Tight Bounds on the Chromatic Sum of a Connected Graph

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ABSTRACT

The chromatic sum of a graph is introduced in the dissertation of Ewa Kubicka. It is the smallest possible total among all proper colorings of G using natural numbers. In this article we determine tight bounds on the chromatic sum of a connected graph with *e* edges.

1. THE CHROMATIC SUM

A *proper coloring* of the vertices of the graph G must assign different colors to adjacent vertices. The *chromatic number* $\chi(G)$ is just the smallest number of colors in any proper coloring of G. The chromatic number is well known and much studied. The reader may seek background in any graph theory text, for example, Chartrand and Lesniak [1]. The *chromatic sum* $\Sigma(G)$ is a recent variation introduced in the dissertation of Ewa Kubicka [2]. It is defined as the smallest possible total over all vertices that can occur among all proper colorings of G using natural numbers for the colors (as is customary). It is tempting to suspect that we will attain the minimum sum by first selecting a coloring that achieves the chromatic number and then arranging the color classes so that the largest is color 1, the next largest is color 2, and so on. But it is shown in [2] that even among trees (whose chromatic number is clearly 2) the chromatic sum sometimes requires the use of more than 2 colors; in fact, it is shown that for every positive integer k nearly all trees require the use of at least k colors to attain the chromatic sum. Thus, in the long run, we cannot expect a coloring

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that achieves the chromatic number to provide the chromatic sum as well. In fact, it is also shown in [2] that, just as in determining the chromatic number, computing the chromatic sum for arbitrary graphs is an NP-complete problem, but in the case of trees, a linear algorithm has been found.

We wish to bound the chromatic sum for graphs with e edges. One bound is based on n, the number of vertices, as well as e. The *degree* d_i of vertex i is just the number of edges incident at i. For a specified ordering of the vertices, we define the *lower degree* l_i to be the number of lower indexed neighbors of i, while the *upper degree* u_i is the number of higher indexed neighbors. Of course, this means that $l_i + u_i = d_i$. Since each edge is counted only once among the lower degrees, we have $e = \sum l_i = \sum u_i = \frac{1}{2}\sum d_i$.

Lemma 1. For any graph G, the chromatic sum is bounded by $\Sigma(G) \le n + e$.

Proof. Consider a coloring C that attains the minimum sum. We write ΣC for the sum over all vertices of the colors used in C. Order the vertices so that the colors are in nondecreasing order, that is, if c_i denotes the color on vertex i, the ordering requires that $c_1 \le c_2 \le c_3 \le \cdots \le c_n$. Think of assigning the colors in the order given. When we reach vertex i, it is joined to l_i earlier vertices, so at most l_i small colors are forbidden. Thus at least one color less than or equal to $1 + l_i$ is available. Indeed, if no such available color is used in the minimum coloring, we could substitute it to reduce the sum. Thus, we have shown that $c_i \le 1 + l_i$ for each vertex i. We sum over all vertices to obtain $\Sigma(G) = \Sigma C = \sum c_i \le \sum(1 + l_i) = n + e$. ∎

Observe that the upper bound is attained if and only if each vertex i is colored with $c_i = 1 + l_i$ because it is joined to exactly one vertex of each smaller color. In particular, any graph with all components complete will attain the bound.

However, our goal is to bound the chromatic sum for all graphs in terms of the number of edges only. We begin with connected graphs. It is then an easy corollary to modify the upper bound for arbitrary disconnected graphs.

Theorem 2. For any connected graph G with e edges, the chromatic sum is bounded by

$$\lceil\sqrt{8e}\,\rceil \le \Sigma(G) \le \lfloor \tfrac{3}{2}(e + 1)\rfloor .$$

Moreover, for each value of e there exist graphs that attain these bounds.

Proof. First, we shall show that for each e, the lower bound can be achieved by a bipartite graph (bigraph) using only 2 colors. Then we proceed to identify which bigraphs are minimal. Among all graphs with e edges and minimum value for the chromatic sum, select G and its optimal coloring C to have the largest number of vertices with color 1. In fact, for each i let a_i denote the number of vertices with color i in C. We shall show that G must be bipartite.

Suppose a minimum coloring C uses color 3, and let j be a vertex of color 3. Now delete vertex j and insert a new vertex of color 1 and another of color 2. Join the new vertices to all vertices of different color to form a new graph G'. We already have G' colored with the sum $\Sigma(G)$. But how many edges does G' have? We deleted at most $n - a_3$ edges and then added $n - 1 - a_1$ edges and $n - a_2$ edges. The net increase is at least $(n - 1 - a_1 - a_2) + a_3$. But the terms in parentheses have a nonnegative total because $n = \Sigma a_i \geq a_1 + a_2 + a_3 \geq a_1 + a_2 + 1$. That is, G' has at least a_3 edges more than G. Delete edges until we form a graph G'' with exactly e edges. Now G'' is already colored with sum $\Sigma(G)$, so its chromatic sum is bounded above by $\Sigma(G)$. Since G gives the minimum $\Sigma(G)$ among all graphs with e edges, we must have $\Sigma(G'') = \Sigma(G)$. But the coloring we have produced uses one more vertex of color 1, contradicting the selection of G. Hence, the assumption that G might have a vertex with color 3 is false. Since G must be 2 colored, it must be a bigraph as claimed. However, it should be noted that sometimes nonbigraphs attain the same sum. For example, when $e = 2a^2 + 1$, the minimum sum of $4a + 1$ is attained both by subgraphs of $K_{2a+1,a}$ and by the graph $K_{2a,a}$ with one edge added to the smaller set to form a tripartite graph.

Now among all bigraphs B with e edges and using $a = a_1$ vertices of color 1 and $a_2 = n - a$ vertices of color 2 in the minimum coloring, the chromatic sum is unchanged upon adding edges to form a complete bigraph $K_{a,n-a}$. Thus, for $e \leq a(n - a)$ and $a \geq (n/2)$, we know that $\Sigma(B)$ is at most $\Sigma(K_{a,n-a}) = a + 2(n - a) = 2n - a$. For a chosen value of a, the minimum occurs for $n - a = \lceil e/a \rceil$, giving $\Sigma(B) = \min_{n/2 \leq a \leq n}\{a + 2\lceil e/a \rceil\}$. Ignoring the ceiling function yields the function $f(a) = a + (2e)/a$, which attains its minimum value of $\sqrt{8e}$ by choosing $a = \sqrt{2e}$. Thus we have verified the lower bound $\lceil \sqrt{8e} \rceil \leq \Sigma(B)$ as required in the theorem.

We proceed to the upper bound. For any proper coloring $C = \{c_1, c_2, c_3, \ldots, c_n\}$, the modified coloring C_{ij} is obtained by interchanging colors i and j wherever they occur in C. Recall that ΣC is the sum over all vertices of the colors in C. The first part of the theorem follows from the lemma below plus the observation that

$$\Sigma(G) \leq \min\{\Sigma C, \Sigma C_{12}\} \leq \frac{1}{2} \cdot 3(e + 1).$$

Lemma 3. For any connected graph G with e edges, there exists a proper coloring C such that $\Sigma C + \Sigma C_{12} \leq 3(e + 1)$.

Proof. We prove the lemma by induction on the number of edges. The lemma is trivial if G is the unique connected graph with no edges. Let G be an arbitrary connected graph with e edges and select a non-cutvertex, say vertex i. The inductive hypothesis assures that the connected graph $G - i$ has a coloring C^* with $\Sigma C^* + \Sigma C^*_{12} \leq 3(e - d_i + 1)$. For $d_i \geq 2$, we achieve the desired coloring C of G by coloring vertex i with the smallest available color common to both C^* and C^*_{12}. This increases $\Sigma C^* + \Sigma C^*_{12}$ by at most $2(d_i + 1)$ so that

$$\Sigma C + \Sigma C_{12} \leq 3(e - d_i + 1) + 2(d_i + 1) = 3(e + 1) + 2 - d_i \leq 3(e + 1).$$

On the other hand, for $d_i = 1$, we may color vertex i with one of 1 or 2 in C^* (and with the other in C^*_{12}) so that

$$\Sigma C + \Sigma C_{12} \leq 3(e - 1 + 1) + 1 + 2 = 3(e + 1). \quad \blacksquare$$

It remains to be shown how to achieve the bounds in the theorem for each value of e. It is routine to verify that the upper bound is attained by all paths P_n, odd cycles C_{2m+1}, and also by the graph of order $3m + 1$ formed by taking m copies of K_3 and joining one vertex in each copy to a new vertex to form a graph in which $e = 4m$ while $\Sigma(G)$ can be shown to be $6m + 1$.

One bigraph B with e edges achieving the lower bound is found by setting b to be the closest integer to $\sqrt{e/2}$. That is, let $b = \sqrt{e/2} + \varepsilon$ where ε lies between $-.5 < \varepsilon < .5$ and is chosen to make b an integer. Set a to be the integer $2b - \lfloor 4\varepsilon \rfloor$. The graph $K_{a,b}$ has chromatic sum $\Sigma(K_{a,b}) = a + 2b = \sqrt{8e} + 4\varepsilon - \lfloor 4\varepsilon \rfloor$. Since this differs from $\sqrt{8e}$ by less than 1, it must be equal to $\lceil \sqrt{8e} \rceil$. But how many edges does it have? By considering values of ε in each of the four subintervals $(-.5, -.25)$, $[-.25, 0)$, $[0, .25)$, and $[.25, .5)$, it is easy to verify that in each case the graph $K_{a,b}$ has $ab > e - 1$ edges. Since ab is an integer, we conclude that $ab \geqslant e$. Finally, we set B to be any subgraph of $K_{a,b}$ with exactly e edges. This completes the proof of Theorem 2. $\blacksquare$

It is now easy to bound the chromatic sum for disconnected graphs.

Corollary 4. For any graph G with no isolated vertices and e edges,

$$\lceil \sqrt{8e} \rceil \leq \Sigma(G) \leq 3e.$$

Proof. The lower bound has already been established in the main theorem and is never improved by having more than one connected component. For the upper bound, the connected components G_i have the ratio

$$\frac{\Sigma(G_i)}{e_i} \leq \frac{\left\lfloor \frac{3}{2}(e_i + 1) \right\rfloor}{e_i}.$$

The largest value of 3 occurs precisely when $e_i = 1$. Thus, choosing $G = eK_2$, that is e independent edges, gives the largest possible chromatic sum, namely $3e$.

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