Vienna Circle and Logical Analysis of Relativity Theory

H. Andréka, J. X. Madarász, I. Németi, P. Németi and G. Székely

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1 INTRODUCTION

In this paper we present some of our school's results in the area of building up relativity theory (RT) as a hierarchy of theories in the sense of logic. We use plain first-order logic (FOL) as in the foundation of mathematics (FOM) and we build on experience gained in FOM. The main aims of our school are the following: We want to base the theory on simple, unambiguous axioms with clear meanings. It should be absolutely understandable for any reader what the axioms say and the reader can decide about each axiom whether he likes it. The theory should be built up from these axioms in a straightforward, logical manner. We want to provide an analysis of the logical structure of the theory. We investigate which axioms are needed for which predictions of RT. We want to make RT more transparent logically, easier to understand, easier to change, modular, and easier to teach. We want to obtain deeper understanding of RT.

Our work can be considered as a case-study showing that the Vienna Circle's (VC) approach to doing science is workable and fruitful when performed with using the insights and tools of mathematical logic acquired since its formation years at the very time of the VC activity. We think that logical positivism was based on the insight and anticipation of what mathematical logic is capable when elaborated to some depth. Logical positivism, in great part represented by VC, influenced and took part in the birth of modern mathematical logic. The members of VC were brave forerunners and pioneers.

Let's see what was available before or during the VC activity, and what was not available for the members of VC but available for us now. The VC activities in the strict sense fall into the period of 1922-1936. The beginning of intensive development of FOL coincides with this period. The elements of first-order language FOL (propositional logic, quantifiers) were available based on the works of Boole, Peirce, Schröder, Frege, Russel (roughly 1860-1910). However, the completeness and incompleteness theorems, compactness theorem, semantics for FOL, model theory, proof theory and definability theory were not there before VC. Many of these became available as works of people influenced by or working in VC. Here is a brief chronological order for some turning-points in this development: 1929 Gödel's completeness theorem, 1930 Tarski's decision method for the elementary theory of reals, 1931 Gödel's incompleteness theorem, 1933 Tarski mathematical definition of truth, 1936 Tarski concept of semantical consequence relation, definition of model theoretic semantics. Proof theory developed only later than around 1940, and model theory developed only later than around 1950. Beth's definability theorem was anticipated by Hans Reichenbach 1924 (motivated by RT), Tarski 1936, but became available only in 1952. The authors of the standard Model Theory book, Chang and Keisler, are both students of Tarski. Two chapters of model theory, the theory of semantics and the theory of definability are of great importance for our work. So what was/is FOL used for? Some dates showing the emergence of paradoxes in mathematics necessitating an axiomatic approach, experiments with formal languages to describe, found and unify mathematics and then all science are as follows: 1897 Burali-Forti Paradox, 1900 Hilbert's Program, 1901 Russel's Paradox, 1908 Zermelo's Axiom system for Set Theory, 1910 Russel-Whitehead Principia Mathematica, 1918 Löwenhein-Skolem "Paradox," 1922 Fraenkel's addition to Zermelo-Fraenkel Set theory, 1924 Tarski-Banach "Paradox," 1926 Tarski's axiomatization of Euclidean geometry in FOL, 1930 the addition of the Axiom of Regularity to Set Theory, beginning with 1935 the Bourbaki group's work formalizing and uniting mathematics, 1937 Tarski role of logic in scientific studies. Success story of FOM. Gödel-Bernays and von Neumann's Set Theory. We will see in this paper many of the above results used in our work. Gödel, Tarski, Reichenbach, Hilbert, Russel, and Einstein were all connected to VC in some way or other.

Why relativity theory? Logical positivism is a philosophy which holds that the only authentic knowledge is that based on observation, experience, experiment through formal logic. Einstein's relativity theory transforms space and time from being a priori Euclidean and absolute things to something which emerges from experience and experiments, thus claiming the subjects of space and time from the realm of metaphysics for science. Einstein's thought-experiments served to bring logic into the picture. Formalizing relativity theory in logical language was a primary interest in logical positivism, see Reichenbach's works. Relativity theory also led to Modern High-Precision Cosmology as a branch of hard-core physics (and not part of metaphysics). Since space-time is the arena in which most processes studied by modern science unfold, a logic based foundation for RT (the theory of space-time) might be a natural starting point for a foundation and unification of the whole of science (a VC goal [8], [29]).

Our group investigates a hierarchy of relativity theories, weaker and stronger theories. We not only give axiom systems and prove their completeness with respect to their intended models, but we also derive RT's main predictions, ask ourselves which axioms play the key role in their derivations, we make "reverse relativity" in analogy with "reverse mathematics," and we analyze the theories in many ways. In this paper we present three of our main axiom systems (i.e., theories) with just stating some of their most important properties. These three theories have the same language. We begin with introducing this language.

2 THE COMMON LANGUAGE OF THE THEORIES PRESENTED HERE

We will use FOL. FOL can be viewed as a fragment of natural language with unambiguous syntax and semantics. One of acknowledged traits of using FOL is that it helps eliminate tacit assumptions, one of VC's maxim. The most important decision in writing up an axiom system in FOL is to choose the vocabulary, or primitive symbols of our language, i.e., what objects and what relations between them will belong to the language we will use.

We want to talk about space and time as relativity theory conceives them. We will talk about space-time as experienced through motion. We represent motion as changing spatial location in time. We will call the entities that do the motion "test-particles." Sometimes, for using a shorter word, we will call them "bodies" but in reality they can be anything that move, e.g., they can be coordinate systems or electromagnetic waves, or light signals or centers of mass.[1] To talk about spatial locations and time we will use quantities

[1]Note about extended bodies: We concentrate on test-particles and regard test-particles as spatially point-like, i.e., of size zero. As far as we are aware of it, this idealization is harmless from the point of view of the goals of relativity theory. If we want to treat an extended body in our theory (as we do in the theory AccRel of accelerated observers), we represent it as a "cloud" of test-particles. This is consistent with the spirit of standard

arranged in a (space-time) coordinate system, and we will have a basic relation, the so-called worldview relation, which tells us which test-particles are present in which locations at which instants. We will think of the quantities as the real numbers (i.e., the number-line), so we will use a "less than" relation and two operations, addition and multiplication, on them. In this paper to axiomatize special relativity theory, we will use two more primitive notions, namely that of "inertial test-particles" and "light-signals" which we will simply call photons.[2]

To concretize what we said so far, let us consider the following two-sorted first-order language:

$$\{\, B, \mathsf{IB}, \mathsf{Ph}, Q, +, \cdot, <, \mathsf{W}\,\},$$

where B (test-particles or bodies) and Q (quantities) are the two sorts, IB (inertial bodies) and Ph (light signals or photons) are unary relation symbols of sort B, $\cdot$ and $+$ are binary function symbols and $<$ is a binary relation symbol of sort Q, and W (the worldview relation) is a 6-ary relation symbol of sort $BBQQQQ$. B and Q can be thought of as the physical and as the mathematical universes.

Atomic formulas $\mathsf{IB}(c)$ and $\mathsf{Ph}(p)$ are translated as "c is an inertial body," and "p is a photon," respectively. We use the worldview relation W to speak about coordinatization by translating $\mathsf{W}(o, b, x, y, z, t)$ as "observer o coordinatizes body b at space-time location $\langle x, y, z, t\rangle$," (i.e., at space location $\langle x, y, z\rangle$ and at instant t). We sometimes use the more intuitive expressions "sees" or "observes" for coordinatizes. We will use the letters, and their

physical worldview of regarding extended bodies as clouds of elementary particles.

[2]To talk about light-signals is not necessary for building up SR. One simple way of avoiding them is defining light-signal as anything that moves with "speed of light." There are deeper ways of avoiding the use of light-signals in building up relativity theory, see, e.g., [5, sec.5].

variants, o, b, p, m, k for variables of sort B, and the letters x, y, z, t and their variants for variables of sort Q. For easier readability, we will use $\bar{x}, \bar{y}$ for sequences of four variables x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4.

We have not introduced the concept of observers as a basic one because it can be defined as follows: an **observer** is nothing else than a body who "observes" (coordinatizes) some other bodies somewhere, this property can be captured by the following first-order formula of our language:

$$\mathsf{Ob}(o) \stackrel{def}{\Longleftrightarrow} \exists b\bar{x} \;\; \mathsf{W}(o, b, \bar{x});$$

and **inertial observers** can be defined as inertial bodies which are observers, formally:

$$\mathsf{IOb}(o) \stackrel{def}{\Longleftrightarrow} \mathsf{IB}(o) \wedge \mathsf{Ob}(o).$$

To abbreviate formulas of FOL we often omit parentheses according to the following convention. Quantifiers bind as long as they can, and $\wedge$ binds stronger than $\rightarrow$. For example, we write $\forall x \;\; \varphi \wedge \psi \rightarrow \exists y \;\; \delta \wedge \eta$ instead of $\forall x\big((\varphi \wedge \psi) \rightarrow \exists y(\delta \wedge \eta)\big)$.

3 AXIOMS OF SPECIAL RELATIVITY

Having specified the language, let us turn to the axioms of our first theory. This will be an axiom system for Special Relativity theory (SR).

AxField**:** The *quantity part* $\langle Q; +, \cdot, < \rangle$ is an ordered field.

For the FOL definition of linearly ordered field see, e.g., [9, p.41]; this is a formulation of some of the most basic properties of addition and multiplication of real numbers. One of these properties is that there is a unique neutral element for addition ($\exists z \forall x \, z + x = x$), we call this element z zero and we denote it with 0.

The next axiom simply states that each inertial observer assumes that it rests at the origin of the space part of its coordinate system. It also can be thought of as expressing that we identify a coordinate system (or reference frame) with a test-particle "sitting" at the origin.

AxSelf: Any inertial observer coordinatizes (observes) itself as "living on the time-axis," i.e., it coordinatizes itself at a coordinate point if and only if the space component of this point is the origin:

$$\forall oxyzt \;\; \mathsf{IOb}(o) \rightarrow \big(\mathsf{W}(o,o,x,y,z,t) \leftrightarrow x = y = z = 0\big).$$

Our next axiom is on the constancy of the speed of light. For convenience, we choose 1 for this speed. This choice physically means using units of distance compatible with units of time, such as light-year, light-second, etc.

AxPh: The speed of light signals is 1 and it is possible to "send out" a photon in any direction, according to any inertial observer:

$$\forall o\bar{x}\bar{x}' \;\; \mathsf{IOb}(o) \rightarrow \big(\exists p(\mathsf{Ph}(p) \wedge \mathsf{W}(o,p,\bar{x}) \wedge \mathsf{W}(o,p,\bar{x}')) \\ \leftrightarrow (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 = (x_4 - x_4')^2\big).$$

This is the most important axiom of SR, it is its "physical" axiom. Axiom AxPh is very well confirmed by experiments, such as the Michaelson–Morley experiment and its variants. The next axiom establishes connections between different coordinate systems. It expresses the idea that all observers "observe" the same outside reality.

AxEv: All inertial observers coordinatize the same "meetings of bodies:"

$$\forall oo'\bar{x} \;\; \mathsf{IOb}(o) \wedge \mathsf{IOb}(o') \rightarrow \exists \bar{x}' \;\; \forall b \;\; \mathsf{W}(o,b,\bar{x}) \leftrightarrow \mathsf{W}(o',b,\bar{x}').$$

We call "meetings of bodies" events. By our next axiom, we assume that inertial observers use the same units of measurement. This is only a "simplifying" axiom.

AxSymd**:** Inertial observers agree as to the spatial distance between events if these events are simultaneous for both of them, formally:

$$\forall o o' \bar{x}\bar{x}'\bar{y}\bar{y}' \ \mathsf{IOb}(o) \wedge \mathsf{IOb}(o') \wedge x_4 = y_4 \wedge x'_4 = y'_4 \wedge$$
$$\forall b \ \big(\mathsf{W}(o,b,\bar{x}) \leftrightarrow \mathsf{W}(o',b,\bar{x}')\big) \wedge \forall b \ \big(\mathsf{W}(o,b,\bar{y}) \leftrightarrow \mathsf{W}(o',b,\bar{y}')\big)$$
$$\rightarrow (x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2 = (x'_1-x'_2)^2+(y'_1-y'_2)^2+(z'_1-z'_2)^2.$$

Let us now introduce our axiom system of SR as the set of the axioms above:

$$\boxed{\mathsf{SpecRel} = \{\, \mathsf{AxField}, \mathsf{AxSelf}, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxSymd} \,\}}.$$

The reader is invited to check that all the axioms of SpecRel are simple, comprehensible and observationally oriented. In setting up an axiom system, we want the axioms be streamlined, economical, transparent and few in number. On the other hand, we want to have all the surprising, shocking, paradoxical predictions of RT as theorems (and not as axioms). We want to have the price-value ratio to be good, where the axioms are on the "cost"-side, and the theorems are on the "gain"-side.

Let us see what theorems we can prove from SpecRel. We will see that we can prove everything from our five axioms that "usual" SR can, but let us proceed more slowly. In the axioms we did not require explicitly, but it can be proved from SpecRel with the rigorous methods of FOL that inertial observers see each other move on a straight line, uniformly (covering the same amount of distance in the same amount of time). For a "fancy theorem" from "plain axioms," let us prove from SpecRel that "no inertial observer can move faster than light.” Below, $\vdash$ denotes derivability in one of FOL’s standard proof systems.

Theorem 1. (NoFTL) In an inertial observer m’s worldview, any inertial observer k moves slower than any light-signal p, i.e., if both k and p move from spatial locations $\langle x_1, x_2, x_3\rangle$ to $\langle y_1, y_2, y_3\rangle$, then for the observer k this trip took more time than for the photon p. Formally:

$$\mathsf{SpecRel} \vdash \forall mkp\bar{x}\bar{y}t \;\; \mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \wedge \mathsf{Ph}(p) \wedge$$
$$\mathsf{W}(m,k,\bar{x}) \wedge \mathsf{W}(m,p,\bar{x}) \wedge \mathsf{W}(m,k,\bar{y}) \wedge \mathsf{W}(m,p,y_1,y_2,y_3,t) \rightarrow y_4 > t.$$

For **proof** see, e.g., [22, Thm.3.2.13]. What the average layperson usually knows about the predictions of relativity is that “moving clocks slow down, moving spaceships shrink, and moving clocks get out of synchronism, i.e., the clock in the nose of a fast moving spaceship is late (shows less time) when compared with the clock in the rear.” See Figure 1. Let’s call these three predictions the “paradigmatic effects” of SR. Now, SpecRel implies all the paradigmatic effects quantitatively, too.[3] From this it follows that the so-called worldview transformations are Poincaré-functions, thus everything follows from our SpecRel what follows from “usual” special relativity theory.

Different observers may observe different spatial distance between the same two events. This is so in Newtonian Kinematics (NK), too. (For example, if I ate a sandwich and later drank a coffee on a train, these two events were at the same place according to me, but according to a coordinate system attached to Earth I ate the sandwich at Budapest and drank the coffee at Vienna.) However, in NK the time-difference between two events is the

[3]They follow from our next theorem. However, in our works we usually prefer proving the paradigmatic effects one-by-one, directly from the axioms of SpecRel because this illuminates or illustrates how we perform our conceptual analysis. These proofs can be found in, e.g., [4, sec.2.4].

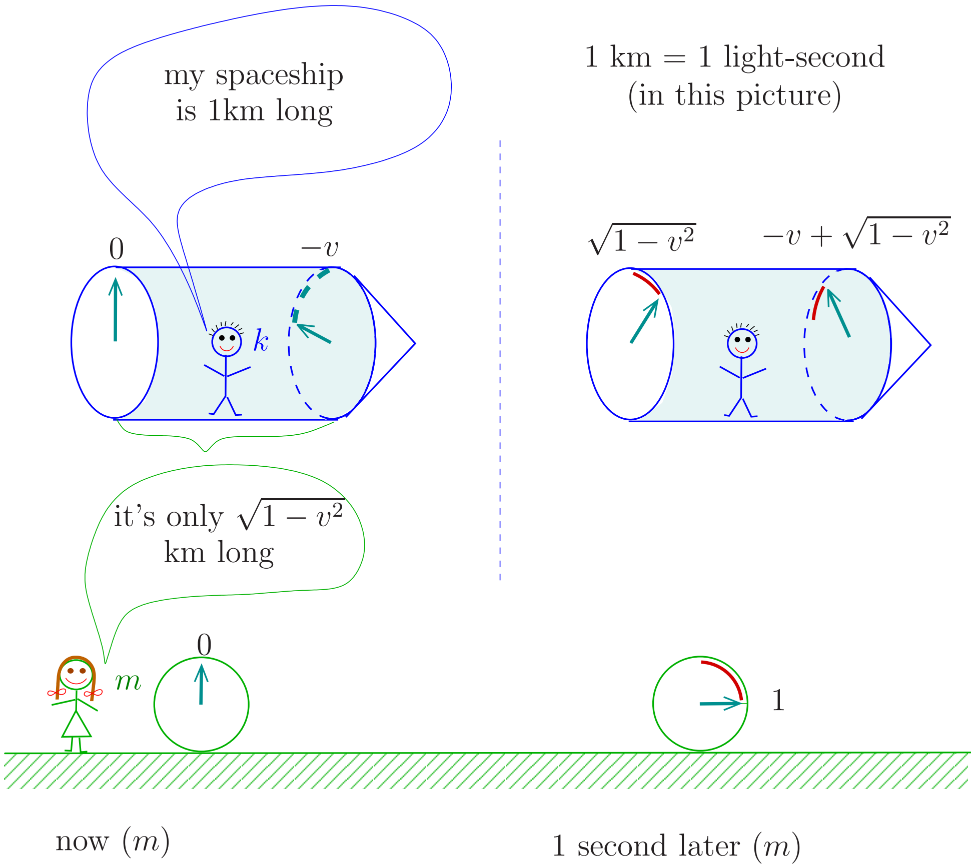

Figure 1: According to m, the length of the spaceship is $\sqrt{1-v^2}$ km, it is 1 km wide and tall, and the clocks in the nose show v less time than those in the rear, where v is the relative velocity of m and k. According to k, the length of the ship is 1 km, it is 1 km wide and tall, and the clocks in the nose and the ones in the rear all show the same time.

same for all observers, it is "absolute." According to the paradigmatic effects, in RT even the time-difference between two events depends on the state of motion of the observer! (The observer moving relative to m will observe less time passed between e and e' because his clock "slowed down.") In this respect, space and time in RT are "more alike" than in NK. Our next theorem states that a certain combination of spatial distance and time-difference is "absolute" in RT, too. The proof of this theorem can be found in, e.g., [4, p.650]. Let us define

$$\mu(\bar{x}, \bar{y}) := (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - (x_4 - y_4)^2.$$

Thus $\mu(\bar{x}, \bar{y})$ is squared spatial distance minus squared time-difference between events e and e' if these events took place at $\bar{x}$ and $\bar{y}$, respectively. This quantity is called (squared) *relativistic distance* between events e and e' (or squared Minkowski-distance between space-time locations $\bar{x}$ and $\bar{y}$). According to the next theorem, relativistic distance is "absolute" in RT. In RT relativistic distance plays the same role as absolute time in NK. Minkowski geometry is based on relativistic distance (in place of Euclidean distance).

Theorem 2.

$$\begin{aligned} \mathsf{SpecRel} \vdash \forall o o' \bar{x}\bar{x}'\bar{y}\bar{y}' \ \ \mathsf{IOb}(o) \land \mathsf{IOb}(o') \land \\ \forall b \ \ (\mathsf{W}(o, b, \bar{x}) \leftrightarrow \mathsf{W}(o', b, \bar{x}')) \land \forall b \ \ (\mathsf{W}(o, b, \bar{y}) \leftrightarrow \mathsf{W}(o', b, \bar{y}')) \\ \to \mu(\bar{x}, \bar{y}) = \mu(\bar{x}', \bar{y}'). \end{aligned}$$

According to the theorem above, relativistic distance between events is an "absolute" property. Clearly, any property defined from it is absolute, too. By the use of modern rigorous logic, it can be stated and proved that the properties definable from relativistic distance are the only absolute properties; and moreover all of SpecRel can be re-constructed from Minkowski geometry (i.e., from the "pseudo-metric" μ). How can one formulate such a statement rigorously in formal logic?

Definability theory is one of the most beautiful parts of modern logic, see [9], [21], [25], [30]. It is about investigating connections between theories formulated in FOL with completely different vocabularies (such as, e.g., Finite Set Theory based on the ϵ relation and Arithmetic based on $+$ and $*$). What would happen if we did not consider, e.g., the quantities and operations on them as primitives of the language? What happens if we are curious about where these primitive notions come from, if we want to give them "operational" meanings? What happens if we choose the so-called causality relation as the only primitive symbol of our language (as, e.g., in Robb [32], Mundy [27])? Can we compare then these theories, can we say that one of these is stronger or weaker than the other, or that two such theories express the same amount of "knowledge" about the world?

Definability theory is strongly related to relativity theory and to positivists ideas. In fact, its existence was initiated by Hans Reichenbach in 1924 [31]. Reichenbach in his works emphasized the need of definability theory and made the first steps in creating it. It was Alfred Tarski who later founded this branch of mathematical logic. Since then it developed to a well-used and powerful theory, in much extent due to the works of Michael Makkai.

Very briefly, the reason for the need of definability theory of logic in relativity theory, as explained by Reichenbach, is as follows. When one sets up a physical theory Th, one wants to use only so-called observational[4] concepts, such as, e.g., "meeting of two particles." While investigating the (observationally based) theory Th (such as our SpecRel), one defines new, so-called "theoretical" concepts, such as, e.g., "relativistic distance" μ. Some defined

[4]This observable/theoretical hierarchy is not perfectly well defined and is known to be problematic, but as Friedman [13] puts it, it is still better than nothing.

concepts then prove to be so useful that one builds a new theory Th' based on the most useful theoretical concepts, and investigates this new theory Th' in its own merits. The new theory Th' usually is simple, streamlined, elegant - built in such a way that we satisfy our aesthetic desires. This is the case with Minkowski geometry. The original theory Th contains its own interpretation, because we tried to use observational concepts. The physical interpretation of the new streamlined theory Th' is its connection with Th. The strongest relationship between two theories in different first-order languages is called definitional equivalence. When two theories are definitionally equivalent, in the rigorous sense of definability theory of FOL, the observational oriented theory Th can be recaptured completely from the theoretical-oriented streamlined theory Th' (and vice versa).

As examples, we can take our axiom system SpecRel for Th, and we can take Minkowski Geometry for Th'. Goldblatt [16, App.A] gave a complete FOL axiom system MG for Minkowski geometry. His theory is based on and is nicely analogous to Tarski's FOL axiomatization for Euclidean geometry (see, e.g., [37]). The definitional equivalence of our present "observationally-oriented" theory SpecRel and the FOL theory of Minkowski geometry MG given by Goldblatt is proved by Madarász [22, Chap.6.2].[5] It is noteworthy to mention that in this application, relativity theory contributed to definability theory once again: for the precise formulation of the equivalence of the two theories we had to elaborate a methodology for how to define new "entities" (such as "events") in addition to the old methods which are about how to define new relations on already existing entities (such as "observer"). This definitional equivalence of the two theories can also be expressed by saying that SpecRel is complete with respect to the Minkowskian model of SR

[5]For proving this equivalence, one has to add extensionality axioms for observers and light-signals to SpecRel, and one has to enrich MG with a "meter-rod."

generalized over ordered fields. Hence everything which can be formulated in our language and true in these Minkowskian models can be proved from our axiom system SpecRel. This is a kind of completeness theorem for the streamlined theory SpecRel with respect to Minkowskian Geometry as the intended model for SR.

Less tight relationships than definitional equivalence between theories are also very useful, these kinds of relationships are called interpretability and duality connections. For an illustration, let us turn to the question of where quantities and coordinate systems come from. The axiom system AxSR of James Ax [1] for SR is based on a first-order language that contains only two unary relation symbols P,S for "particles" and "signals" (corresponding to our "bodies" and "photons"), and two binary relation symbols T,R for "transmitting a signal" and "receiving a signal." One can give an interpretation of our FOL theory SpecRel in Ax's FOL theory AxSR (see [3, proof-outline of Thm.2.1]). This amounts to defining the primitive relations of the language of our SpecRel in terms of the primitives of Ax's AxSR, and then proving from AxSR the translated axioms of SpecRel as theorems. This is an interpretation in the sense of definability theory. Now, this interpretation also can be thought of as giving a kind of operational "definition" for how to set up "operationally" the coordinate systems appearing in SpecRel as primitives. The question of how to give algorithms for setting up coordinate systems in this context is treated in more detail and depth in Szabó [33].

Theories form a rich structure when we investigate their interconnections. Gödel's incompleteness theorem pointed already in the direction of investigating hierarchies of theories rather than single theories. (There is no "strongest" theory for the interesting subjects, there are only stronger and stronger theories.) Answering "why-questions," "reverse mathematics," modularizing our knowledge all point to the study of weaker and weaker theories, and also to studying the interpretations between theories (see [35]). Algebraic logic, developed by Tarski and his followers, is a branch of definability theory which establishes a duality between hierarchies of theories and between classes of algebras (cf., e.g., [18, Chap.4.3], [19], [25]). In modern approaches to logic, theories are considered as dynamic objects as opposed to the more traditional "eternally frozen" idea of theories. For approaches to the dynamic trend in mathematical logic cf., e.g., van Benthem [38], Gabbay [15], and [26]. This new "plurality of theories" or "hierarchy of small theories (as opposed to a single monolithic one)" approach can help realizing the central or essential VC-aims without the old stumble blocks of the original VC attempts. This is a wisdom gained from FOM, see [12].

In the rest of the paper we briefly indicate how to arrive to a transparent FOL axiomatization of general relativity from our SpecRel. By this we realize Einstein's original program formally and literally.

4 FIRST STEP TOWARD GR: EXTENDING THE THEORY TO ACCELERATED OBSERVERS

As a first step toward General Relativity theory (GR), we are going to extend our SpecRel theory with accelerated observers. By accelerated observer we mean any not necessarily inertial observer. Let us first note that none of the axioms of SpecRel speaks about noninertial observers.

Since in the language we have already introduced the concept of arbitrary observer, the only thing we have to do is to assume some axioms about them. Our key axiom to assume about arbitrary observers is the following:

AxCmv: At each moment of its life, any observer coordinatizes ("sees") the nearby world for a short while in the same way as some inertial observer does.

For precise formulation of this axiom in the spirit we formulated the axioms of SpecRel see [4], [23], [35]. Let AccRel^- be the axiom system consisting of AxCmv and all the axioms of SpecRel.

Let us see how strong our theory AccRel^- is. To test its strength we are going to investigate whether the Twin Paradox (TwP) and the gravitational time dilation ("gravity causing slow time") are provable from it.

According to TwP, if a twin makes a journey into space (accelerates), he will return to find that he has aged less than his twin brother who stayed at home (did not accelerate). However much surprising TwP is, it is not a contradiction. It is only a fact showing that the concept of time is not as simple as it seems at first.

A more optimistic consequence of Twp is the following. Suppose you would like to visit a distant galaxy 200 light years away. You are told it is impossible because even light travels there for 200 years. But you do not despair, you accelerate your spaceship nearly to the speed of light. Then you travel there in 1 year subjective time. When you arrive back, you aged only 2 years. So you are happy, but of course you cannot tell the story to your brother, who stayed on Earth. Alas you can tell it to your grand-...-grand-children only.

In the FOL language introduced in this paper we can formulate TwP, see [23], [35]. Let us denote the formulated version of TwP as TwP.

AccRel^- is not yet strong enough to imply TwP. One would think that this is so because we did not state enough properties of the real numbers for speaking about curved lines. However, even assuming $\mathsf{Th}(\mathbb{R})$, i.e., all the FOL formulas valid in the real numbers, together with AccRel^- is not sufficient to prove TwP, see [23], [35]:

Theorem 3. $\mathsf{AccRel}^- \cup \mathsf{Th}(\mathbb{R}) \nvdash \mathsf{TwP}$.

We note that the above theorem is a theorem stating that one cannot prove TwP from $\mathsf{AccRel}^- \cup \mathsf{Th}(\mathbb{R})$, it is not only the case that we are not "clever enough" to find a proof but there is none. Its proof goes via using the completeness theorem of FOL, namely we find a model in which all the formulas in $\mathsf{AccRel}^- \cup \mathsf{Th}(\mathbb{R})$ are true, but in which TwP is not true.

This theorem states that even assuming every first-order formula which is true in $\mathbb{R}$ is not enough for our purposes. At first sight this result suggests that our programme of FOL axiomatization of GR breaks down at the level of TwP. It would be depressing if we were not able to keep our axiomatization within FOL, because there are weighty methodological reasons for staying within it, see, e.g., [5, Appendix], [35, sec.11]. However, we are saved: in our language there is a FOL axiom scheme (nice set of axioms) called IND which is sufficient for our purposes. Axiom scheme IND expresses that every nonempty and bounded subset of the quantities which is parametrically definable in our language has a least upper bound (i.e., supremum). IND is a first-order logic approximation of the second-order logic continuity axiom of the real numbers, and it belongs to the methodology developed in FOM and in reverse mathematics that AxField strengthened with IND are strong enough for a FOL treatment of areas involving the real numbers.

Together with this scheme AccRel^- implies TwP, i.e., the following theorem can be proved, see [23], [35]:

Theorem 4. $\mathsf{AccRel}^- \cup \mathsf{IND} \vdash \mathsf{TwP}$.

How can a FOL axiom scheme be stronger than all the FOL formulas valid in $\mathbb{R}$? The answer is that IND is formulated in a richer language than that of the reals, hence it can state more than the whole FOL theory of $\mathbb{R}$. If we assume IND only for formulas in the language of ordered fields, we get an axiom schema equivalent to $\mathsf{Th}(\mathbb{R})$, see [35].[6] Let us now introduce our axiom system for accelerated observers as:

$$\boxed{\mathsf{AccRel} = \mathsf{SpecRel} \cup \{\mathsf{AxCmv}\} \cup \mathsf{IND}}.$$

Let us continue with the gravitational time dilation. By Einstein's equivalence principle, we can also formulate the statement "gravity causes slow time" (usually called "gravitational time dilation" GTD) in our language. Moreover, the formulated version of this statement is provable from the theory AccRel, see [24], [35]. The AccRel formulation of GTD basically says that in any accelerated spaceship the clocks in the rear run slower than those in the nose. (The effect is increasing with increasing acceleration. Moreover, it approaches infinity as the acceleration does.) So we are able to derive nontrivial predictions about gravity before we have introduced any axiom system of GR.

The theory AccRel is halfway between SR and GR. Einstein used a non-formalized version of AccRel as a heuristic in introducing GR, e.g., when he made predictions about the influence of gravitation on the propagation of light [10], [11, §§18-22].

5 SECOND STEP: "EMANCIPATING" NONINERTIAL OBSERVERS

We are going to modify the axioms of SpecRel and AxCmv one by one and get an axiomatic theory of general relativity. The modification consists of "eliminating the privileged class of inertial reference frames," which was a central idea of Einstein's, see [11, §§18-22], [13]. We replace each axiom of SpecRel by a new one which does not speak about inertiality but otherwise the content of which tries to approximate that of the old one. All the new

[6] Actually, the restriction of IND to fields $\langle Q, +, *, < \rangle$ coincides with Tarski's FOL version of Hilbert's continuity axiom for geometry, cf. [16, p.71, axiom B5].

axioms will be motivated by our theory AccRel. Roughly, each axiom of AccRel will be replaced by a "generalized" version which does not mention inertiality and which is still in the spirit of AccRel.

The generalized version of AxSelf is the following:

AxSelf$^-$: An observer coordinatizes itself on a subset of the time axis:

$$\forall oxzyt \; \mathsf{W}(o, o, x, y, z, t) \to x = y = z = 0.$$

The modified version AxEv$^-$ of AxEv contains the following two statements: (1) any observer coordinatizes the events in which it was observed by some other observer, and (2) if observer o coordinatizes an event which is coordinatized by observer o', then o also coordinatizes the events which are near this event according to o'. This can be summarized as follows:

AxEv$^-$: Any observer coordinatizes the events in which it was observed; and the domains of worldview transformations are open.

The modified versions of AxPh and AxSymd are achieved by localizing and generalizing them, i.e., we get the modified versions by restating these axioms only in infinitesimally small neighborhoods, but for every observer. The idea that "GR is locally SR" also goes back to Einstein. Our symmetry axiom AxSymd has many equivalent versions, see [5, sec.s 2.8, 3.9, 4.2]. We can localize any of these versions and use it in a FOL axiom system for GR. For aesthetic reasons here we localize AxSymt, the version stating that inertial observers see each others' clocks behave the same way. So AxPh$^-$ and AxSymt$^-$ are the formalized versions of the following statements:

AxPh$^-$: The instantaneous velocity of photons is 1 in the moment when they "meet" the observer who coordinatizes them, and any observer can send out photons in any direction with this instantaneous velocity.

AxSymt^-**:** Meeting observers see each other's clocks behaving the same way, at the event of meeting.

For formulation of these axioms and the corresponding concepts in our first-order language, see [35].

Now all the four axioms of SpecRel are modified according to the above requirements. Strictly following these guidelines, AxCmv^- would state that the worldview transformation between observers are differentiable in their meeting-point. To avoid baroque, we state simply differentiability of the worldview transformations. A natural generalization is n-times differentiability (which is natural to consider in view of our wanting to speak about location, speed and acceleration). Each axiom of this series of potential axioms can be formulated in the language above by the techniques used in [4], [23], [35].

$\mathsf{AxDiff_n}$**:** The worldview transformations are n-times differentiable functions.

Let us introduce the following simple axiom systems for general relativity:

$$\boxed{\mathsf{GenRel_n} := \left\{ \mathsf{AxField}, \mathsf{AxSelf}^-, \mathsf{AxPh}^-, \mathsf{AxEv}^-, \mathsf{AxSymt}^-, \mathsf{AxDiff_n} \right\} \cup \mathsf{IND}}.$$

The following theorem illustrates that our axiom system $\mathsf{GenRel_n}$ captures the n-times differentiable standard models of usual GR well.

Lorentzian manifolds are the intended models of GR, much the same way as Minkowski geometry was the intended model of SR. Roughly, a Lorentzian manifold is a geometry which at every of its points locally looks like the Minkowski geometry, cf., e.g., [39, p.23].

Theorem 5. $\mathsf{GenRel_n}$ is complete with respect to the n-times differentiable Lorentzian manifolds over real-closed fields.

There are many interesting GR space-times, black holes, worm-holes, time-warps, etc. The physical relevance of these so called exotic space-times increases with time. For instance, there is a rapidly growing number of experimental evidence for huge slowly rotating black holes, which are the simplest examples of time-warps. By Theorem 5 even the most exotic model of GR is also a model of our GenRel theory. Hence, within GenRel we can investigate the properties of these exotic models.

To ensure that we can do indeed physics in the framework of GenRel_n ($n \geq 3$) we defined in [4], [35] the notion of time-like geodesics in terms of GenRel. These serve as world-lines of inertial bodies. So, though we abandoned inertial observers as primitives, inertial motion becomes accessible/definable as a derived notion (in terms of the primitives of GenRel). Space-time curvature is defined from geodesics the usual way. So, in particular, the outcomes of experiments involving inertial motion can be predicted (e.g., computing the trajectories of bullets or photon geodesics) on the basics of the new, streamlined theory GenRel in a purely logical way.

6 Concluding remarks

As it was the case with SpecRel, cf., [2]-[6], having obtained the streamlined axiomatization GenRel and its completeness for "usual" GR is only a first step towards a logic based conceptual analysis of GR, its predictions, alternatives or variants, answering the why-questions in a spirit which is a natural continuation of the VC programme.

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References

[1] Ax, J., The elementary foundations of spacetime, *Foundations of Physics* 8, 7-8 (1978), pp.507-546.

[2] Andai, A., Andréka, H., Madarász, J. X. and Németi, I., Visualizing some ideas about Gödel-type rotating universes, in: M. Scherfner and M. Plaue, eds., *Gödel-type spacetimes: history and new developments*, Springer, to appear. arXiv:0811.2910v1.

[3] Andréka, H., Madarász, J. X. and Németi, I., Logical axiomatizations of space-time. Samples from the literature, in: A. Prékopa and E. Molnár, eds., *Non-Euclidean geometries*, volume 581 of *Mathematics and Its Applications*. Springer, 2006, pp.155-185.

[4] Andréka, H., Madarász, J. X. and Németi, I., Logic of space-time and relativity theory, in: M. Aiello, I. Pratt-Hartmann and J. van Benthem, eds., *Handbook of Spatial Logics*. Springer, 2007, pp.607-711.

[5] Andréka, H., Madarász, J. X. and Németi, I., *On the logical structure of relativity theories*. E-book, Alfréd Rényi Institute of Mathematics, Budapest, 2002. With contributions from Andai, A., Sági, G., Sain, I. and Tőke, Cs. http://www.math-inst.hu/pub/algebraic-logic/Contents.html. 1312 pp.

[6] Andréka, H., Madarász, J. X., Németi, I. and Székely, G., Axiomatizing relativistic dynamics without conservation postulates, *Studia Logica* 89, 2 (2008), pp.163-186.

[7] Andréka, H., Németi, I. and Wüthrich, C., A twist in the geometry of rotating black holes: seeking the cause of acausality, *General Relativity and Gravitation* 40, 9 (2008), pp.1809-1823.

[8] Carnap, R., Frank, P., Hahn, H., Neurath, O., Joergensen, J. and Morris, C. (eds)., *Unified Science*. These works are translated in *Unified Science: The Vienna Circle Monograph Series Originally Edited by Otto Neurath*, Kluwer, 1987.

[9] Chang, C. C. and H. Keisler, J., *Model theory*. North-Holland 1973, 1977, 1990.

[10] Einstein, A., Über den Einfluss der Schwercraft auf die Ausbreitung des Lichtes, *Annalen der Physik* 35, 10 (1911), pp.898-908.

[11] Einstein, A., Über die Spezielle und die Allgemeine Relativitätstheorie, *Verlag von F. Vieweg and Son, Braunschweig*, 1921.

[12] Friedman, H., On foundational thinking 1, *Posting in FOM (Foundations of Mathematics)*, Archives www.cs.nyu.edu, January 20 2004.

[13] Friedman, M., *Foundations of Space-Time Theories. Relativistic Physics and Philosophy of Science*, Princeton University Press 1983.

[14] Friedman, M., *Reconsidering logical positivism*, Cambridge University Press 1999.

[15] Gabbay, D. M., *Labelled Deductive Systems*, Clarendon Press, 1996.

[16] Goldblatt, R., *Orthogonality and spacetime geometry*, Springer-Verlag, 1987.

[17] Gödel, K., An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation, *Reviews of Modern Physics* 21, 3 (1949), pp.447-450.

[18] Henkin, L., Monk, J. D., and Tarski, A., *Cylindric Algebras Part II*, North-Holland, 1985.

[19] Henkin, L., Monk, J. D., Tarski, A., Andréka, H., and Németi, I., *Cylindric Set Algebras*, Lecture Notes in Mathematics Vol 883, Springer Verlag, 1981.

[20] Hirsch, R., Einstein: Logic, Philosophy, Politics. Abstract for talk in "Logic in Hungary, 2005", Budapest, Hungary August 5-10, 2005. http://atlas-conferences.com/cgi-bin/abstract/caqb-48

[21] Hodges, W., *Model Theory*, Cambridge University Press, 1993.

[22] Madarász, J. X., *Logic and Relativity (in the light of definability theory).* PhD thesis, Eötvös Loránd Univ., Budapest, 2002. http://www.math-inst.hu/pub/algebraic-logic/diszi0226.pdf.gz

[23] Madarász, J. X., Németi, I. and Székely, G., Twin paradox and the logical foundation of relativity theory, *Foundations of Physics* 36, 5 (2006), pp.681-714.

[24] Madarász, J. X., Németi, I. and Székely, G., First-order logic foundation of relativity theories, in: D. M. Gabbay, S. Goncharov and M. Zakharyaschev, eds., *Mathematical problems from applied logic II.* Springer, 2007, pp.217-252.

[25] Makkai, M., *Duality and definability in first order logic*, Memoirs of the American Mathematical Society No 503, Providence, Rhode Island, 1993.

[26] Mikulás, Sz., Németi, I. and Sain, I., Decidable Logics of the Dynamic Trend, and Relativized Relation Algebras. In: L. Csirmaz, D. M. Gabbay and M. de Rijke, eds., *Logic Colloquium'92*, Studies in Logic, Language and Computation, CSLI Publications, 1995. pp.165-175.

[27] Mundy, B., Optical axiomatization of Minkowski space-time geometry, *Philosophy of Science* 53, 1 (1986), pp.1-30.

[28] Németi, I. and Dávid, Gy., "Relativistic computers and the Turing barrier", in: *Applied Mathematics and Computation* 178, 1 (2006), pp.118-142.

[29] Neurath, O., Carnap, R. and Morris, C. (eds), *Foundations of the Unity of Science.* The university of Chicago Press, Series: Foundations of the Unity of Science: Toward an International Encyclopedia of Unified Science. Vol.s 1 and 2. 1971.

[30] Pambuccian, V., Alexandrov-Zeeman type theorems expressed in terms of definability, *Aequationes Mathematicae* 74, 3 (2007), pp.249-261.

[31] Reichenbach, M., *Axiomatization of the theory of relativity*, Univeversity of California Press, Berkeley, 1969. Translated by M. Reichenbach. Original German edition published in 1924.

[32] Robb, A. A., *A Theory of Time and Space.* Cambridge University Press, 1914.

[33] Szabó, L. E., Empirical foundation of space and time. In: M. Suárez, M. Dorato and M. Rédei, eds., *EPSA; Epistemology and Methodology: Launch of the European Philosophy of Science Association Proceedings of the First Conference of the European Philosophy of Science Association, Madrid, Spain, November 14-17, 2007*, Springer, 2009.

[34] Székely, G., On why-questions in physics, in this volume.

[35] Székely, G., *First-Order Logic Investigation of Relativity Theory with an Emphasis on Accelerated Observers*, PhD thesis, Eötvös Loránd Univ., Budapest, 2009.

[36] Tarski, A., What is elementary geometry?, In: L. Henkin, P. Suppes and A. Tarski, eds., *The axiomatic method with special reference to geometry and physics*, North-Holland, 1959, pp.16-29.

[37] Tarski, A., and Givant, S. R., Tarski's system of geometry, *Bulletin of Symbolic Logic*, 5, 2 (1999), pp.175-214.

[38] van Benthem, J. F. A. K., *Exploring logical dynamics*, Sudies in Logic, Language and Information, CSLI Publications, 1996.

[39] Wald, R. M., *General Relativity*, The University of Chicago Press, 1984.

Alfréd Rényi Institute of Mathematics
of the Hungarian Academy of Sciences
Budapest P.O. Box 127, H-1364 Hungary.
andreka@renyi.hu, madarasz@renyi.hu, nemeti@renyi.hu, nemetip@gmail.com, turms@renyi.hu