function for which $i(t) = (t, 0, \ldots, 0)$ for every $t \in \mathsf{F}$. This notion covers our intuition when $\mathsf{tr}_k(k) = \bar{t} \cap \mathsf{Dom}(\mathsf{f}_{km})$ and f_{km} is a partial function on $\bar{t} \cap \mathsf{Dom}(\mathsf{f}_{km})$. Then $\mathsf{Tr}_m(k)$ is the life-line $\mathsf{tr}_m(k)$ parametrized by the "proper time" of k.

Lemma 5.7. Assume that m and k are co-moving observers at $q \in {}^n\mathsf{F}$ and $\mathsf{tr}_m(m) = \bar{t} \cap \mathsf{Dom}(\mathsf{f}_{mm})$. Then the life-curve $\gamma = \mathsf{Tr}_m(k)$ of k is a local injection at q, it is differentiable at $t = q_1$ (in the ordinary sense) and $\gamma'(t) = (1, 0, \ldots, 0)$. I.e. $\mathsf{Tr}_m(m)$ and $\mathsf{Tr}_m(k)$ are curves in a neighbourhood of t, they have the same tangent, and the inner clocks of m and k tick with the same rate at t.

We are ready now for stating our main axiom for accelerated observers.

Axg At any point of the life-line of an observer k there is a co-moving inertial observer, i.e.

$$(\forall k \in \mathsf{Ob})(\forall q \in \mathsf{tr}_k(k))(\exists m \in \mathsf{Ob} \cap \mathsf{Ib})\mathsf{comove}(k, m, q) \ .$$

Claim 5.8. Assume $\mathsf{Specrel}$. The co-moving inertial observer is unique in the following sense. Assume that m and k are co-moving inertial observers at a point q. Then the life-curves of m and k co-incide (i.e. m and k are always at the same place [at the origin] and their clocks tick the same way), and moreover their coordinatizations of the world differ only in an isometry of ${}^n\mathsf{F}$.

Lemma 5.9. Assume $\mathsf{Specrel} + \mathbf{Axg}$. Assume that h, k are co-moving observers at q. Then, for any inertial observer m, the life-curves $\gamma = \mathsf{Tr}_m(h)$ and $\eta = \mathsf{Tr}_m(k)$ of h and k respectively are "touching" curves at $t = q_1$ with the same derivative $\gamma'(t) = \eta'(t) = v$ where the Minkowski-length of v is 1, i.e. $|v_1^2 - v_2^2 - \cdots - v_n^2| = 1$.

We note that v above is basically the so-called four-velocity (of e.g. k). By the above lemma, the co-moving inertial observer is in intimate connection with the *four-velocity.* (The four-velocity of an observer k at point q is defined as the space-time displacement per unit of proper time along a straight-line approximation of the life-curve, see [18, pp.49,50].)

$\mathsf{Specrel} + \mathbf{Axg}$ is at the heart of the theory of accelerated observers, in some sense, cf. e.g. [2, chapter 8] together with [18, chapter 6]. Some interesting statements of relativity can be derived already from $\mathsf{Specrel} + \mathbf{Axg}$. Let $\mathbf{Ax3}^+$ be the extension of $\mathbf{Ax3}$ to all observers, i.e. $\mathbf{Ax3}^+$ says that $\mathsf{tr}_k(k) \subseteq \bar{t}$ for all observers k. Now, the *Twin Paradox* can be naturally formulated and proved in $\mathsf{Specrel} + \mathbf{Ax3}^+ + \mathbf{Axg}$, if we assume that our field F is the real line. Generalizing the last condition is up to future work. (Note that the *Twin Paradox* does involve accelerated observers.) Similar observation applies to the effect of gravity on clocks (also known as the *Tower Paradox*). For the time being, more concrete formulations, details etc. are available in [2] using some axioms extra to our present $\mathsf{Specrel} + \mathbf{Ax3}^+ + \mathbf{Axg}$. We conjecture that the axioms extra to $\mathsf{Specrel} + \mathbf{Ax3}^+ + \mathbf{Axg}$ can be either eliminated or significantly simplified. (Concerning the above effect of gravity on clocks, we note that this effect is at the heart of the "science-fiction"-like behaviour of certain black holes.)

Acknowledgements. We would like to thank János Makowsky and Ildikó Sain for helpful comments on an earlier version. Thanks go to Victor Pambuccian for helpful suggestions and pointers to the literature.

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