A LOGIC ROAD FROM SPECIAL RELATIVITY TO GENERAL RELATIVITY

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ABSTRACT. We present a streamlined axiom system of special relativity in first-order logic. From this axiom system we "derive" an axiom system of general relativity in two natural steps. We will also see how the axioms of special relativity transform into those of general relativity. This way we hope to make general relativity more accessible for the non-specialist.

INTRODUCTION

In axiomatizing physical theories, we follow in the footsteps of many great predecessors. Logical axiomatization of physics, especially that of relativity theory, is not at all a new idea. It goes back to such leading mathematicians and philosophers as Hilbert, Gödel, Tarski, Reichenbach, Carnap, Suppes and Friedman. It also has an extensive literature, see, e.g., the references of [2], [4]. Our aims go beyond these approaches, because we not only axiomatize relativity theories, but also analyze their logical and conceptual structures.

There are many examples showing the benefits of using the axiomatic method in the foundations of mathematics. The success story of the axiomatic method in the foundations of mathematics suggests that it is worth to apply this method in the foundations of spacetime theories.

For good reasons, foundations of mathematics was carried through strictly within first-order logic (FOL). For the same reasons, foundations of spacetime theories are best developed within FOL. For example, in any foundational work it is essential to avoid tacit assumptions, and one acknowledged feature of using FOL is that it helps to eliminate tacit assumptions. That is only one of the many reasons why we work within FOL. For further reasons, see [2, §Why FOL?], [23, §11].

In physics, the same way as in mathematics, we do not address the question whether the axioms are true or not, we just postulate them. The reason for this in mathematics is that we want to give a tool that is usable in all applications where the axioms are true. However, in physics the statements of the theories are closely related to the real

physical world, thus the application area is fixed in a way. Therefore, the role of the axioms (the role of statements that we assume without proofs) in physics is more fundamental than in mathematics. That is why we aim to formulate simple, logically transparent and intuitively convincing axioms. All the surprising or unusual predictions of a physical theory should be provable as theorems and not assumed as axioms. For example, the prediction "no observer can move faster than light" is a theorem in our approach and not an axiom, see Theorem 2.1.

Some of the questions we study when investigating the logical structure of relativity theories are: – What is believed and why? – Which axioms are responsible for certain predictions? – What happens if we discard some axioms? – Can we change the axioms and at what price?

First-order logic can be viewed as a fragment of natural language with unambiguous syntax and semantics. Being a *fragment of natural language* is useful in our project because one of our aims is to make relativity theory accessible to a broad audience. *Unambiguous syntax and semantics* are important for the same reason, because they make it possible for the reader to always know what is stated and what is not stated by the axioms. Therefore they can use the axioms without being familiar with all the tacit assumptions and rules of thumb of physics (that one usually learns via many, many years of practice).

A novelty in this paper is that we concentrate on the transition from special relativity to general relativity, we try to keep this transition logically transparent and illuminating for the non-specialist. We are going to "derive" the axioms of general relativity from those of special relativity in two natural steps. In the first step we will extend our FOL axiom system of special relativity of inertial observers to accelerated observers. This step will provide us a FOL theory of accelerated observers, which implies the usual predictions about them, such as the twin paradox, see Theorem 3.1. In the second step we will eliminate the difference between inertial and noninertial observers on the level of axioms. By these two natural steps, we will get a FOL axiomatization of the spacetimes of *general relativity* suitable for further study. All these three theories (special relativity, theory of accelerated observers, general relativity) will be formulated in the same streamlined FOL language.

1. The FOL language of our three theories

The first important decision in writing up an axiom system in FOL is to choose the vocabulary (set of basic symbols) of our language, i.e., what objects and relations between them we will use as basic concepts. We will have to stick to this language while writing up the axioms and investigating our theory. However, later we can change the vocabulary of our language, and we can write up a new theory (or a new version of our theory) in this new language. Then we can investigate the logical

connections between these theories built up in different FOL languages. This way we can investigate the role of having chosen the particular basic concepts of our theory, see [5, second half of §3].

In this paper we will use the following two-sorted FOL language:

$$\{\, B, \mathsf{IB}, \mathsf{Ph},\ Q, +, \cdot,\ \mathsf{W}\,\},$$

where B (bodies[1]) and Q (quantities) are the two sorts, IB (inertial bodies) and Ph (light signals or photons[2]) are one-place relation symbols of sort B, $+$ and $\cdot$ are two-place function symbols of sort Q, and W (the worldview relation) is a $2+4$-place relation symbol the first two arguments of which are of sort B and the rest are of sort Q.

Atomic formulas $\mathsf{IB}(b)$ and $\mathsf{Ph}(p)$ are translated as "*b is an inertial body*," and "*p is a photon*," respectively. We use the worldview relation W to speak about coordinatization by translating $\mathsf{W}(o,b,x,y,z,t)$ as "*body o coordinatizes body b at space-time location $\langle x,y,z,t\rangle$*," (i.e., at space location $\langle x,y,z\rangle$ and at instant t). Sometimes we use the more picturesque expressions *sees* or *observes* for coordinatizes. However, these "seeing" and "observing" have nothing to do with visual seeing or observing they only mean associating coordinate points to bodies.

The above, together with statements of the form $x = y$ (read as x *equals* y) are the so-called *atomic formulas* of our FOL language, where x and y can be arbitrary variables of the same sort, or terms built up from variables of sort Q by using the two-place operations $\cdot$ and $+$. The *formulas* of our FOL language are built up from these atomic formulas by using the logical connectives *not* ($\neg$), *and* ($\wedge$), *or* ($\vee$), *implies* ($\rightarrow$), *if-and-only-if* ($\leftrightarrow$) and the quantifiers *exists* ($\exists$) and *for all* ($\forall$). For the precise definition of the syntax and semantics of FOL, see, e.g., [9, §1.3], [12, §2.1, §2.2], or [14, pp.39–46].

For example, the formula $\forall btxzy\ \mathsf{W}(b,b,x,y,z,t) \rightarrow x = y$ abbreviates the natural language sentence "*For all b, t, x, y, z it is true that b observes b at t, x, y, z implies that x equals y*," or in a more readable form "*If b sees itself at t, x, y, z, then x equals y; and this is true for all b, t, x, y, z.*"

To abbreviate formulas of FOL we often omit parentheses according to the following convention. Quantifiers bind as long as they can, and $\wedge$ binds stronger than $\rightarrow$. For example, we write $\forall x\ \varphi \wedge \psi \rightarrow \exists y\ \delta \wedge \eta$ instead of $\forall x\big((\varphi \wedge \psi) \rightarrow \exists y(\delta \wedge \eta)\big)$.

We will use the letters p, b, m, k, h and their variants for variables of sort B, and the letters x, y, z, t, v, w, c and their variants for variables

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[1]By bodies we mean anything which can move, e.g., test-particles, reference frames, electromagnetic waves, centers of mass, etc.

[2]Here we use light signals and photons as synonyms because it is not important here whether we think of them as particles or electromagnetic waves. The only thing that matters here is that they are things that can move. So they are bodies in the sense of our FOL language.

of sort Q. For easier readability, we will use $\bar{x}$, $\bar{y}$, etc. for sequences of variables. The i-th element of the sequence $\bar{x}$ is denoted by x_i.

2. An axiomatization of special relativity in FOL

Having fixed our language, we now turn to formulating an axiom system for special relativity in this language. The first axiom states some usual properties of addition $+$ and multiplication $\cdot$ true for real numbers.

<u>AxFd:</u> The quantity part $\langle Q, +, \cdot \rangle$ is a Euclidean field, i.e.,
  - $\langle Q, +, \cdot \rangle$ is a field in the sense of abstract algebra,
  - the relation $\leq$ defined by $x \leq y \ \overset{d}{\Leftrightarrow} \ \exists z \ x + z^2 = y$ is a linear ordering on Q, and
  - $\forall x \ \exists y \ x = y^2 \vee -x = y^2$, i.e., positive elements are squares.[3]

The field-axioms (see, e.g., [9, pp.40–41], [15, p.38]) say that $+$, $\cdot$ are associative and commutative, they have neutral elements 0, 1 and inverses $-$, $/$ respectively, with the exception that 0 does not have an inverse with respect to $\cdot$, as well as $\cdot$ is additive with respect to $+$. We will use 0, 1, $-$, $/$, $\sqrt{\ }$ as derived (i.e., defined) operation symbols. As usual, x^2 denotes $x \cdot x$.

AxFd is a "mathematical" axiom in spirit. However, it has physical (even empirical) relevance. Its physical relevance is that we can add and multiply the outcomes of our measurements and some basic rules apply to these operations. Physicists use all properties of the real numbers tacitly, without stating explicitly which property is assumed and why. The two properties of real numbers which are the most difficult to defend are the Archimedean property, see [20], [21, §3.1], and the supremum property,[4] see the remark after the introduction of Cont on p.11.

In special relativity, we will not need more properties of the real numbers than stated in AxFd. In the next two sections, we will use more properties of the real numbers in our next two theories for relativity, but we will state exactly and explicitly how much we will use.

Euclidean fields got their names after their role in Tarski's FOL axiomatization of Euclidean geometry [25]. By AxFd we can reason about the Euclidean structure of a coordinate system the usual way, we can introduce Euclidean distance, talk about straight lines, etc. In particular, we will use the following notation for $\bar{x}, \bar{y} \in Q^n$ if $n \geq 1$:

$$|\bar{x}| = \sqrt{x_1^2 + \cdots + x_n^2}, \quad \text{and} \quad \bar{x} - \bar{y} \overset{d}{=} \langle x_1 - y_1, \ldots, x_n - y_n \rangle.$$

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[3]We note that the second statement in the definition of AxFd can be replaced with $x^2 + y^2 + z^2 = 0 \to x = 0$.

[4]The supremum property (i.e., every nonempty and bounded subset of the real numbers has a least upper bound) implies the Archimedean property. So if we want to get ourselves free from the Archimedean property, we have to leave this property, too.

The rest of our axioms speak about the worldviews of inertial observers. We have not introduced the concept of observers as a basic one because it can be defined as follows: an *observer* is nothing else than a body who "observes" (coordinatizes) some other bodies somewhere, this property can be captured by the following formula of our language:

$$\mathsf{Ob}(m) \quad \overset{d}{\Leftrightarrow} \quad \exists b\bar{x} \;\; \mathsf{W}(m, b, \bar{x});$$

and *inertial observers* can be defined as inertial bodies which are observers, formally:

$$\mathsf{IOb}(m) \quad \overset{d}{\Leftrightarrow} \quad \mathsf{IB}(m) \wedge \mathsf{Ob}(m).$$

We will also use the following two notations:

$$\bar{x}_s \overset{d}{=} \langle x_1, x_2, x_3 \rangle \quad \text{and} \quad x_t \overset{d}{=} x_4$$

for the *space component* and the *time component* of $\bar{x} \in Q^4$, respectively.

Our next axiom is the key axiom of SpecRel, it has an immediate physical meaning. This axiom is the outcome of the Michelson-Morley experiment. It has been continuously tested ever since then. Nowadays it is tested by GPS technology.

AxPh: For any inertial observer, the speed of light is the same everywhere and in every direction, and it is finite. Furthermore, it is possible to send out a light signal in any direction. Formally:

$$\forall m \;\; \exists c_m \;\; \forall \bar{x}\bar{y} \;\; \mathsf{IOb}(m) \rightarrow$$
$$\left(\exists p \;\; \mathsf{Ph}(p) \wedge \mathsf{W}(m, p, \bar{x}) \wedge \mathsf{W}(m, p, \bar{y}) \right) \leftrightarrow |\bar{y}_s - \bar{x}_s| = c_m \cdot |y_t - x_t|.$$

Let us note here that AxPh does not require that the speed of light is the same for every inertial observer or that it is nonzero. It requires only that the speed of light according to a fixed inertial observer is a quantity which does not depend on the direction or the location.

Our next axiom connects the worldviews of different inertial observers by saying that all observers observe the same "external" reality (the same set of events). Intuitively, by the event occurring for m at $\bar{x}$, we mean the set of bodies m observes at $\bar{x}$. Formally:

$$\mathsf{ev}_m(\bar{x}) \overset{d}{=} \{ b : \mathsf{W}(m, b, \bar{x}) \}.$$

AxEv: All inertial observers coordinatize the same set of events, i.e.,

$$\forall mk \;\; \mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \;\; \rightarrow \;\; \forall \bar{x} \;\; \exists \bar{y} \;\; \forall b \;\; \mathsf{W}(m, b, \bar{x}) \leftrightarrow \mathsf{W}(k, b, \bar{y}).$$

Hereafter, we will use $\mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{y})$ to abbreviate the subformula $\forall b \;\; \mathsf{W}(m, b, \bar{x}) \leftrightarrow \mathsf{W}(k, b, \bar{y})$ of AxEv.

Our two remaining axioms are simplifying ones. We could leave them out without losing the essence of our theory, only the formalizations of the theorems would become much more complicated.

<u>AxSf:</u> Any inertial observer sees himself as standing still at the origin:

$$\forall m \ \mathsf{IOb}(m) \rightarrow \ \big(\forall \bar{x} \ \mathsf{W}(m, m, \bar{x}) \leftrightarrow x_1 = 0 \wedge x_2 = 0 \wedge x_3 = 0\big).$$

Our last axiom is a symmetry axiom saying that all observers use the same units of measurement.

<u>AxSm:</u> Any two inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them; furthermore, the speed of light is 1 for all observers:

$$\forall mk \ \mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \rightarrow \forall \bar{x} \bar{y} \bar{x}' \bar{y}' \ x_t = y_t \wedge x'_t = y'_t \wedge$$
$$\mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}') \wedge \mathsf{ev}_m(\bar{y}) = \mathsf{ev}_k(\bar{y}') \rightarrow |\bar{x}_s - \bar{y}_s| = |\bar{x}'_s - \bar{y}'_s|, \text{ and}$$

$$\forall m \ \mathsf{IOb}(m) \ \rightarrow \ \exists p \ \mathsf{Ph}(p) \wedge \mathsf{W}(m, p, 0, 0, 0, 0) \wedge \mathsf{W}(m, p, 1, 0, 0, 1).$$

We introduce an axiom system for special relativity as the collection of these five axioms:

$$\mathsf{SpecRel} \overset{d}{=} \{\mathsf{AxFd}, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxSf}, \mathsf{AxSm}\}.$$

The so-called worldline of body b according to observer m is defined as follows:

$$\mathsf{wl}_m(b) \overset{d}{=} \{\bar{x} : \mathsf{W}(m, b, \bar{x})\}.$$

To abbreviate formulas, we will use bounded quantifiers in the following way: $\exists x \ \varphi(x) \wedge \psi$ and $\forall x \ \varphi(x) \rightarrow \psi$ are abbreviated to $\exists x \in \varphi \ \psi$ and $\forall x \in \varphi \ \psi$, respectively. For example, $\forall \bar{x}, \bar{y} \in \mathsf{wl}_m(b) \ \psi$ abbreviates $\forall \bar{x} \bar{y} \ \mathsf{W}(m, b, \bar{x}) \wedge \mathsf{W}(m, b, \bar{y}) \rightarrow \psi$.[5]

In an axiom system, the axioms are the "price" we pay, and the theorems are the "goods" we get for them. Therefore, we strived for putting only simple, transparent, easy-to-believe statements in our axiom system. We want to get all the hard-to-believe predictions as theorems. For example, now we are going to prove from SpecRel that it is impossible for inertial observers to move faster than light relative to each other. This theorem is a generic example for a "fancy theorem" following from "plain axioms."

Theorem 2.1. (no faster than light inertial observers)

$$\mathsf{SpecRel} \vdash \forall mk \ \forall \bar{x}, \bar{y} \in \mathsf{wl}_m(k)$$
$$\bar{x} \neq \bar{y} \wedge \mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \rightarrow |\bar{y}_s - \bar{x}_s| < |y_t - x_t|.$$

Intuitively, no observer can travel faster than light relative to another. Let us see the axioms in action: now we will prove Theorem 2.1 paying a close attention to the axioms used in each step.

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[5]Both $b \in \mathsf{ev}_m(\bar{x})$ and $\bar{x} \in \mathsf{wl}_m(b)$ represent the same atomic formula of our FOL language, namely: $\mathsf{W}(m, b, \bar{x})$.
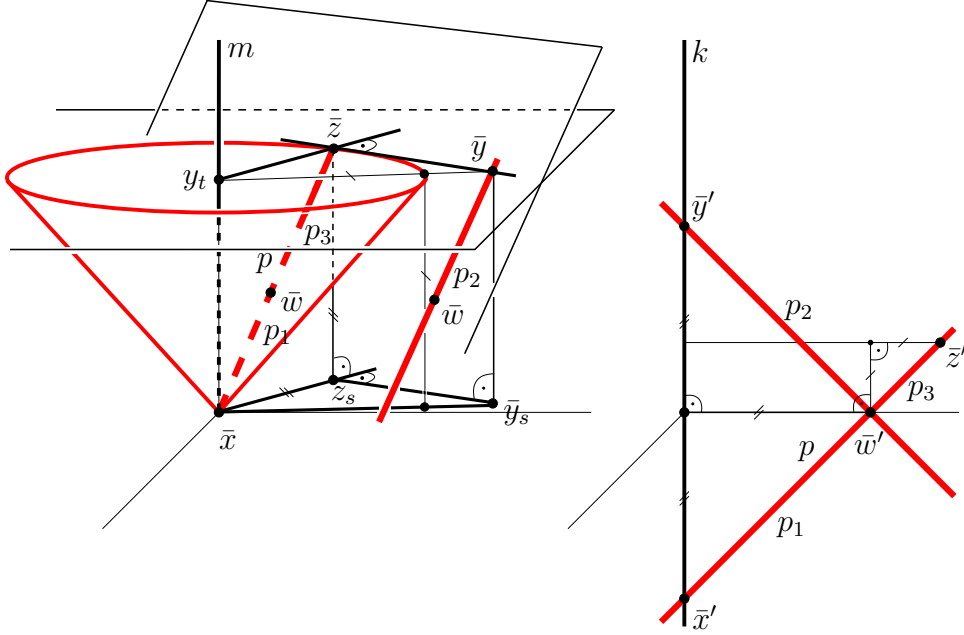
FIGURE 1.   Illustration for the proof of Theorem 2.1

Proof. Let m and k be inertial observers and let $\bar{x}, \bar{y} \in \mathsf{wl}_m(k)$ such that $\bar{x} \neq \bar{y}$. By AxFd, $\leq$ is a total order, so there are three possibilities only: $|\bar{y}_s - \bar{x}_s| < |y_t - x_t|$, $|\bar{y}_s - \bar{x}_s| > |y_t - x_t|$ or $|\bar{y}_s - \bar{x}_s| = |y_t - x_t|$. We will prove $|\bar{y}_s - \bar{x}_s| < |y_t - x_t|$ by excluding the other two possibilities.

Let us first prove that $|\bar{y}_s - \bar{x}_s| > |y_t - x_t|$ cannot hold. Figure 1 illustrates this proof.[6] So, let us assume that $|\bar{y}_s - \bar{x}_s| > |y_t - x_t|$, we will derive a contradiction. By AxFd, there is a coordinate point $\bar{z}$ such that $|\bar{z}_s - \bar{x}_s| = |z_t - x_t| \neq 0$, $z_t = y_t$ and $\bar{z}_s - \bar{x}_s$ is orthogonal to $\bar{z}_s - \bar{y}_s$ if $x_t \neq y_t$, and $|\bar{z}_s - \bar{x}_s| = |z_t - x_t| \neq 0$ and $\bar{z}_s - \bar{x}_s$ is orthogonal to $\bar{y}_s - \bar{x}_s$ if $x_t = y_t$ (here we used that $|\bar{y}_s - \bar{x}_s| > |y_t - x_t|$). Any choice of such a $\bar{z}$ implies that any line of slope 1 in the plane $\bar{x}\bar{y}\bar{z}$ is parallel to the line $\bar{x}\bar{z}$ (because the plane $\bar{x}\bar{y}\bar{z}$ is tangent to the light cone through $\bar{z}$). To choose one concrete $\bar{z}$ from the many, let

$$\bar{w}_s \stackrel{d}{=} \frac{\bar{y}_s - \bar{x}_s}{|\bar{y}_s - \bar{x}_s|}, \quad \bar{w}_s^{\perp} \stackrel{d}{=} \frac{\langle y_2 - x_2, x_1 - y_1, 0 \rangle}{\sqrt{(y_2 - x_2)^2 + (x_1 - y_1)^2}}.$$

Then, if $x_t = y_t$, let

$$\bar{z}_s \stackrel{d}{=} |\bar{y}_s - \bar{x}_s| \cdot \bar{w}_s^{\perp} + \bar{x}_s, \quad z_t \stackrel{d}{=} |\bar{y}_s - \bar{x}_s| + x_t,$$

and, if $x_t \neq y_t$, let

$$\bar{z}_s \stackrel{d}{=} \frac{|y_t - x_t|^2}{|\bar{y}_s - \bar{x}_s|} \cdot \bar{w}_s + \frac{|y_t - x_t| \cdot \sqrt{|\bar{y}_s - \bar{x}_s|^2 - |y_t - x_t|^2}}{|\bar{y}_s - \bar{x}_s|} \cdot \bar{w}_s^{\perp}, \quad z_t \stackrel{d}{=} y_t.$$

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[6]To simplify the figure, we have drawn $\bar{x}$ to the origin. This is not used in the proof but it can be assumed without losing generality.
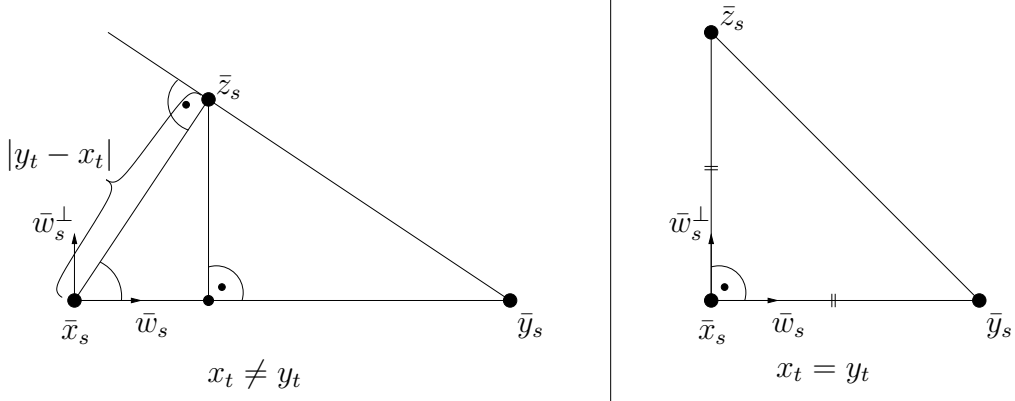
FIGURE 2.   Illustration for the proof of Theorem 2.1

See Figure 2. It is easy to see that this $\bar{z}$ has the required properties.

Then by AxPh and AxSm, there is a photon p such that $p \in \mathsf{ev}_m(\bar{x}) \cap \mathsf{ev}_m(\bar{z})$ since $|\bar{z}_s - \bar{x}_s| = |z_t - x_t|$. Also by AxPh, AxSm and AxFd, there are photons showing that inertial observers see distinct events in distinct points, so $\mathsf{ev}_m(\bar{x})$, $\mathsf{ev}_m(\bar{y})$ and $\mathsf{ev}_m(\bar{z})$ are distinct. By AxEv, there are coordinate points $\bar{x}'$, $\bar{y}'$ and $\bar{z}'$ such that $\mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}')$, $\mathsf{ev}_m(\bar{y}) = \mathsf{ev}_k(\bar{y}')$ and $\mathsf{ev}_m(\bar{z}) = \mathsf{ev}_k(\bar{z}')$. We have that $\bar{x}'$, $\bar{y}'$ and $\bar{z}'$ are distinct since $\mathsf{ev}_m(\bar{x}')$, $\mathsf{ev}_m(\bar{y}')$ and $\mathsf{ev}_m(\bar{z}')$ are so. By AxSf, $\bar{x}'_s = \bar{y}'_s = \langle 0,0,0 \rangle$. By AxPh and AxSm, $|\bar{z}'_s - \bar{x}'_s| = |z'_t - x'_t|$. By AxFd, there is a coordinate point $\bar{w}'$ on the line $\bar{x}'\bar{z}'$ such that $|\bar{x}'_s - \bar{w}'_s| = |x'_t - w'_t|$, $|\bar{y}'_s - \bar{w}'_s| = |y'_t - w'_t|$ and $|\bar{z}'_s - \bar{w}'_s| = |z'_t - w'_t|$. By AxPh and AxSm, there are photons p_1, p_2 and p_3 such that $\bar{x}', \bar{w}' \in \mathsf{wl}_k(p_1)$, $\bar{y}', \bar{w}' \in \mathsf{wl}_k(p_2)$ and $\bar{w}', \bar{z}' \in \mathsf{wl}_k(p_3)$. By AxEv, there should be a coordinate point $\bar{w}$ such that $\mathsf{ev}_m(\bar{w}) = \mathsf{ev}_k(\bar{w}')$. By AxPh and AxSm, this $\bar{w}$ should be on the line $\bar{x}\bar{z}$, since by AxFd there is no nondegenerate triangle whose sides are of slope 1 (this fact can be shown by proving that if we project a triangle of this kind vertically, we get another triangle whose one side is the sum of the other two). Specially, $\bar{w}$ should be in the plane $\bar{x}\bar{y}\bar{z}$. By AxPh and AxSm, this $\bar{w}$ should also be on a line of slope 1 through $\bar{y}$. Since $\bar{w}$ is in the plane $\bar{x}\bar{y}\bar{z}$, line $\bar{w}\bar{y}$ has to be parallel to the line $\bar{x}\bar{z}$. However, distinct parallel lines do not intersect. Thus this $\bar{w}$ cannot exist. That contradicts AxEv, so $|\bar{y}_s - \bar{x}_s| > |y_t - x_t|$ cannot hold.

Let us now prove that $|\bar{y}_s - \bar{x}_s| = |y_t - x_t|$ cannot hold, either. If $|\bar{y}_s - \bar{x}_s| = |y_t - x_t|$, then by AxPh and AxSm, there is a photon p, such that $p \in \mathsf{ev}_m(\bar{x}) \cap \mathsf{ev}_m(\bar{y})$. By AxPh, AxSm and AxFd, $\mathsf{ev}_m(\bar{x}) \neq \mathsf{ev}_m(\bar{y})$ since $\bar{x} \neq \bar{y}$. So by AxEv, there are distinct coordinate points $\bar{x}'$ and $\bar{y}'$ such that $\mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}')$ and $\mathsf{ev}_m(\bar{y}) = \mathsf{ev}_k(\bar{y}')$. By AxSf, $\bar{x}'_s = \bar{y}'_s = \langle 0,0,0 \rangle$. So $|\bar{y}'_s - \bar{x}'_s| = 0$. By AxPh and AxSm, $|\bar{y}'_s - \bar{x}'_s| = |y'_t - x'_t|$. Hence $y'_t = x'_t$, too. Thus $\bar{y}' = \bar{x}'$. That contradicts the fact that $\bar{x}'$ and $\bar{y}'$ are distinct.

The only remaining possibility is that $|\bar{y}_s - \bar{x}_s| < |y_t - x_t|$ which was to be proved.   □

It is easy to see that AxSm was not fully used in the proof above. We only used its second part, i.e., that the speed of light (the c_m in AxPh) is 1. Even the use of this part of AxSm was not essential. By slightly changing the proof, we could get basically the same result using only that $c_m \neq 0$, but then we also need to mention the speed c_m of light (according to m) explicitly in the formalization of the theorem.

In relativity theory we are often interested in comparing the world-views of two different observers. To do so, we introduce the worldview transformation between observers m and k (in symbols, w_{mk}) as the following binary relation:

$$\mathsf{w}_{mk}(\bar{x}, \bar{y}) \quad \overset{d}{\Leftrightarrow} \quad \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{y}) \neq \emptyset.$$

By the following theorem, the worldview transformations between inertial observers in the models of SpecRel are not only binary relations but very special transformations. For the definition of a Poincaré transformation (which is a Lorentz transformation composed with a translation) we refer to [10, p.110] or to [19, pp.66–69]. For the proof of the next theorem, see [3, Thm.11.10, p.640] or [23, Thm.3.2.2, p.22].

Theorem 2.2.

SpecRel $\vdash \forall m, k \ \ \mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \rightarrow \mathsf{w}_{mk}$ is a Poincaré transformation.

Every Poincaré transformation is an affine[7] one, specially it takes lines to lines. So by AxSf, Theorem 2.2 implies that $\mathsf{wl}_m(k)$ is a line for any inertial observers m and k. Thus Theorem 2.2 implies Theorem 2.1 since a Poincaré transformation cannot take a line of slope less than 1 (slower than light) to a line of slope more than 1 (faster than light). In the proof of Thm.2.2 the symmetry axiom, AxSm has to be fully used.[8] Therefore, using Theorem 2.2 to prove that "no inertial observer can move faster than light" does not show the roles of the particular axioms in the proof, e.g., it does not reveal that the first part of AxSm plays no role in proving the no faster than light theorem.

By Theorem 2.2, SpecRel implies the paradigmatic effects of special relativity, i.e., "moving clocks slow down," "moving meter-rods shrink" and "moving pairs of clocks get out of synchronism." However, we prefer proving these effects directly from the axioms, see, e.g., [3, Thm.11.6, p.631], [7].

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[7]Let us recall that an affine transformation is the composition of a linear mapping and a translation.

[8]Without AxSm there are other possible worldview transformations, such as dilations (i.e., rescaling of the units of measurement). For a complete characterization of the possible worldview transformations without AxSm, see, e.g., [4, Thm.1.2(i)].

3. Axioms for accelerated observers in FOL

In SpecRel we restricted our attention to inertial observers. It is a natural idea to generalize the theory by including accelerated observers as well. We will refer to such a generalized theory as a theory of accelerated observers. It is explained in the classic textbook [19, pp.163–165] that the study of accelerated observers can be regarded as a natural first step (from special relativity) towards general relativity. The theory of accelerated observers can also be used to explain how the relativistic paradigmatic effects of special relativity develop.

The most important axiom for accelerated observers will state that at each moment of his life-time, an accelerated observer coordinatizes the world near him for a short while as some inertial observer does. How can we formalize this idea? Saying that the worldview transformation w_{mk} is the identity function in a neighborhood would state that m and k totally agree in this neighborhood. This statement would connect the worldviews too rigidly. We want to state a somewhat looser connection between m and k. So, instead, we will state that the identity function approximates the worldview transformation w_{mk} at the spacetime point in question.

Let $f, g : Q^n \to Q^m$, $n, m \geq 1$ be partial[9] mappings and $\bar{x} \in Q^n$. We say that f *approximates* g *at* $\bar{x}$, in symbols $f \sim_{\bar{x}} g$, if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall \bar{y} \ |\bar{y} - \bar{x}| \leq \delta \to |f(\bar{y}) - g(\bar{y})| \leq \varepsilon \cdot |\bar{y} - \bar{x}|.$$

Let us recall that for partial functions f and g, the formula $|f(\bar{y}) - g(\bar{y})| \leq z$ is true iff both $f(\bar{y})$ and $g(\bar{y})$ are defined and the inequality holds. Thus $f \sim_{\bar{x}} g$ implies that $\bar{x}$ has a neighborhood where both f and g are defined; and also it implies that $f(\bar{x}) = g(\bar{x})$.

Let us note that approximation at a given point is an equivalence relation on functions, and if two affine mappings approximate each other, then they are equal. These can be proved from AxFd.

Let Id denote the identity function from Q^4 to Q^4, i.e., $\mathsf{Id}(\bar{x}) = \bar{x}$ for all $\bar{x} \in Q^4$.

AxCm: At each moment of his life, observer k "sees" (i.e., coordinatizes) the nearby world for a short while as an inertial observers m does, i.e., the identity map Id approximates w_{mk} at this moment:

$$\forall k \in \mathsf{Ob} \ \forall \bar{x} \in \mathsf{wl}_k(k) \ \exists m \in \mathsf{IOb} \ \mathsf{w}_{mk} \sim_{\bar{x}} \mathsf{Id}.$$

Let us note here that AxCm is true for inertial observers in a stronger form, i.e., $\mathsf{SpecRel} \vdash \forall k \ \mathsf{IOb}(k) \to \mathsf{w}_{kk} = \mathsf{Id}$. Axiom AxCm ties the behavior of accelerated observers to those of inertial ones. Justification of this axiom is given by experiments. If $\mathsf{w}_{mk} \sim_{\bar{x}} \mathsf{Id}$, we say that m and k *comove* at $\bar{x}$. If k is an accelerated spaceship, we can think of a dropped spacepod as a comoving inertial observer (comoving at the

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[9]Partial means that f and g are not necessarily everywhere defined on Q^n.

moment of dropping). Or, if k switches off his engines at $\bar{x}$, he will move on as a comoving inertial observer would.

Assuming SpecRel, AxCm implies that the worldlines of m and k meet and are tangent at $\bar{x}$ in the worldviews of all other inertial observers. Moreover, any body b present in the event at $\bar{x}$, $\mathsf{wl}_m(b)$ and $\mathsf{wl}_k(b)$ are tangent at $\bar{x}$; intuitively we could say that the whole worldviews of m and k are tangent at $\bar{x}$.

Our next two axioms ensure that the worldviews of accelerated observers are big enough. They are generalized versions of the corresponding axioms for inertial observers, but now postulated for all observers.

<u>AxEv$^-$</u>: If m sees k participate in an event, then k cannot deny it, i.e.,

$$\forall m, k \in \mathsf{Ob} \ \ \mathsf{W}(m, k, \bar{x}) \to \exists \bar{y} \ \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{y}).$$

<u>AxSf$^-$</u>: The worldline of any observer is an interval of the time-axis, in his own worldview:

$$\forall m \in \mathsf{Ob} \ \ \forall \bar{x} \ \ \mathsf{W}(m, m, \bar{x}) \to x_1 = x_2 = x_3 = 0 \quad \text{and}$$
$$\forall \bar{x}, \bar{y} \in \mathsf{wl}_m(m) \ \ \forall t \ \ x_t < t < y_t \to \mathsf{W}(m, m, 0, 0, 0, t).$$

Our last two axioms will ensure that the worldlines of accelerated observers are "tame" enough, e.g., they have velocities at each moment. In SpecRel, the worldview transformations between inertial observers are affine functions, the next axiom will state that the worldview transformations between accelerated observers are approximately affine, wherever they are defined.

<u>AxDf</u>: The worldview transformations have affine approximations at each point of their domain (i.e., they are differentiable):

$$\forall m, k \in \mathsf{Ob} \ \ \forall \bar{x} \in \mathsf{Dom}(\mathsf{w}_{mk}) \ \ \exists \text{ affine } A \ \ \mathsf{w}_{mk} \sim_{\bar{x}} A, {}^{10}$$

where $\mathsf{Dom}(R)$, the domain of a binary relation R, is defined as:

$$\mathsf{Dom}(R) \overset{d}{=} \{ x \ : \ \exists y \ \langle x, y \rangle \in R \}.$$

We note that AxDf implies that the worldview transformations are functions with open domains. However, if the numberline has gaps, still there can be crazy motions. Our last assumption is an axiom scheme supplementing AxDf by excluding these gaps.

<u>Cont</u>: Every subset of Q which is definable, bounded and nonempty has a supremum.

In Cont "definable" means "definable in the language of AccRel, parametrically." For a precise formulation of Cont, see [17, p.692] or [23, §10.1].

That Cont requires the existence of supremum only for sets definable in the language of AccRel instead of every set is important not only

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[10]The quantifier "$\exists$ affine A" looks like a second-order logic one, but truly it is a first-order logic quantifier because every affine map from Q^4 to Q^4 can be represented by a 4×4 matrix, i.e., 16 elements of Q, together with an $\bar{x} \in Q^4$.

because by this trick we can keep our theory within FOL (which is crucial in our foundational setting), but also because it makes this postulate closer to the physical/empirical level. This is true because Cont does not speak about "any fancy subset" of the quantities, but just about those "physically meaningful" sets which can be defined in the language of our (physical) theory.

Our axiom scheme of continuity (Cont) is a "mathematical axiom" in spirit. It is Tarski's first-order logic version of Hilbert's continuity axiom in his axiomatization of geometry, see [13, pp.161-162], fitted to the language of AccRel.

When Q is the usual real number-line, Cont is automatically true.

Adding this five axioms to SpecRel, we get the following axiom system for accelerated observers:

$$\mathsf{AccRel} \overset{d}{=} \mathsf{SpecRel} \cup \{\mathsf{AxCm}, \mathsf{AxEv}^-, \mathsf{AxSf}^-, \mathsf{AxDf}\} \cup \mathsf{Cont}.$$

The explicit introduction and development of AccRel as a theory in its own right is a contribution of our group. As an example we show that the so-called *twin paradox* can be naturally formulated and proved in AccRel. More importantly, the details of the twin paradox (e.g., who sees what, when) can be analyzed with the clarity of logic, see [2, pp.139–150] for part of such an analysis.

According to the twin paradox, if a twin makes a journey into space (accelerates), he will return to find that he has aged less than his twin brother who stayed at home (did not accelerate). We formulate the twin paradox in our FOL language as follows.

TwP: Every inertial observer m measures at least as much time as any other observer k between any two events e_1 and e_2 in which they meet; and they measure the same time iff they have encountered the very same events between e_1 and e_2:

$$\forall m \in \mathsf{IOb} \ \ \forall k \in \mathsf{Ob} \ \ \forall \bar{x}\bar{x}'\bar{y}\bar{y}' \ \ x_t < y_t \wedge x'_t < y'_t \wedge$$
$$m, k \in \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}') \wedge m, k \in \mathsf{ev}_m(\bar{y}) = \mathsf{ev}_k(\bar{y}') \to y'_t - x'_t \le y_t - x_t$$
$$\wedge \left(y'_t - x'_t = y_t - x_t \iff enc_m(\bar{x}, \bar{y}) = enc_k(\bar{y}', \bar{y}') \right),$$

where $enc_m(\bar{x}, \bar{y}) = \{\mathsf{ev}_m(\bar{z}) : \mathsf{W}(m, m, \bar{z}) \wedge x_t \le z_t \le y_t\}$.

Theorem 3.1.

(i) $\mathsf{AccRel} \models \mathsf{TwP}$.
   Moreover, $\mathsf{AccRel} - \mathsf{AxDf} \models \mathsf{TwP}$.

(ii) $\mathsf{AccRel} - \mathsf{Cont} \not\models \mathsf{TwP}$.
   Moreover, for any Euclidean ordered field F different from the field of real numbers there is a model $\mathfrak{M}$ such that its field reduct $\langle Q, +, \cdot \rangle$ is F and $\mathfrak{M} \models (\mathsf{AccRel} - \mathsf{Cont})$, but $\mathfrak{M} \not\models \mathsf{TwP}$.

For the proof of Theorem 3.1, see [17] or [23, §7].

The second part of Theorem 3.1 implies that Cont cannot be replaced with the whole FOL theory of real numbers in AccRel if we do not want to lose TwP from its consequences.
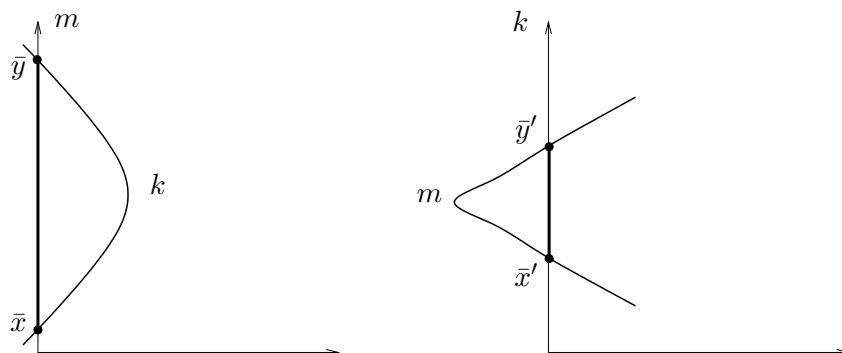
m: "non-moving" (inertial) brother k: traveling (accelerated) brother

FIGURE 3.   The "twin paradox"

Adding the axiom schema Cont to our axioms in AccRel represents a first step in the direction pursued in the so-called nonstandard-time logics of time approach represented by [1], [22].

All this enables us to "import" just as much of any field of mathematics, e.g., mathematical analysis into our first-order theory AccRel of accelerated observers as we need. Explicit and detailed elaborations of these ideas to situations similar to our present one (theory of accelerated observers) can be found in the above quoted [1], [22] and in the works quoted therein. For developing AccRel further, it is a distinct possibility to adopt the methods of the works just quoted to the framework of AccRel (extended with first-order logic sorts to treat second-order logic objects, just like in Henkin-style second-order logic).

4. An axiomatization of general relativity in FOL

The theory of accelerated observers AccRel speaks about two kinds of observers, inertial ones and accelerated ones. Some of the axioms are postulated for inertial observers only (such is, e.g., AxSm), some of the axioms apply to all observers (such is, e.g., AxSf⁻), and there is one axiom, AxCm, which talks about both of them. We get the axiom system GenRel for general relativity by stating the axioms of AccRel in a generalized form in which they are postulated for all observers, inertial and accelerated ones equally. In other words, we will change all axioms of AccRel in the same spirit as AxSf⁻ and AxEv⁻ were obtained from AxSf and AxEv, respectively. This kind of change AccRel ↦ GenRel can be regarded as a "democratic revolution" with the slogan "all observers

are equivalent, the same laws should apply to all of them." Here "law" translates as "axiom." This idea originates with Einstein (cf. his book [11, Part II, ch.18]). In [11, pp.58(ch.18),88(ch.28)], Einstein calls this idea of "democratic revolution of observers" the "General Principle of Relativity." Below, we implement Einstein's idea in logic (particularly in FOL).

For simplicity, we will use an equivalent version of the symmetry axiom AxSm (see [2, Thm.2.8.17(ii), p.138] or [23, Thm.3.1.4, p.21]), and we will require the speed of photons to be 1 in AxPh$^-$ (as opposed to requiring it in AxSm$^-$).

We will need the notion of velocity for curved worldlines. This notion will be based on the usual notion of the affine approximation. First we define the affine approximation (also called linear approximation or differential) of f at $\bar{x}$, denoted by $\mathsf{Apr}(f, \bar{x})$.

$$\mathsf{Apr}(f, \bar{x}) = g \quad \overset{d}{\Leftrightarrow} \quad f \sim_{\bar{x}} g \text{ and } g \text{ is affine.}$$

Let $f : Q \to Q^n$ be an affine map. We will use the following auxiliary notation for the velocity of f:

$$v(f) \overset{d}{=} \begin{cases} f(1) - f(0) & \text{if } n = 1, \\ \frac{f(1)_s - f(0)_s}{f(1)_t - f(0)_t} & \text{if } n > 1 \text{ and } f(1)_t \neq f(0)_t, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

To define the velocity of body b according to observer m, first we define the time-parameterized worldline of b (parameterized by the time of m):

$$\mathsf{wl}_m(b)(t) = \bar{x} \quad \overset{d}{\Leftrightarrow}$$
$$\mathsf{W}(m, b, \bar{x}) \wedge x_t = t \wedge \left(\forall \bar{y} \ \mathsf{W}(m, b, \bar{y}) \wedge y_t = t \to \bar{x} = \bar{y} \right).$$

By this definition, the time-parameterized worldlines are partial functions from Q to Q^4. Now we can define the velocity of body b according to observer m as follows:

$$v_m(b, \bar{x}) \overset{d}{=} v(\mathsf{Apr}(\mathsf{wl}_m(b), \bar{x})).$$

The behavior of observer k's clock as seen by observer m is defined as follows:

$$\mathsf{cl}_m(k)(t) = t' \quad \overset{d}{\Leftrightarrow} \quad \exists \bar{x} \bar{x}' \ \mathsf{W}(m, k, \bar{x}) \wedge \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{x}') \wedge$$
$$t = x_t \wedge t' = x'_t \wedge \left(\forall \bar{z} \ \mathsf{W}(m, k, \bar{z}) \wedge z_t = t \to \bar{x} = \bar{z} \right).$$

By this definition, $\mathsf{cl}_m(k)$ is a function relating t' to t if k's clock shows t' when m's clock shows t. That is, $\mathsf{cl}_m(k)$ is the time k's clock shows at t according to m's clock. Thus, e.g., $v(\mathsf{Apr}(\mathsf{cl}_m(k), t)) = 2$ means that at t (according to m's clock) k's clock runs twice as fast as m's. Now we are ready to state our axioms for general relativity.

AxPh$^-$: The velocity of photons an observer "meets" is 1 when they meet, and it is possible to send out a photon in each direction where the observer stands, i.e.,

$$\forall k \ \ \forall p \in \mathsf{Ph} \ \ \forall \bar{x} \in \mathsf{wl}_k(k) \cap \mathsf{wl}_k(p) \ \ |v_k(p, \bar{x})| = 1 \quad \text{and}$$

$$\forall k \ \ \forall \bar{x} \in \mathsf{wl}_k(k) \ \ \forall v \in Q^3 \ \ |v| = 1 \rightarrow \exists p \in \mathsf{Ph} \cap \mathsf{ev}_k(\bar{x}) \ \ v_k(p, \bar{x}) = v.$$

AxSm$^-$: Meeting observers see each other's clocks slow down with the same rate, i.e.,

$$\forall mk\bar{x}\bar{y} \quad m, k \in \mathsf{ev}_m(\bar{x}) = \mathsf{ev}_k(\bar{y})$$
$$\rightarrow \ v\big(\mathsf{Apr}\big(\mathsf{cl}_m(k), x_t\big)\big) = v\big(\mathsf{Apr}(\mathsf{cl}_k(m), y_t)\big).$$

We introduce an axiom system for general relativity as the collection of the following axioms:

$$\mathsf{GenRel} \overset{d}{=} \{\mathsf{AxFd}, \mathsf{AxPh}^-, \mathsf{AxEv}^-, \mathsf{AxSf}^-, \mathsf{AxSm}^-, \mathsf{AxDf}\} \cup \mathsf{Cont}.$$

Axiom system GenRel contains basically the same axioms as SpecRel, the difference is that they are assumed only locally but for all the observers. Axiom AxDf also fits into this picture. By Theorem 2.2, the worldview transformations between inertial observers are affine ones, and AxDf is the localization of this statement assumed for all the observers. Cont is a property of real numbers that we could have assumed in SpecRel but we did not assume it because it was not needed.

The following theorem states that our axiom system GenRel captures general relativity in that its models are exactly the spacetimes of usual general relativity. For the notion of a Lorentzian manifold we refer to [10, p.55], [19, p.241] and [3, sec.3.2].

Theorem 4.1 (Completeness theorem). GenRel *is complete with respect to its standard models, i.e., to Lorentzian Manifolds over real closed fields.*

This theorem can be regarded as a completeness theorem in the following sense. Let us consider Lorentzian manifolds as intended models of GenRel. How can we do that? We give a method for constructing a model of GenRel from each Lorentzian manifold; and conversely, we show that each model of GenRel is obtained in this way from a Lorentzian manifold. By the above, we defined what we mean by a formula φ in the language of GenRel being valid in a Lorentzian manifold, or in all Lorentzian manifolds. Then completeness means that for any formula φ in the language of GenRel, we have GenRel $\vdash \varphi$ iff φ is valid in all Lorentzian manifolds over real closed fields. That is completely analogous to how Minkowskian spacetimes were regarded as intended models of SpecRel in the completeness theorem of SpecRel, see [3, Thm.11.28, p.681] and [16, §4]. For more on the proofidea for our completeness theorem for GenRel, cf. also [3, items 11.28-11.30, pp.681-2].

Our theory GenRel was obtained from AccRel by getting rid of the concept of inertiality on the level of our axioms. However, we can recover this concept. We call the worldline of observer m *timelike geodesic*, if each of its points has a neighborhood within which this observer "maximizes measured time" between any two encountered events, i.e.,

$$\forall \bar{z} \in \mathsf{wl}_m(m) \ \exists \delta \ \ \delta > 0 \wedge$$
$$\forall k \bar{x} \bar{y} \ \ |\bar{x} - \bar{z}| < \delta \wedge |\bar{y} - \bar{z}| < \delta \wedge \mathsf{Ob}(k) \wedge \bar{x}, \bar{y} \in \mathsf{wl}_m(m) \cap \mathsf{wl}_m(k)$$
$$\wedge \left(\forall \bar{w} \in \mathsf{wl}_m(k) \ \ |\bar{w} - \bar{z}| < \delta \right) \to |x_t - y_t| \geq |\mathsf{w}_{mk}(\bar{x})_t - \mathsf{w}_{mk}(\bar{y})_t| \, .$$

In this case we also say that observer m is an *inertial* body (but not necessarily an inertial observer). This definition is justified by the twin paradox theorem of AccRel, see Theorem 3.1. This theorem says that in the models of AccRel the worldlines of *inertial* observers are timelike geodesics in the above sense.

According to the definition above, if there are only a few observers, then it is not a big deal that the worldline of m is a time-like geodesic (it is easy to be maximal if there are only a few to be compared to). To generate a real competition for the rank of having a timelike geodesic worldline, we postulate the existence of many observers by the following axiom scheme of comprehension.

Compr: For any parametrically definable timelike curve in any observer's worldview, there is another observer whose worldline is the range of this curve.

A precise formulation of Compr can be obtained from that of its analogue in [3, p.679].

The assumption of axiom schema Compr guarantees that our definition of geodesic coincides with that of the literature on Lorentzian manifolds. Therefore we also introduce the following theory:

$$\mathsf{GenRel}^+ \stackrel{d}{=} \mathsf{GenRel} \cup \mathsf{Compr}.$$

So in our theory GenRel$^+$, our notion of timelike geodesic coincides with its standard notion in the literature on general relativity. All the other key notions of general relativity, such as curvature or Riemannian tensor field, are definable from timelike geodesics. Therefore we can treat all these notions (including the notion of metric tensor field) in our theory GenRel$^+$ in a natural way.

In general relativity Einstein's equations give the connection between the geometry of the spacetime and the energy-matter distribution (given by the energy-momentum tensor field). Since in GenRel$^+$ all the geometric notions of the spacetime are definable, we can use Einstein's equation as a definition of the energy-momentum tensor, see, e.g., [8] or [10, §13.1, p.169], or we can extend the language of GenRel$^+$ by the concept of energy-momentum tensor and assume Einsten's equations as axioms. As far as we do not assume anything more

from the energy-momentum tensor than its connection to the geometry described by Einstein's equations, there is no real difference in these two approaches. In both approaches we can add any extra condition about the energy-momentum tensor to our theory, e.g., the dominant energy condition or that the spacetimes are vacuum solutions.

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