New Challenges in the Axiomatization of Relativity Theory[1]

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Abstract: **Einstein's theory of relativity not just had but still has a great impact on science. It has an impact even on military sciences, e.g., via GPS technology (which cannot exist without relativity theory). Any theory with such an impact is also interesting from the point of view of axiomatic foundations. The aim of this paper is to outline the new challenges of the axiomatic approach to relativity theory developed by our research group led by Hajnal Andréka and István Németi.**

1. Cutting edge engineering based on the two theories of relativity

Einstein's formulated his theory of special relativity in 1905 a decade later he generalized special relativity and introduced his theory of general relativity in 1915. Even Einstein's special theory of relativity radically changed our way of thinking about space and time, because among other things it states that there are no such things as observer independent concepts of time and space. However, the two theories of relativity not just had a great impact on our way of thinking about space and time but on engineering sciences and even on our every day life.

GPS technology is a famous cutting edge technology of today, which greatly depends on the two theories of relativity, which were theories of pure basic research a hundred years ago.

It depends on special relativity because according to special relativity relatively moving clocks slow down and a typical NAVSTAR GPS satellite orbits around the Earth with speed 3 km/s, see [1]. According to special relativity, if a clock is moving with relative speed v, it slows down by factor $\sqrt{1-v^2/c^2}$, where c = 299792 km/s is the speed of light in vacuum. So the clock of a typical GPS satellite slows down by 4 μs every day.

It also depends on general relativity because of the gravitational potential difference between the ground level and the level of the GPS satellites. According to general relativity gravitation slows clocks down. Therefore, the clocks of GPS satellites are faster because they are farther from the source of gravitation. The ratio of the time differences between two clocks at distance R_0 and R_1 from the center of a planet of mass M is given by the following formula:

$$\frac{\Delta T_1}{\Delta T_0} = \sqrt{\frac{1-\frac{2GM}{R_1c^2}}{1-\frac{2GM}{R_0c^2}}},$$

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where G is the gravitational constant and *c* is the speed of light, see section 9.4 of [10]. A typical NAVSTAR GPS satellite orbits around the Earth at altitude 21000 km, see [1]. Therefore, the clocks of GPS satellites are 46 μs ahead of time every day because of this effect.

Consequently, GPS satellites have to make 42 μs correction every day because of these two effects of relativity theory. 42 microsecond correction seems to be a tiny amount at first sight. However, it is more than 10 km in distance, which is a huge correction in navigation. Obviously, without these relativistic corrections, the GPS navigation system would be completely useless.

GPS technology has a great many applications in areas ranging from agriculture to architecture through aviation. So via GPS technology, relativity theory has an effect even on our everyday life.

Application of relativity theory in GPS technology is a good example to that something which was pure basic research 100 years ago can be a fundamental basis of todays cutting edge engineering technology.

2. Why do we axiomatize relativity theories?

Relativity theory had and still has a great impact on several areas of science. Any theory with such an impact is also interesting from the point of view of axiomatic foundations. Therefore, it is not surprising that there are several axiomatizations of both special and general relativity in the literature, see, e.g., the references of [3], [13]. Some of the axiomatizations are based on mathematical logic or even first-order logic, which is considered the best logic for logical foundations, see, e.g., [4], [13]. However, as far as we know, all of them stops at presenting the axioms and none of them goes further doing a logical and conceptual analysis of the theory and none of them tries to connect the axioms of general relativity to that of special relativity.

A novelty in our approach is that we try to keep the transition from special relativity to general relativity logically transparent and illuminating. We “derive” the axioms of general relativity from that of special relativity in two natural steps. First we extend our axiom system of special relativity to accelerated observers. Then we eliminate the differences between inertial and accelerated observers in the level of axioms, see [5].

Another thing why relativity theory is interesting from the point of view of logic is that it has many models with rather fancy and surprising properties, e.g., black holes, Gödel's rotating universe, worm holes, etc. Several of these interesting spacetimes contain closed timelike curves (possibility of time-travel) another interesting class of spacetime models makes hypercomputation possible, i.e., in these spacetimes it is possible that a programmer may wait only a couple of hours for the result of an infinitely long computation, see, e.g., [7]. We are interested in the logical investigation of these spacetimes not because we want to build time-machines or hypercomputers in the near future, but because we would like to understand these spacetimes from the point of view of logic, e.g., the possible resolution of the paradoxes risen by them or the feedback that spacetime theory could give to logic by “deciding” some undecidable questions (such as the consistency of mathematics) by hypercomputers.

Axiomatic foundation of physical theories is important not only because they lead to a deeper logical understanding of the theory but also because in physics the role of the axioms (statements that we assume without proofs) is more fundamental than in mathematics. That is why we aim to axiomatize relativity theory using simple, comprehensible and logically transparent axioms only.

In some sense the axioms are only our beliefs on which we base our theories. That is why, we aim to prove all the surprising or unusual predictions of spacetime theories from a few convincing axioms. For example, the statement “no observer can move faster than light” is a theorem in our approach and not an axiom, see, e.g., [5].

Some of the questions we study to clarify the logical structure of relativity theory are:

- What is believed and why?
- Which axioms are responsible for certain predictions?
- What happens if we discard some axioms?
- Can we change the axioms and at what price?

Let us note here that a similar kind of axiomatic investigation led to the discovery of the famous Bolyai-Lobachevsky Geometry; namely the investigation of the logical connection of the axiom of parallels and the rest of the assumptions. This is a good example showing that basic axiomatic investigations can lead to discoveries of new, interesting and physically relevant theories.

3. An axiom system of special relativity

To axiomatize any theory, first we have to fix a language (set of basic concepts) which will be used to formulate our axioms. Here we will use the following two-sorted[2] first-order logic language:

$$\{ Q, +, *, <; B, IOb, Ph; W \},$$

where Q is the sort of quantities and B is the sort of bodies; * and + (multiplication and addition) are binary function symbols and < (ordering) is a binary relation symbol of sort Q; IOb (inertial observers) and Ph (light signals or photons) are unary relation symbols of sort B; and W (the worldview relation) is a 6-ary relation symbol of the type BBQQQQ. Relations IOb(o) and Ph(b) are translated as “o is an inertial observer,” and “b is a light signal,” respectively. The worldview relation W is used to speak about coordinatization by translating W(o,b,x,y,z,t) to natural language as “observer o coordinatizes body b at spacetime location (x,y,z,t),” (that is, at space location (x,y,z) and at instant t).

Of course, this choice of basic concepts is only one of the many possibilities. One important method of conceptual analysis of relativity theory is axiomatizing it choosing different sets of basic concepts and then logically comparing these different axiomatizations. These kinds of investigations also lead to a deeper understanding of relativity theories. For investigation of the logical connections between axiom systems of special relativity in the language above and

2 That our theory is two-sorted means only that there are two types of basic objects (bodies and quantities) as opposed to, e.g., set theory where there is only one type of basic objects (sets).

in the language of Minkowski geometry, see [11].

Because of lack of space, here we do not present the formalized versions of our axioms just their informal readings. However, all the axioms can be formalized in the simple language above, see, e.g., [5].

Here we will base special relativity on the light axiom which is a consequence of Einstein's two original (non-formalized) postulates, see [9].

- ***AxPh:*** For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction everywhere.

Axiom ***AxPh*** is well supported by experiments, such as the Michelson-Morley experiment. Moreover, it has been continuously tested ever since then. Nowadays it is tested by GPS technology.

To make the statement of ***AxPh*** unambiguous we have to state explicitly what we assume for the structure of quantities which ensures that the concepts of distance, speed, etc. are meaningful and behave as we desire them to. Therefore, we assume some usual properties of addition + and multiplication * true for real numbers by the next axiom.

- ***AxFd:*** The quantity part (Q, +, *, <) is a Euclidean ordered field, i.e., it is an ordered field in the sense of algebra in which every positive number has a square root.

Our next axiom, called the event axiom, is also needed to make the light axiom meaningful.

- ***AxEv:*** All inertial observers coordinatize the same set of events.

This axiom is very natural and tacitly assumed in the non-axiomatic approaches to special relativity, too.

In principle, we do not need more axioms for capturing the essence of special relativity, but let us introduce two more simplifying ones. We could leave them out without losing the essence of our theory but they make life easier.

- ***AxSf:*** Any inertial observer sees himself on the time axis.

The role of axiom ***AxSf*** is nothing more than that it makes easier to speak about the motion of reference frames via the motion of their time axes. Identifying the motion of reference frames with the motion of their time axes is a standard simplification in literature. ***AxSf*** is a way to formally capture this simplifying identification.

Our last axiom is a symmetry axiom saying that all inertial observers use the same units of measurements.

- ***AxSm:*** Any two inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them. Furthermore, the speed of light is 1:

Let us see how axiom ***AxSm*** states that “all inertial observers use the same units of measurements.” That “the speed of light is 1” means only that observers are using units measuring time distances compatible with the units measuring spatial distances, such as light years or light seconds. The first part of axiom ***AxSm*** means that different observers use the same unit measuring spatial distances. This is so because, if two events are simultaneous for both observers, they can measure their spatial distance and the outcome of their measurements are the same if and only if the two observers are using the same units to measure spatial distances.

Our axiom system for special relativity contains these five simple axioms only:

SpecRel = { ***AxFd, AxPh, AxEv, AxSf, AxSm*** **}**.

In an axiom system, the axioms are the “price” we pay, and the theorems are the “goods” we get for them. Therefore, we strive for putting only simple, transparent, easy-to-believe statements in our axiom systems. We want to get all the hard-to-believe predictions as theorems. For example, we prove from **SpecRel** that it is impossible for inertial observers to move faster than light relative to each other, see [5].

The five simple axioms of **SpecRel** perfectly capture the kinematics of special relativity. This is so because, it implies that the transformations connecting the worldviews of different inertial observers (reference frames) are Poincaré transformations, see, e.g, [2], [3]. An important corollary of this theorem is that all the predictions (concerning kinematics) of the official version of special relativity are provable from the five simple axioms of **SpecRel**. So **SpecRel** implies that “relatively moving clocks get out of synchronism,” “relatively moving clocks slow down by the Lorentzian time dilation factor,” etc. Nevertheless, in our approach the direct proofs of these predictions from the axioms are also important because they show the roles of the particular axioms. For direct proofs of these predictions from the axioms, see, e.g., [2].

If we would like to capture more from special relativity (not only the kinematics), we have to add new basic concepts and assume axioms for these concepts. For example, in the case of relativistic dynamics this kind of extension of **SpecRel** is already started in [6], where we have introduced an axiom system **SpecRelDyn** containing axioms formulated in the spirit of the axioms of **SpecRel**. The axioms of **SpecRelDyn** are strong enough to imply the relativistic formula connecting the rest and the relativistic masses and to give a support to Einstein's famous insight $E=mc^2$.

4. An axiom system of general relativity

Using the same set of basic concepts we extend **SpecRel** to a theory of accelerated observers called **AccRel**. Our key axiom on accelerated observers is the following:

- ***AxCmv:*** At each moment of his life, every accelerated observer sees (coordinatizes) the nearby world for a short while in the same way as an inertial observer does.

For the formalized version of axiom ***AxCmv***, see [13]. **AccRel** is **SpecRel** extended with ***AxCmv*** and some auxiliary axioms:

AccRel = SpecRel ∪ { ***AxCmv, AxEv-, AxSf-, AxDf, AxCont*** **}**.

This extension (**AccRel**) can be considered as a first step toward providing an axiom system of general relativity, which is only one natural step from **AccRel**. The only thing we have to do with **AccRel** to get an axiom system of general relativity (**GenRel**) is eliminating the differences between inertial and accelerated observers in the level of axioms. The natural idea making all the observers equal in the level of basic assumptions goes back to Einstein, see [8]. The elimination of the differences between inertial and accelerated observers is done by searching consequences of **AccRel** corresponding to each axiom of **SpecRel** which are true for all the observers. After this process, we get generalized (localized) versions of the axioms of **SpecRel** assumed for all the observers (and not just for inertial observers).

Let us now see the axioms of **GenRel**. Its first axiom is the localized version of the light axiom of **SpecRel**.

- ***AxPh-:*** The velocity of photons an observer "meets" is 1 when they meet, and it is possible to send out a photon in each direction where the observer "stands."

The second axiom is the localized version of the event axiom of **SpecRel** stating only that, if an observer is seen in an event, he hast to encounter that event.

- ***AxEv-:*** If an observer is seen in a certain event, he also has to see this event.

For practical reasons, we localize an equivalent version of the symmetry axiom of **SpecRel**.

- ***AxSm-:*** Meeting observers see each other's clocks behaving in the same way.

The next axiom is the localized version of the self axiom of **SpecRel**.

- ***AxSf-:*** Any observer sees himself in an interval of the time axis.

In **SpecRel,** it was a theorem that the worldview transformations are affine transformations. In our axiom system for general relativity we assume the localized version of this theorem as an axiom.

- ***AxDf:*** The worldview transformations have affine approximations at each point of their domains (i.e., they are differentiable).

In **SpecRel** the simple assumption ***AxFd*** was perfectly enough, surprisingly even in **AccRel** we need to assume more about the quantities, see [13]. The assumption we need is the following first-order logic approximation of the supremum axiom of real numbers.

- ***AxCont:*** Every definable, bounded and nonempty subset of the quantities has a supremum.

Our axiom system **GenRel** contains these seven basic assumptions only.

GenRel = { ***AxFd, AxPh-, AxEv-, AxSf-, AxSm-, AxDf, AxCont*** **}**.

Let us note that there is no axiom requiring the existence of observers in **GenRel**. Sometimes for example, if we would like to investigate geodesics, we require the existence of several observers by the following assumption:

- ***AxCompr:*** For any parametrically definable timelike curve in any observer's worldview, there is another observer whose worldline is the range of this curve.

Axiom system ***GenRel*** together with ***AxCompr*** is strong enough to capture Einstein's field equations, see. [5].

5. New challenges in our axiomatic approach

We have laid down the first brick of the logical and conceptual analysis of relativity theories, i.e., we have presented axiom systems for special and general relativity. Moreover, we have "derived" the axioms of general relativity from that of special relativity. However, this is only the first step in the logical investigation of spacetime theories.

This first step generates many new and interesting questions, which is important because science not only advances by new answers but also by new questions. Some of these new questions are:

- Which axioms are responsible for a certain theorem?
- How are the possible axioms/axiomatizations related to each other?
- How can these axiomatizations be extended, e.g., towards Quantum Theory?
- How are the independent statements of our axiomatizations related to each other?

Let us note here that investigating these kinds of questions is not just a beloved game of logicians. The question which have led to the discovery of hyperbolic geometry was also a question of logical independence of some axioms, namely the question whether the axiom of parallels is independent or not from the other axioms of Euclid. The logical fact that the axiom of parallels is independent from the rest of the axioms of Euclid made the discovery of Bolyai and Lobachevsky possible.

Investigating the axioms that are responsible for a certain theorem or prediction is a way to answer why-type questions of relativity theory. Fore more details on the subject of why-type questions in relativity, see [4], [12].

The investigation of these questions may give us a better and deeper logical understanding of the fundamental concepts and assumptions of spacetime theories and the logical connections between these concepts and assumptions. It may also help in the unification of relativity theory with quantum theory.

A new challenge in the axiomatization of spacetime theories is to continue building our axiom systems by extending and generalizing them continuously investigating the logical connections of the new axiom systems and the old ones. A high reaching new aim of our research project is building a flexible hierarchy of axioms systems (instead of one axiom system only) and analyzing the logical connections between the different axioms and axiomatizations.

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