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ON THE NUMBER OF DISTINCT INDUCED SUBGRAPHS OF A GRAPH

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Let G be a graph on n vertices, $i(G)$ the number of pairwise non-isomorphic induced subgraphs of G and $k \geq 1$. We prove that if $i(G) = o(n^{k+1})$ then by omitting $o(n)$ vertices the graph can be made (l, m)-almost canonical with $l + m \leq k + 1$.

0. Introduction

We need some notation to state our main result.

Definition 1. $G = \langle V, E\rangle$ is l-canonical if there is a partition $\langle A_i : 0 \leq i < l\rangle$ of the vertex set V such that for $i, j < l$, $x, x' \in A_i$, $y, y' \in A_j$

$$\{x, y\} \in E \Leftrightarrow \{x', y'\} \in E.$$

Definition 2. For $G = \langle V, E\rangle$, $G' = \langle V, E'\rangle$ put $G\Delta G' = \langle V, E\Delta E'\rangle$, the symmetric difference of G and G'.

Definition 3. For $G = \langle V, E\rangle$ set $i(G) = |\{G[W] : W \subset V\}/\cong|$ i.e. denote by $i(G)$ the number of pairwise non-isomorphic induced subgraphs of G.

Definition 4. $G = \langle V, E\rangle$ is (l, m)-almost canonical if there is an l-canonical graph $G_0 = \langle V, E_0\rangle$ such that all the components of $G\Delta G_0$ have sizes at most m.

During the Cambridge Combinatorial conference held in March 1988 the second author stated the following conjecture.

Assume $i(G) = o(n^2)$. Then one can omit $o(n)$ vertices of G in such a way that the remaining graph is either complete or empty.

This was proved later independently by the two of us and by Alon and Bollobás [1]. We can actually prove the following stronger result.

Theorem 1. $\forall \varepsilon > 0 \forall k \geq 1 \exists \delta > 0 \forall n \forall G$ *with* n *vertices* $i(G) \leq \delta n^{k+1} \Rightarrow \exists W \subset V$,

$|W| \leq \varepsilon n$, such that $G[V \setminus W]$ is (l, m)-almost canonical for some l, m satisfying $l + m \leq k + 1$.

Note first that this implies the conjecture, as $l + m \leq 2$ implies $l = m = 1$. We would like to mention that this strong formulation of the theorem was inspired by a result of Zs. Nagy, who proved and strengthened a conjecture of the second author concerning infinite graphs. He proved that if for a graph $G = \langle \omega, E \rangle$, where ω is the set of natural numbers, $i(G)$ is less than the continuum, then for some $l, m < \omega$, the graph G is (l, m)-almost canonical. His result extends to weakly compact cardinals κ in place of ω. This result will be published elsewhere.

The main aim of this paper is to prove Theorem 1. This will be done in Section 1. In Section 2 we will discuss some further results and problems.

1. Proof of Theorem 1.

First we list our notation. Most of it is standard; we list it for the convenience of the reader. However, we will point out that, applying double-think, we use the convention $n = \{0, \ldots, n-1\}$ whenever it is convenient for us.

(1) For a set A, $[A]^2 = \{\{u, v\} : u, v \in A \wedge u \neq v\}$, the set of unordered pairs of A; $G[W] = \langle V, E \cap [W]^2 \rangle$ is the subgraph of $G = \langle V, E \rangle$ induced by W.

(2) For $A, B \subset V$ with $A \cap B = \emptyset$, $[A, B] = \{\{u, v\} : u \in A \wedge v \in B\}$; $G[A, B] = \langle A \cup B, E \cap [A, B] \rangle$ is the bipartite subgraph of G induced by A and B.

(3) $\bar{G}$ is the complement of G, i.e. $\bar{G} = \langle V, [V]^2 \setminus E \rangle$.

(4) For $x \in V$, $A \subset V$, $\Gamma(x, A) = \{y \in A : \{x, y\} \in E\}$, and $\Gamma(x) = \Gamma(x, V)$; $d(x, A) = |\Gamma(x, A)|$, $d(x, V) = d(x)$. We let $\bar{\Gamma}$, $\bar{d}$ denote the same functions for $\bar{G}$.

(5) $(A)^r$ is the set of sequences of length r formed for the elements of A. For $x \in (A)^r$ and $i < r$, x_i is the ith member of the sequence. For $r = 0$, $(A)^r = \{\emptyset\}$. For $x \in (V)^r$, $\varphi \in (2)^r$ put

$$\Gamma(x, \varphi) = \{z \in V : \forall i < r \, (\{z, x_i\} \in E \Leftrightarrow \varphi_i = 0)\}.$$

Note that $\Gamma(\langle u \rangle, \langle 0 \rangle) = \Gamma(u)$, $\Gamma(\langle u \rangle, \langle 1 \rangle) = \bar{\Gamma}(u)$ for $u \in V$, and $\Gamma(\emptyset, \emptyset) = V$.

(6) $\Delta(G) = \max\{d(x) : x \in V\}$; $\Delta(G, A, B) = \max\{d(x, B) : x \in A\}$.

(7) For $A \cap B = \emptyset$, $U, W \subset A \cup B$ put $G[U] \cong_{A,B} G[W]$ if there is an isomorphism π between $G[U]$ and $G[W]$ such that $\pi(U \cap A) = \pi(W \cap A)$.

(8) For $A \cap B = \emptyset$ we write

$$i(G, A, B) = |\{G[W] : W \subset A \cap B\}| \cong_{A,B}|,$$

i.e. the number of the equivalence classes with respect to the equivalence relation $\cong_{A,B}$. We will often use the fact that

$$i(G, A, B) \geq i(G[A, B]).$$

Our proof of Theorem 1 is quite lengthy. First, by proving a sequence of easy lemmas, we will establish that the theorem is (almost) true without the restriction $l+m \leqslant k+1$. This will be done in Lemma 9.

Then, in Lemma 10, we prove that this implies the theorem. We would like to point out that our proof yields a similar result in case k tends to infinity slowly (e.g. if $k = o(\log_3(n))$), but we do not go into the technical details.

First we give a rough estimate for $i(G)$ in the case of a disconnected graph.

Lemma 0. *Assume G has r components of sizes $n_i : i < r$. Then*
(a) $i(G) \geqslant (r!)^{-1} \prod_{i<r} n_i$
(b) *If $n_i \geqslant l$ for $i < r$ then $i(G) \geqslant \binom{r}{l}^l$.*

Lemma 1. *Assume $\{x_i : i < l\}$, $A_i : i < l$ are pairwise disjoint subsets of V, $[\{x_i : i < l\}]^2 \cap E = \emptyset$, $U_{i<l} A_i = A$, $[A]^2 \subset E$, $\Gamma(x_i, A) = A_i$ and $|A_i| \geqslant t$ for $i < l$. Then*

$$i(G) \geqslant \binom{t}{l}.$$

Lemma 2. *For every k there is an l such that whenever $\Delta(G) = o(n)$ and $i(G) \leqslant O(n^k)$ then there is a $W_n \subset V$, $|W_n| = o(n)$ such that*

$$\Delta(G[V \setminus W_n]) \leqslant l.$$

Lemma 2 is an important tool in our proof but we can only prove it later, after the proof of Lemma 8. First we prove a consequence of it.

Lemma 3. *For every k there is an l such that whenever $c > 0$; $A, B \subset V$; $A \cap B = \emptyset$; $|A|, |B| \geqslant cn$, $\Delta(G, A, B) = o(n)$ and $i(G, A, B) = O(n^k)$ then there is a $W_n \subset V$, $|W_n| = o(n)$ such that*

$$\Delta(G[A \setminus W_n, B \setminus W_n]) < l.$$

Proof. By omitting $o(n)$ vertices, we may assume $\Delta(G[A, V]) = o(n)$. By averaging we can see that for $C \subset A$ or $C \subset B$, $C \neq \emptyset$ and for every integer m

$$\left|\left\{y \in B : d(y, C) \geqslant \frac{1}{m}|C|\right\}\right| = o(n)$$

and

$$\left|\left\{y \in A : d(y, C) \geqslant \frac{1}{m}|C|\right\}\right| = o(n).$$

Using these, we can either pick, for every m and for sufficiently large n, an induced subgraph of $G[A, B]$ with m components, each having size at least $\frac{1}{m} n^{\frac{1}{2}}$, or we can omit $o(n)$ vertices from $A \cup B$ so that for the remaining graph

$G[A', B']$ we have $\Delta(G[A', B']) \leq n^{\frac{1}{2}}$. In the first case, by Lemma 0.a, we have

$$i(G[A, B]) \geq O\Big(\frac{l}{m!m^m} n^{m/2}\Big) \quad \text{for every } m.$$

In the second case, if the conclusion of Lemma 3 does not hold for an l, we can choose an induced subgraph of $G[A', B']$ having at least $n^{\frac{1}{2}}/2l$ components of sizes l. Then, by Lemma 0.b,

$$i(G[A, B]) \geq \frac{n^{l/2}}{(2l)^l}.$$

Hence $l \geq 2k+2$ satisfies the requirements of the Lemma. □

Lemma 4. *Assume* $r \geq 1$, $x \in (V)^r$, $\varphi_0 \neq \varphi_1 \in (2)^r$. *Let* $A = \Gamma(x, \varphi_0)$, $B = \Gamma(x, \varphi_1)$. *Then*

$$i(G) \geq i(G, A, B)n^{-r}.$$

Proof. Assume that $W_i \subset A \cup B$ for $i \leq n^r$ and that the $G[W_i]$ are pairwise non-equivalent with respect to $\cong_{A,B}$. We claim that the graphs

$$G_i = [W_i \cup \{x_\nu : \nu < r\}], \quad i \leq n^r$$

are not pairwise isomorphic. Indeed, otherwise for some $i \neq j \leq n^r$ there is an isomorphism π of G_i and G_j with $\pi(x_\nu) = x_\nu$ for $\nu < r$. Then π maps $W_i \cap A$ onto $W_j \cap A$, a contradiction. □

Lemma 5. *Let* $c > 0$, $r, l \geq 1$, $y \in V$, $x \in (V)^r$, $x_i \notin \Gamma(y)$ *for* $i < r$. *Assume further that there are* $\varphi_j \in (2)^r$, $j < l$ *such that*

$$|\Gamma(y) \cap \Gamma(x, \varphi_j)| \geq cn \quad \text{for } j < l.$$

Then

$$i(G) \geq (nr!)^{-1}(cn)^l.$$

Proof. For each sequence $\nu \in (cn)^l$ let W_ν be a set such that

$$\{y\} \cup \{x_i : i < r\} \subset W_\nu \subset \{y\} \cup \{x_i : i < r\} \cup \bigcup_{j<l} \quad (\Gamma(y) \cap \Gamma(x, \varphi_j))$$

and

$$|W_\nu \cap \Gamma(y) \cap \Gamma(x, \varphi_j)| = \nu_j, \quad \text{for} \quad j < l.$$

If $nr!+1$ of the different $G[W_\nu]$ are isomorphic, then $r!+1$ are pairwise isomorphic by isomorphisms keeping y fixed. Such an isomorphism keeps the set $\{x_i : i < r\}$ fixed. Hence there are $\nu \neq \nu'$ and an isomorphism π of $G[W_\nu]$ and $G[W_{\nu'}]$ such that $\pi(y) = y$, and $\pi(x_i) = x_i$ for $i < r$. But for any such π

$$\pi(\Gamma(y) \cap \Gamma(x, \varphi_j) \cap W_\nu) = \Gamma(y) \cap \Gamma(x, \varphi_j) \cap W_{\nu'} \quad \text{for} \quad j > l.$$

Hence $\nu = \nu'$, a contradiction. □

Lemma 6. *Assume* $x \in (V)^l$. *For* $y \in V$ *let*

$$f_x(y) = \max\{\min\{d(y, \Gamma(x, \varphi)), \bar{d}(y, \Gamma(x, \varphi))\} : \varphi \in (2)^l\}$$

and

$$g_x(n) = \max\{f_x(y) : y \in V \setminus \{x_i : i < l\}\}.$$

Assume $g_x(n) = o(n)$. *Then there are* $W_n \subset V$ *and* G_0 *such that* $|W_n| = o(n)$, G_0 *is* $\leq 2^l$*-canonical on* $V \setminus W_n$ *and* $\Delta(G[V \setminus W_n]\Delta G_0) = o(n)$. *Moreover, each of the classes of the canonical partition coincides with some* $\Gamma(x, \varphi) \setminus W_n$.

Proof. Put $A_\varphi = \Gamma(x, \varphi)$. We claim that we can omit $o(n)$ vertices W_n so that for $A'_\varphi = A_\varphi \setminus W_n$

$$\min\{\Delta(G, A'_\varphi, A'_\psi), \Delta(\bar{G}, A'_\varphi, A'_\psi)\} = o(n)$$

and

$$\min\{\Delta(G[A_\varphi]), \Delta(\bar{G}[A'_\varphi])\} = o(n),$$

holds for $\varphi \neq \psi \in (2)^l$. Indeed if for example the first of these claims is false for some $\varphi \neq \psi \in (2)^l$, then for some $c > 0$ and infinitely many n, we would have say

$$|\{x \in A'_\varphi : d(x, A'_\psi) \geq cn\}| \geq cn$$

and

$$|\{x \in A'_\varphi : \bar{d}(x, A'_\psi) \geq cn\}| \geq cn.$$

Then, by the assumption, for infinitely many n,

$$|\{x \in A'_\varphi : d(x, A'_\varphi) > \tfrac{3}{4}|A'_\psi|\}| \geq cn$$

and

$$|\{x \in A'_\varphi : \bar{d}(x, A'_\psi) > \tfrac{3}{4}|A'_\psi|\}| \geq cn$$

hence for some $y \in A'_\psi$, $f_x(y) > \frac{c}{2} n$ for infinitely many n, a contradiction. □

Lemma 7. *For every* k *there is an* l *such that whenever* $y \in V$, $A \subset \Gamma(y)$, $B \subset \bar{\Gamma}(y)$, $c > 0$, $|A|$, $|B| \geq cn$ *and* $i(G) \leq O(n^k)$ *then there are* $W_n \subset V$ *and a* G_0 *for which* $|W_n| = o(n)$, G_0 *is* l*-canonical on* $(A \cup B) \setminus W_n$ *and*

$$\Delta(G[A \setminus W_n, B \setminus W_n]\Delta G_0) \leq l.$$

Proof. We use the notation f_x, g_x introduced in the proof of Lemma 6 for the graph $G' = G[A, B]$ with $V' = A \cup B$. For an $x \in (V)^r$ and $i \leq r$ we denote the restriction of x to i by $x \mid i$. For every fixed l and for every $n \geq l$ we define a sequence $\langle x_i : i < l\rangle$ by recursion on i, using a greedy algorithm: we let x_i be an element of $V' \setminus \{x_j : j < i\}$ satisfying

$$f_{x|i}(x_i) = g_{x|i}(n).$$

We now claim that $g_x(n) = o(n)$ for an $x \in (V')^{l_1}$ with $l_1 \leq 2k + 3$. Indeed if

$g_x(n) \geq c_1 n$ for some $c_1 > 0$ for infinitely many n, then for all these n we have

$$\forall_i < l_1 \exists \varphi \in (2)^i (d(x_i, \Gamma(x \mid i, \varphi)) \geq c_1 n \wedge \bar{d}(x_i, \Gamma(x \mid i, \varphi) \geq c_1 n).$$

Then either there is a subsequence $\{x_{i_\nu} : \nu < k+2\} \subset A$ such that for $k+2$ functions $\psi \in (2)^{k+2}$ we have

$$|B \cap \Gamma(\langle x_{i_\nu} : \nu < k+2 \rangle, \psi)| \geq c_1 n$$

or the same holds when the roles of A and B are interchanged. This however, by Lemma 5, contradicts our assumption. This proves the claim. The claim and Lemma 6 imply that there is a 2^{l_1}-canonical graph G_0 and $W'_n \subset V$ such that $|W'_n| = o(n)$ and

$$\Delta(G'[V' \setminus W'_n] \,\Delta G_0) = o(n).$$

Let $\{A_j : j < 2^{l_1}\}$ be the canonical classes of G_0. We may assume (increasing l_1 to $2l_1$), that $A_j \subseteq A$ or $A_j \subset B$, hence we may assume that $G_0[A_j] = G'[A_j]$ has no edges. By Lemmas 3 and 4, using the last clause of Lemma 6, we can omit W_n, $|W_n| = o(n)$ vertices in such a way that $\Delta(G'[V \setminus W_n] \Delta G_0[V' \setminus W_n]) \leq l$ with $l \leq l_1 + 2k + 2 \leq 4k + 5$. □

Lemma 8. *For all k there exists an l such that whenever there are disjoint subsets* $\{x_i : i < l\}$, $A_i : i < l$ *and* $c > 0$ *satisfying* $[\{x_i : i < l\}]^2 \cap E = \emptyset$ *and*

$$A = \bigcup_{i<l} A_i, \quad \Gamma(x_i, A) = A_i; |A_i| \geq cn \quad \text{for} \quad i > l$$

then $i(G) \geq c_1 n^k$ *for some* $c_1 \geq 0$ *infinitely often.*

Proof. Assume that $\{x_i : i < l\}$ and $\{A_i : i < l\}$ are as above. We prove that $i(G) \geq c_1 n^k$ holds for some $c_1 > 0$ infinitely often, provided l is large enough. By Lemma 7, there exists an l_1 and l_1-canonical graphs $G_i : < l$ such that

$$\Delta\Big(G\Big[A_i, \bigcup_{j \neq i, j < l} A_i\Big] \Delta G_i\Big) \leq l_1.$$

Using a Ramsey type argument we can select a subsequence $\{x_{i_j} : j < l_2\}$, $c_2 > 0$ and $A'_{i_j} \subset A_j$ such that by putting $y_j = x_{i_j}$, $A''_j = A'_{i_j}$ we have $|A''_j| \geq c_2 n$ and either

(1) $\quad [A''_j, A''_t] \subset E, \quad$ for $\quad j < t < l_2$

or

(2) $\quad [A''_j, A''_t] \cap E = \emptyset, \quad$ for $\quad j < t < l_2$,

provided l is large enough compared to k, l_1, and l_2. If case (2) holds, by Lemma 0(a) we have

$$i(G) \geq i\Big(G\Big[\{x_j : j < l_2\} \cup \bigcup_{j<l_2} A''_j\Big]\Big) \geq c_3 n^{l_2}$$

for some $c_3 > 0$. If case (1) holds, then either for some $c_4 > 0$ and for more than

$l_2/2$ values of j, $\bar{G}[A_j'']$ has a component of size at least $c_4 n^{\frac{1}{2}}$ and in this case Lemma 0(a) implies that $i(\bar{G}) \geq c_5 n^{l_2/4}$ for some $c_5 > 0$, or else we may assume that for more than $l_2/2j$, the components of $\bar{G}[A_j'']$ have sizes at most k. This follows from Lemma 0(b). Then for some $c_6 > 0$ we can choose $\tilde{A}_j \subset A_j''$, $|\tilde{A}_j| \geq c_6 n$ for more than $l_2/2$ values of $j < l_2$ in such a way that $[\tilde{A}_j]^2 \subset E$. By Lemma 1, we have

$$i(G) \geq \binom{c_6 n}{l_2/2}. \quad \square$$

We are now in a position to prove Lemma 2.

Proof of Lemma 2. Just as in the proof of Lemma 3, if the lemma fails with $l = 2k+2$, then we may assume that omitting $o(n)$ vertices W_n arbitrarily, $\Delta(G[V \setminus W_n]) \geq n^{\frac{1}{2}}$ holds and that for every $A \subset V$, $A \neq \emptyset$ and for every m, $|\{x \in V : d(x, A) \geq 1/m\, |A|\}| = o(n)$. Using these, for every m and sufficiently large n, we can choose disjoint sets $\{x_i : i < m\}$, $A_i : i < m$ in such a way that $[\{x_i : i < m\}]^2 \cap E = \emptyset$ and for $A = \bigcup_{i>m} A_i$, $\Gamma(x_i, A) = A_i$ and $|A_i| \geq 1/m n^{\frac{1}{2}}$ hold for $i < m$. Now applying Lemma 8 for the graphs $G[\{x_i : i < m\} \cup A]$ we get a contradiction. $\square$

Now we can prove our main lemma.

Lemma 9. *Assume $i(G) = o(n^{k+1})$, $k \geq 1$. Then there are $W_n \subset V$, l and a G_0 such that $|W_n| = o(n)$, G_0 is l-canonical on $V \setminus W_n$ and*

$$\Delta(G[V \setminus W_n] \Delta G_0) \leq l.$$

Proof. We use the notation f_x, g_x introduced in Lemma 6 and we repeat the greedy algorithm described in the proof of Lemma 7, i.e. for every fixed l and for every $n \geq l$ we define a sequence $\{x_i : i < l\}$ by recursion on $i < l$ as follows: x_i is an element of $V \setminus \{x_j : j < i\}$ satisfying $f_{x|i}(x_i) = g_{x|i}(n)$. If for some l we have $g_x(n) = o(n)$, then by Lemma 6 there are $W_n' \subset V$, l_1 and G_0 such that $|W_n'| = o(n)$, G is 2^{l_1}-canonical on $V \setminus W_n'$ and

$$\Delta(G[V \setminus W_n'] \Delta G_0) = o(n).$$

Then, by Lemmas 2, 3, and 4, we can omit W_n, $|W_n| = o(n)$ vertices so that for some l

$$\Delta(G[V \setminus W_n] \Delta G_0[V \setminus W_n]) \leq l.$$

Hence we may assume that the following holds infinitely many n:

(*) There is a sequence $\{x_i : i < l\}$ of distinct elements such that

$$\forall i < l \exists \varphi \in (2)^i\, d(x_i, \Gamma(x \mid i, \varphi) \geq cn) \wedge \bar{d}(x_i, \Gamma(x_i, \varphi) \geq cn)$$

for some $c > 0$.

We may as well assume that (*) holds for all n and prove that if (*) holds for large enough l, then $i(G) \geq c_0 n^{k+1}$ for some $c_0 > 0$ infinitely often.

First remark that (*) holds for any subsequence of $\langle x_i : i < l \rangle$. Now, by Lemma 5, we may assume that

$$(\forall \varepsilon < 2) \mid \{0 < i < l : x_i \in \Gamma(\langle x_0 \rangle, \langle \varepsilon \rangle) \wedge \exists \varphi (\varphi(o) = 1 - \varepsilon \wedge d(x_i, \Gamma(x \mid i, \varphi)) \geq cn \wedge \bar{d}(x_i, \Gamma(x \mid i, \varphi))) \geq cn\} \mid \leq k + 1,$$

as otherwise we are done.

It follows that for either the graph or its complement the following statement is true.

There is a set

$$T \subset l - \{0\}, \qquad |T| \geq \frac{l}{2(k+1)2^{k+1}} \geq \frac{l}{5^{k+1}}$$

such that $\{x_i : i \in T\} \subset \Gamma(x_0)$, and we can omit W_n vertices, $|W_n| = o(n)$, of $\bar{\Gamma}(x_0)$ in such a way that for all $i, j \in T$ and for all $z \in \bar{\Gamma}(x_0) \setminus W_n$, $\{z, x_i\} \in E \Leftrightarrow \{z, x_j\} \in E$. Now by a repeated application of this argument we obtain that if $l > 4.5^{l_1(k+1)}$ then for either the graph or its complement the following holds:

(1) There is a set $Y = \{y_i : i < l_1\}$, $[Y]^2 \subset E$, a $c_1 > 0$ and a sequence of pairwise disjoint subsets of V such that

$$|A_i| \geq c_1 n, A_i \subset \bar{\Gamma}(y_i) \quad \text{for} \quad i < l_1;$$

$$A_i \subset \Gamma(y_i) \wedge A_i \cap \Gamma(y_{i+1}) = A_i \cap \Gamma(y_j) \quad \text{for} \quad i < j < l_1;$$

and either $A_i \cap \bar{\Gamma}(y_{i+1}) \geq c_2 n$ for $i + 1 < l_1$ or $A_i \subset \Gamma(y_j)$ for $i < j < l_1$, for $c_2 > 0$.

We will assume that (1) holds for G. If in the last statement the first alternative holds, then applying Lemma 5 with $y = y_{l_1 - 1}$ we get that

$$i(G) \geq c_2 n^{l_1 - 3} \quad \text{with some} \quad c_3 > 0.$$

Thus we may assume that $A_i \subset \Gamma(y_j)$ for $i < j < l_1$. However, in this case Lemma 8 yields $i(\bar{G}) \geq c_0 n^{k+1}$ provided l_1 is large enough. □

To conclude the proof of Theorem 1, it remains only to prove the following.

Lemma 10. *Assume G has n vertices, $i(G) = o(n^{k+1})$ for some $k \geq 1$. Assume further that l is minimal with respect to the following property:*

(*) There are $c > 0$ and s and an l-canonical graph $G_0 = \langle V, E_0 \rangle$ with canonical classes $\langle A_i : i < l \rangle$, $|A_i| \geq cn$ for $i < l$ and $\Delta(G \Delta G_0) \leq s$.

Then $l \leq k$ and we can find $W_n \subset V$, $|W_n| = o(n)$ such that setting $G_1 = G \Delta G_0$ all components of $G_1[V \setminus W_n]$ have size at most $m = k + 1 - l$.

Proof. Set $m = k + 1 - l$ if $l \leq k$ and $m = 0$ otherwise. Assume for a contradiction that the claim is not true. Then for some $c_1, c_2 > 0$, $c_2 < \frac{1}{4} c_1$ we can find pairwise disjoint sets $\{A'_i : i < l\}$ and a set B such that

(1) $|A'_i| = c_1 n$, $A'_i \subset A_i$ for $i < l$.

(2) For $A = \bigcup_{i<l} A'_i$, $B \subset A$, $|B| = c_2 n$.

(3) $G_1[B]$ consists of components of size $m+1$, and $G_1[A]$ has only edges contained in $G_1[B]$.

We claim that $i(G[A]) \geq c_3 n^{l+m}$ for some $c_3 > 0$. Let $A''_i = A'_i \setminus B$ for $i < l$. Then $|A''_i| \geq 3/4 c_1 n$. Let now $X, Y \subset A$ and let π be an isomorphism of $G[X]$ and $G[Y]$. Assume further that $|X \cap A''_i| \geq c_1/2$ for $i < l$.

For $u \in X$ set $\bar{\pi}(u) = j$ if $\pi(u) \in A'_j$. Using $|X \cap A''_i| \geq 2|B|$, for large enough n there are $l+1$ elements of $X \cap A''_i$ with image in $A \setminus B$, hence we can choose $x_i \neq y_i \in X \cap A''_i$ with $\pi(x_i)$, $\pi(y_i) \in A \setminus B$ and $\bar{\pi}(x_i) = \bar{\pi}(y_i)$, for $i < l$. Then the minimality of l implies that $\bar{\pi}(x_i) \neq \bar{\pi}(x_j)$ for $i \neq j < l$. Using again the minimality of l and the fact that

$$\{x_i : i < l\} \cup \{\pi(x_i) : i < l\} \subset A \setminus B$$

we get that if $u, v \notin \{x_i : i < l\}$ then $u, v \in A'_\nu$ for some $\nu < l$ if and only if

$$(\forall i < l)(\{u, x_i\} \in E \Leftrightarrow \{v, x_i\}) \in E$$

and also that if $u, v \notin \{\pi(x_i) : i < l\}$ then $u, v \in A'_\nu$ for some $\nu < l$ if and only if

$$(\forall i < l)(\{u, \pi(x_i)\} \in E \Leftrightarrow \{v, \pi(x_i)\}) \in E.$$

Now for each $u \in A'_i \cap X$, $\pi(u) \in A_{\bar{\pi}(x_i)}$. Indeed, for $u \in A'_i \cap X$, $u \neq x_i, y_i$ we have

$$\begin{aligned} &(\forall i < l)(\{u, x_i\} \in E \Leftrightarrow \{y_i, x_i\} \in E) \\ &\quad \Leftrightarrow (\forall i < l)(\{\pi(u), \pi(x_i)\} \in E \Leftrightarrow \{\pi(y_i), \pi(x_i)\} \in E) \\ &\quad \Leftrightarrow \pi(u) \in A'_{\bar{\pi}(y_i)} \\ &\quad \Leftrightarrow \bar{\pi}(u) = \bar{\pi}(x_i). \end{aligned}$$

It follows that

$$(4) \qquad \pi(A'_i \cap X) = A_{\bar{\pi}(x_i)} \cap Y \quad \text{for} \quad i < l.$$

Now, for each $i < l$, $G_1[A_i] = G[A_i]$ or $G_1[A_i] = \bar{G}[A_i]$. Also, for each $i < j < l$, $G_1[A_i, A_j] = G[A_i, A_j]$ or $G_1[A_i, A_j] = \bar{G}[A_i, A_j]$. Considering this, (4) implies that π is an isomorphism of $G_1[X]$ onto $G_1[Y]$. In the case $m = 0$, (4) implies that $i(G) \geq c_3 n^l$ for some $c_3 > 0$. In the case $m > 0$ and all the components $G_1[X \cap B]$ have size at least two, then

$$\pi(X \cap B) = Y \cap B \quad \text{and} \quad \pi(A''_i \cap X) = A''_i \cap Y \quad \text{for} \quad i < l.$$

As there are $c_4 n$ ways to choose the cardinalities $|B \cap A''_i|$ for $i < l$, and since $G_1[B]$ has $c_5 n^m$ pairwise nonisomorphic subgraphs each having no isolated points, for some $c_4, c_5 > 0$, we are done. □

2. One more result and some problems

One may conjecture that if G is a strong Ramsey example, then G is close to a random graph, hence $i(G)$ is very large, say exponential. As is shown by the attempt described in [1], this will be difficult to prove. We only have one result pointing in this direction.

Theorem 2. *Assume G is a graph with n-vertices $c>0$, $k>2c\log 2$ and*

$$K_{c\log n, c\log n}\not\subset G, \bar{G}.$$

Then, for every sufficiently large n, $i(G)\geq 2^{n/4k}$.

Proof. We may assume that there is an $x\in V$ with

$$d(x)\geq (n/\log^2 n),\ \bar{d}(x)\geq \tfrac{1}{2}n.$$

Let $A\subset\Gamma(x)$, $B\subset\bar{\Gamma}(x)$ with $|A|=\lfloor(n/\log^2 n)\rfloor$, $|B|=\lfloor\frac{n}{2}\rfloor$. Let $\mathscr{F}=\{\Gamma(x)\cap A: x\in B'\}$, $|B'|>\frac{n}{3}$, $B'\subset B$. Assume first $|\mathscr{F}|>\frac{n}{3k}$. Let $C\subset B'$, $|C|=\lfloor(n/3k)\rfloor$ be such that $\Gamma(y)\cap A\neq\Gamma(z)\cap A$ for $y\neq z\in C$. Consider the graphs $G[\{x\}\cup A\cup Y]$ for $Y\subset C$. If $n\cdot|A|!+1$ of them are pairwise isomorphic, then there are two, say

$$G[\{x\}\cup A\cup Y_0]\quad\text{and}\quad G[\{x\}\cup A\cup Y_1]$$

which are isomorphic by an isomorphism π keeping x and the elements of A fixed. Clearly such a π must keep the elements of Y_0 fixed, hence $Y_0=Y_1$. It follows that in this case

$$i(G)\geq 2^{\lfloor n/3k\rfloor}\cdot(n\cdot n^{n/\log^2 n})^{-1}>2^{n/4k}$$

holds for sufficiently large n. Hence we may assume that there is a sequence $B_i: i\leq l$ of pairwise disjoint subsets of B such that $|B_i|=k$ and $\Gamma(y)\cap A=\Gamma(z)\cap A$ whenever $y,z\in B_i$ for $i<l$, for an l satisfying $k\cdot l>2c\log n$, i.e. for an $l=\lfloor c_1(\log n/\log 2)\rfloor$ with $c_1<1$.

Let $D=\cup_{i<l}B_i$. It now follows that there is an $E\subset A$, $|E|\geq|A|\cdot 2^{-c_1(\log n/\log 2)}\geq n^{1-c_1}\cdot(\log n)^{-2}$ such that $\Gamma(u)\cap D=\Gamma(v)\cap D$ for $u,v\in E$. As $n^{1-c_1}\cdot(\log n)^{-2}>c\log n$ for sufficiently large n, this contradicts the assumptions of the theorem. □

Clearly, the above computation can be slightly improved, but we have examples to show that the assumptions of Theorem 2 do not imply $i(G)>2^{(2n\log k/k)}$.

At present we are unable to extend Theorem 2 to graphs G for which

$$K_{c\log n, c\log n, c\log n}\not\subset G, \bar{G}.$$

Reference

[1] N. Alon, B. Bollobás, Graphs with a small number of distinct induced subgraphs, this issue.