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Radius, Diameter, and Minimum Degree

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We give asymptotically sharp upper bounds for the maximum diameter and radius of (i) a connected graph, (ii) a connected trangle-free graph, (iii) a connected C_4-free graph with n vertices and with minimum degree δ, where n tends to infinity. Some conjectures for K_r-free graphs are also stated.

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For any $x, y \in V(G)$ let $d_G(x,y)$ denote the *distance* between x and y, i.e., the minimum length of an $x-y$ path in G. The *diameter* and the *radius* of G are defined as

$$\operatorname{diam} G = \max_{x\ y \in V(G)} d_G(x,y),$$

$$\operatorname{rad} G = \min_{x \in V(G)} \max_{y \in V(G)} d_G(x,y).$$

The following theorem answers a question of Gallai [6].

THEOREM 1. *Let G be a connected graph with n vertices and with minimum degree $\delta \geqslant 2$. Then*

(i) $$\operatorname{diam} G \leqslant \left[\frac{3n}{\delta+1}\right] - 1.$$

(ii) $$\operatorname{rad} G \leqslant \frac{3}{2}\frac{n-3}{\delta+1} + 5.$$

Furthermore, (i) *and* (ii) *are tight apart from the exact value of the aditive constants, and for every $\delta > 5$ equality can hold in* (i) *for infinitely many values of n.*

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Proof. Let G be a graph of diameter $d>1$ and minimum degree δ, and asume that it is *saturated*; i.e., the addition of any edge results in a graph with smaller diameter. Let x and y be two vertices with $d_G(x,y)=d$, and put $S_i=\{v\in V(G): d_G(x,v)=i\}$ for any $0\leqslant i\leqslant d$. Then $|S_0|=|S_d|=1$ and by the condition on the minimum degree

$$|S_{i-1}|+|S_i|+|S_{i+1}|\geqslant\delta+1 \qquad \text{for all}\quad 0\leqslant i\leqslant d,$$

where $S_{-1}=S_{d+1}=\varnothing$. It can readily be checked by distinguishing cases according to the residue class of $d \bmod 3$ that if $d>2$ then this implies

$$n=\sum_{i=0}^{d}|S_i|\geqslant\left(\left[\frac{d}{3}\right]+1\right)(\delta+1)+\varepsilon_d, \tag{1}$$

where ε_d denotes the remainder of d upon division by 3. This yields (i). Further, it is easily seen that equality can be attained in (1) for any pair $d\geqslant 2, \delta\geqslant 2$.

Note that (i) is tight, e.g., for the following graph. Let $k>1, \delta>5$, and $V(G)=V_0\cup V_1\cup\cdots\cup V_{3k-1}$, where

$$|V_i|=\begin{cases}1 & \text{if}\quad i\equiv 0 \text{ or } 2 \pmod 3,\\ \delta & \text{if}\quad i=1 \text{ or } 3k-2,\\ \delta-1 & \text{otherwise}\end{cases}$$

Let two distinct vertices $v\in V_i$, $v'\in V_j$ be joined by an edge of G if and only if $|j-i|\leqslant 1$.

To prove (ii), let us fix a *center* x of G, i.e., a point for which $\max_{y\in V(G)} d_G(x,y)=\operatorname{rad} G=r$, and put $S_i=\{v\in V(G): d_G(x,v)=i\}$ for $0\leqslant i\leqslant r$. Given any $v\in S_i$, pick a point $v'\in S_{i-1}$ such that $vv'\in E(G)\ (1\leqslant i\leqslant r)$. The collection of the edges $\{vv': v\in V(G)-\{x\}\}$ obviously defines a spanning tree $T\subseteq G$ with the property that

$$d_T(x,y)=d_G(x,y) \qquad \text{for all}\quad y\in V(G).$$

Let $T(x,y)$ denote the path connecting x and y in T. Further, put

$$S_{\leqslant j}=\bigcup_{0\leqslant i\leqslant j} S_i, \qquad S_{\geqslant j}=\bigcup_{j\leqslant i\leqslant r} S_i.$$

Fix a point $y'\in S_r$. A vertex $y''\in V(G)$ is said to be *related to* y', if one can find $\bar{y}'\in T(x,y')\cap S_{\geqslant 5}$ and $\bar{y}''\in T(x,y'')\cap S_{\geqslant 5}$ such that

$$d_G(\bar{y}',\bar{y}'')\leqslant 2. \tag{2}$$

There are two cases to consider.

Case A. There exists a point $y'' \in S_{\geqslant r-5}$ which is not related to y'.

For any i, let S'_i (and S''_i) denote the set of all elements in S_i whose distance from at least one point of $T(x, y') \cap S_{\geqslant 5}$ (one point of $T(x, y'') \cap S_{\geqslant 5}$, resp.) is at most 1 in G. Using the fact that y' and y'' are not related,

$$\left(\bigcup_{i=4}^{r} S'_i\right) \cap \left(\bigcup_{i=4}^{r} S''_i\right) = \varnothing.$$

On the other hand, by the condition on the minimum degree,

$$|S'_{i-1}| + |S'_i| + |S'_{i+1}| \geqslant \delta + 1 \qquad \text{for all} \quad 5 \leqslant i \leqslant r,$$
$$|S''_{i-1}| + |S''_i| + |S'_{i+1}| \geqslant \delta + 1 \qquad \text{for all} \quad 5 \leqslant i \leqslant s,$$

where $s = d_G(x, y'') \geqslant r - 5$. Similarly to (1), we now obtain

$$\begin{aligned} n &\geqslant |S_{\leqslant 3}| + \sum_{i=4}^{r} |S'_i| + \sum_{i=4}^{s+1} |S''_i| \\ &\geqslant \delta + 2 + \left\{\sum_{i=5}^{r} \frac{1}{3}(|S'_{i-1}| + |S'_i| + |S'_{i+1}|) + 1\right\} \\ &\quad + \left\{\sum_{i=5}^{s} \frac{1}{3}(|S''_{i-1}| + |S''_i| + |S''_{i+1}|) + 1\right\} \\ &\geqslant \delta + 4 + \frac{1}{3}(r-4)(\delta+1) + \frac{1}{3}(s-4)(\delta+1) \geqslant \frac{1}{3}(2r-10)(\delta+1) + 3, \end{aligned}$$

whence (ii) follows immediately.

Case B. Every point $y'' \in S_{\geqslant r-5}$ is related to y'.

Let x' denote the only element of $T(x, y')$ which belongs to S_5. Then, for any $y \in S_{\leqslant r-6}$,

$$d_G(x', y) \leqslant d_G(x', x) + d_G(x, y) \leqslant 5 + r - 6 = r - 1.$$

On the other hand, every $y'' \in S_{\geqslant r-5}$ is related to y', therefore by (2)

$$\begin{aligned} d_G(x', y'') &\leqslant d_G(x', \bar{y}') + d_G(\bar{y}', \bar{y}'') + d_G(\bar{y}'', y'') \\ &\leqslant (d_G(x, \bar{y}') - 5) + 2 + (r - d_G(x, \bar{y}'')) \\ &\leqslant r - 3 + d_G(\bar{y}', \bar{y}'') \leqslant r - 1. \end{aligned}$$

Thus, $d_G(x', y) \leqslant r - 1$ for every $y \in V(G)$, contradicting our assumption that rad $G = r$. This completes the proof of (ii). ∎

THEOREM 2. *Let G be a connected triangle-free graph with n vertices, and with minimum degree $\delta \geqslant 2$. Then*

$$\text{(i)} \qquad \operatorname{diam} G \leqslant 4\left\lceil \frac{n-\delta-1}{2\delta} \right\rceil.$$

$$\text{(ii)} \qquad \operatorname{rad} G \leqslant \frac{n-2}{\delta} + 12.$$

Furthermore, (i) *and* (ii) *are tight apart from the exact value of the additive constant, and for every $\delta \geqslant 2$ equality can hold in* (i) *for infinitely many values of n.*

Proof. Let x and y be two vertices of G with $d_G(x, y) = \operatorname{diam} G = d$, and put $S_i = \{v \in V(G) \colon d_G(x, v) = i\}$ for any $0 \leqslant i \leqslant d$.

For every i exactly one of the following two possibilities occurs. Either S_i does not span any edge of G and then

$$|S_{i-1}| + |S_{i+1}| \geqslant \delta, \tag{3}$$

or $vv' \in E(G)$ for some $v, v' \in S_i$, and then the neighborhoods of v and v' are disjoint. Therefore

$$|S_{i-1}| + |S_i| + |S_{i+1}| \geqslant 2d. \tag{4}$$

Note that (3) and (4) immediately imply that

$$|S_{i-1}| + |S_i| + |S_{i+1}| + |S_{i+2}| \geqslant 2\delta \qquad \text{for every} \quad 0 \leqslant i \leqslant d-1, \tag{5}$$

where $S_{-1} = S_{d+1} = \varnothing$. Indeed, if S_i or S_{i+1} contains an edge, then (5) follows from (4). Otherwise, by (3), $|S_{i-1}| + |S_{i+1}| \geqslant \delta$ and $|S_i| + |S_{i+2}| \geqslant \delta$; hence (5) is true again.

Now easy calculations show that

$$n \geqslant \left(\left[\frac{d}{4}\right] + 1\right) 2\delta - 1 + \begin{cases} -\delta + 2 & \text{if} \quad d \equiv 0 \pmod 4, \\ 1 & \text{if} \quad d \equiv 1 \pmod 4, \\ 2 & \text{if} \quad d \equiv 2 \pmod 4) \\ 3 & \text{if} \quad d \equiv 3 \pmod 4, \end{cases}$$

and equality can hold for every pair $d, \delta \geqslant 2$. This yields (i). Note that (i) is tight, e.g., for the following graphs. Let $V(G) = V_0 \cup V_1 \cup \cdots \cup V_{4k}$ with

$$|V_i| = \begin{cases} 1 & \text{if} \quad i \equiv 0 \text{ or } 1 \pmod 4 \text{ and } i \neq 1, \\ \delta & \text{if} \quad i = 1 \text{ or } 4k-1, \\ \delta - 1 & \text{otherwise,} \end{cases}$$

and assume that V_i and V_{i+1} induce a complete bipartite subgraph of G for every i.

The proof of the second part of the theorem is very similar to that of Theorem 1 (ii). We use the same notation and terminology as there, with the following modification. Fix a point $y' \in S_r$. A vertex $y'' \in V(G)$ is now said to be *related to* y', if there exist $\bar{y}' \in T(x, y') \cap S_{\geqslant 9}$ and $\bar{y}'' \in T(x, y'') \cap S_{\geqslant 9}$ such that

$$d_G(\bar{y}', \bar{y}'') \leqslant 4. \tag{2'}$$

Case A. There exists a point $y'' \in S_{\geqslant r-9}$ which is not related to y'.

For any i, let S_i' (S_i'') denote the set of all elements of S_i whose distance from at least one point of $T(x, y') \cap S_{\geqslant 9}$ ($T(x, y'') \cap S_{\geqslant 9}$, resp.) is at most 2. Then

$$\left(\bigcup_{i=7}^{r} S_i'\right) \cap \left(\bigcup_{i=7}^{r} S_i''\right) = \varnothing,$$

and by an argument similar to the proof of (5) we obtain

$$|S_{i-1}'| + |S_i'| + |S_{i+1}'| + |S_{i+2}'| \geqslant 2\delta \qquad \text{for all} \quad 8 \leqslant i \leqslant r-1,$$

$$|S_{i-1}''| + |S_i''| + |S_{i+1}''| + |S_{i+2}''| \geqslant 2\delta \qquad \text{for all} \quad 8 \leqslant i \leqslant s-1,$$

where $s = d_G(x, y'') \geqslant r - 9$. This yields

$$n \geqslant |S_{\leqslant 6}| + \sum_{i=7}^{r} |S_i'| + \sum_{i=7}^{s+1} |S_i''| \geqslant (r-12)\,\delta + 2$$

and (ii) follows.

Case B. Every point of $S_{\geqslant r-9}$ is related to y'.

A slight modification of the argument which settled the corresponding case in Theorem 1 shows that this cannot occur. ▮

THEOREM 3. *Let $\delta \geqslant 2$ be a fixed integer, and let G be a connected, C_4-free graph with n vertices and with minimum degree δ. Then*

(i) $$\operatorname{diam} G \leqslant \frac{5n}{\delta^2 - 2[\delta/2] + 1}.$$

(ii) $$\operatorname{rad} G \leqslant \frac{5n}{2(\delta^2 - 2[\delta/2] + 1)}.$$

Furthermore, if δ is large, then these bounds are almost tight. More precisely, if $\delta + 1$ is a prime power, then there exists a graph G with the above properties and

(iii) $$\operatorname{diam} G \geqslant \frac{5n}{\delta^2 + 3\delta + 2} - 1.$$

Proof. Let $x_0x_1x_2 \cdots x_d$ be a chordless path of length $d = \operatorname{diam} G$ in G. Put $S_{\leqslant 2}(x) = \{v \in V(G): d_G(x, v) \leqslant 2\}$ for any $x \in V(G)$. Since G does not contain C_4,

$$|S_{\leqslant 2}(x)| \geqslant \delta^2 - 2\left[\frac{\delta}{2}\right] + 1 \qquad \text{for every} \quad x \in V(G).$$

In view of the fact that

$$S_{\leqslant 2}(x_{5i}) \cap S_{\leqslant 2}(x_{5j}) = \varnothing \qquad \text{for all} \quad 0 \leqslant i \neq j \leqslant d/5,$$

we obtain

$$n \geqslant \left(\left[\frac{d}{5}\right] + 1\right)\left(\delta^2 - 2\left[\frac{\delta}{2}\right] + 1\right),$$

which proves (i). From here (ii) follows in exactly the same way as before.

To establish (iii), set $q = \delta + 1$ and let H denote the following graph discovered by Brown [4] and Erdős and Rényi [5]. Let $V(H)$ consist of all ordered triples $\underline{x} = (x_1, x_2, x_3) \neq \underline{0}$ whose elements are taken from $GF(q)$, where two triples $\underline{x}$ and $\underline{x}'$ are considered identical if $\underline{x}' = \lambda \underline{x}$ for some $\lambda \in GF(q)$, $\lambda \neq 0$. Let $\underline{x}\underline{y} \in E(H)$ if and only if $\underline{x} \cdot \underline{y} = 0$. Clearly, H is C_4-free and has $q^2 + q + 1$ vertices, each of degree q or $q + 1$.

Let us fix distinct $\mathbf{u}, \mathbf{v}, \mathbf{z} \in V(H)$ satisfying $\mathbf{u} \cdot \mathbf{z} = \mathbf{v} \cdot \mathbf{z} = \mathbf{z} \cdot \mathbf{z} = 0$. Let $\mathbf{u}_0 = \mathbf{z}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q$, and $\mathbf{v}_0 = \mathbf{z}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ denote the neighbors of $\mathbf{u}$ and $\mathbf{v}$, respectively. For every i $(1 \leqslant i \leqslant q)$ there is a uniquely determined $j(i)$ $(1 \leqslant j(i) \leqslant q)$ such that $\mathbf{u}_i\mathbf{v}_{j(i)} \in E(H)$. On the other hand, no $\mathbf{u}_i$ or $\mathbf{v}_j$ $(1 \leqslant i, j \leqslant q)$ is adjacent to $\mathbf{z}$ in H.

Let H_0 denote the graph obtained from H after the removal of the vertex $\mathbf{z}$ and all edges of the form $\mathbf{u}_i\mathbf{v}_{j(i)}$, $1 \leqslant i \leqslant q$. It is clear that $d_{H_0}(\mathbf{u}, \mathbf{v}) = 4$, and the minimum degree of the vertices of H_0 is $q - 1 = \delta$.

Let G be defined as the union of k disjoint isomorphic copies $H_0^{(1)}, H_0^{(2)}, \ldots, H_0^{(k)}$ of H_0, and let us make it connected by adding the edges $\mathbf{v}^{(t)}\mathbf{u}^{(t+1)}$ for every $1 \leqslant t < k$. Then $|V(G)| = n = k(q^2 + q) = k(\delta^2 + 3\delta + 2)$ and

$$\operatorname{diam} G = 5k - 1 = \frac{5n}{\delta^2 + 3\delta + 2} - 1. \quad \blacksquare$$

Conjecture. Let $r, \delta > 1$ be fixed natural numbers, and let G be a connected graph with n vertices and with minimum degree δ.

(i) If G is K_{2r}-free and δ is a multiple of $(r-1)(3r+2)$, then

$$\operatorname{diam} G \leqslant \frac{2(r-1)(3r+2)}{(2r^2-1)\delta} n + O(1) \qquad \text{while} \quad n \to +\infty.$$

(ii) If G is K_{2r+1}-free and δ is a multiple of $3r-1$, then

$$\operatorname{diam} G \leqslant \frac{(3r-1)}{r\delta} n + O(1) \qquad \text{while} \quad n \to +\infty.$$

These bounds, if valid, are asymptotically sharp, as is shown by the following graphs.

(i) Let $V(G) = \bigcup_{i=0}^{k} \bigcup_{j=1}^{r(i)} V_{ij}$, where $r(i) = r$ or $r-1$ depending on whether i is even or odd, and let

$$|V_{ij}| = \begin{cases} r\,\delta/(r-1)(3r+2) & \text{if} \quad i \neq 0, k \text{ is even} \\ (r+1)\,\delta/(r-+1)(3r+2) & \text{if} \quad i \neq 0, k \text{ is odd,} \end{cases}$$

and $|V_{0j}| = |V_{kj}| = \delta$ for every j. Let two vertices $v \in V_{ij}$ and $v' \in V_{i'j'}$ be joined by an edge if and only if (a) $|i - i'| = 1$ or (b) $i = i'$ and $j \neq j'$. Then G is obviously K_{2r}-free.

(ii) Let $V(G) = \bigcup_{i=0}^{k} \bigcup_{j=1}^{r} V_{ij}$, where $|V_{ij}| = \delta/(3r-1)$ if $i \neq 0, k$ and $|V_{0j}| = |V_{kj}| = \delta$ $(1 \leqslant j \leqslant r)$. Let the edge set of G be defined by the same rule as above. Then G is K_{2r+1}-free.

For an extensive survey of problems and results on the relations between the degrees, the radius, and the diameter of a graph see Chapter 4 in Bollobás [3], or Bermond and Bollobás [2]. A statement essentially equivalent to part (i) of Theorem 1 already appears in [1].

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