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# ON THE ITERATES OF THE ENUMERATING FUNCTION OF FINITE ABELIAN GROUPS

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*Abstract. The function $a^{(r)}(n)$, which represents the r-th iteration of $a(n)$ (the number of non-isomorphic Abelian groups with n elements), is studied. Upper bounds for $a^{(r)}(n)$ are established, as well as the asymptotic formula for sums of $K(n)$, where $K(n)=\min\{r: a^{(r)}(n)=1\}$. Connections with analogous problems for the iterations of $d(n)$ (the number of divisors of n) are discussed.*

## 1. *Introduction*

Let $a(n)$ denote the number of non-isomorphic Abelian (commutative) groups with $n$ elements. It is well known that (see [5]) $a(n)$ is a multiplicative function of $n$ ($a(mn)=a(m)a(n)$ for coprime $m$ and $n$) such that $a(p^k)=P(k)$ for every prime $p$ and integer $k\geq 1$ (here and later $p, p_1, p_2, \ldots$ denote primes), where $P(k)$ is the number of unrestricted parititons of $k$. Hence $P(1)=1$, $P(2)=2$, $P(3)=3$, $P(4)=5$, and as $k\to\infty$

$$P(k)=(1+o(1))\,(4\sqrt{3}k)^{-1}\exp\{\pi(2k/3)^{1/2}\},$$

which is a classical formula due to Hardy and Ramanujan (see [13], p. 240). The values of $a(p^k)$ do not depend on $p$ but only on $k$, so that $a(n)$ is a „prime-independent" multiplicative function satisfying $a(p)=1$ for every prime $p$. One can easily exhibit other integer valued arithmetic functions with similar properties, and one such function is $S(n)$, the number of non-isomorphic finite semisimple rings with $n$ elements (see [10]). Thus in what follows one could easily generalize the problems and results to a suitable class of non-negative, prime independent, integer valued multiplicative functions such that $f(p)=1$ for every prime $p$. However, in order to keep the exposition clear and simple, we shall deal only with the case $f(n)=a(n)$.

From known results on $a(n)$ we mention

$$\limsup_{n\to\infty} \frac{\log a(n) \log\log n}{\log n} = \frac{\log 5}{4}, \tag{1.2}$$

which was proved by E. Krätzel [12] ($\log n = \ln n$ is the natural logarithm of $n$), and

$$\sum_{n\le x} a(n) = \sum_{m=1}^{3} A_m x^{1/m} + R(x),\ A_m = \prod_{k=1,\, k\ne m}^{\infty} \zeta(k/m). \tag{1.3}$$

Here $\zeta$ is the Riemann zeta-function, and $R(x)$ is the error term in the asymptotic formula (1.3), for which the best published estimate $R(x) \ll x^{97/381}\log^{35}x$ is due to G. Kolesnik [11] (here and later $f(x) \ll g(x)$ and $f(x)=O(g(x))$ both mean $|f(x)| \le Cg(x)$ for $x \ge x_0$, and $C, C_1, C_2, \ldots$ are some (unspecified) positive constants). For other recent results on $a(n)$ the reader is referred to [6], [7], [9] and Ch. 14 of [8]. The aim of this paper is the study of the iterates of $a(n)$. For any arithmetic function $f: N \to N$, and any integer $r \ge 1$ one can define

$$f^{(r)}(n) = \underbrace{f(f(\ldots f}_{r \text{ times}}(n)\ldots))$$

as the $r$-th iterate of $f$, so that in this notation $f^{(1)}(n) = f(n)$. If $f(n)$ is multiplicative, then in general already $f^{(2)}(n)$ is not multiplicative, which makes the study of the iterates of multiplicative functions difficult. If $r \ge 2$ is fixed, then two among the most natural problems concerning $f^{(r)}(n)$ are the evaluation of sums of $f^{(r)}(n)$ and the determination of the maximal order of $f^{(r)}(n)$. In the case of $f(n) = d(n) = \sum_{ab=n} 1$ (the number of divisors of $n$), these problems were treated by Erdős and Kátai [2], [3]. In [3] it was proved that

$$\sum_{n\le x} d^{(r)}(n) = (1+o(1)) A_r\, x \log_r x \qquad (A_r > 0,\ x \to \infty) \tag{1.4}$$

holds for $r=4$, which was shown earlier by I. Kátai to be true for $r=2,3$ also. An old conjecture of Bellman and Shapiro (see [1]) states that (1.4) holds for any fixed $r \ge 2$ ($\log_r x = \log(\log_{r-1} x)$ is the $r$-fold iterated logarithm). On the other hand, Erdős and Kátai proved in [2] that for every $\varepsilon > 0$

$$d^{(r)}(n) \quad \ll \quad \exp\{(\log n)^{1/l_r + \varepsilon}\} \tag{1.5}$$

and that

$$d^{(r)}(n) \quad > \quad \exp\{(\log n)^{1/l_r - \varepsilon}\} \tag{1.6}$$

holds for infinitely many $n$, which means that they have essentially determined the maximal order of $d^{(r)}(n)$. Here $l_r$ is the $r$-th Fibonacci number: $l_{-1}=0$, $l_0=1$, $l_r = l_{r-1} + l_{r-2}$ for $r \ge 1$.

When one considers the above two problems for $a^{(r)}(n)$, then it turns out that the situation is in a certain sense opposite to the one for $d^{(r)}(n)$, where (1.4) is known only for $r \leq 4$, but (1.5) (up to „ε") is the best possible. Namely, it was proved by A. Ivić [7] that

$$\sum_{n \leq x} a(a(n)) = \sum_{n \leq x} a^{(2)}(n) = Cx + O(x^{1/2} \log^4 x) \tag{1.7}$$

for a suitable $C > 0$, and since trivially $a^{(r)}(n) \leq a(n)$ the method of [7] obviously gives also

$$\sum_{n \leq x} a^{(r)}(n) = B_r x + O(x^{1/2} \log^4 x) \qquad (B_r > 0) \tag{1.8}$$

for any fixed $r \geq 1$, which can be compared to (1.4). In particular, (1.8) shows that $a^{(r)}(n)$ possesses a positive mean value for any fixed $r \geq 1$, and the error term (uniform in $r$) in (1.8) is sharp. Thus this problem is satisfactorily resolved, but determining the maximal order of $a^{(r)}(n)$ turns out to be difficult. The methods of [2] which yield (1.5) seem to be of no avail here, and we are at present unable to determine precisely the maximal order of $a^{(r)}(n)$. There is, however, another problem involving $a^{(r)}(n)$ which is somewhat different from the corresponding problem for $d^{(r)}(n)$, and with which we can deal successfully. Since $a(p) = 1$ and $d(p) = 2$ for all primes $p$, it makes sense to define

$$k(n) = \min \{r : d^{(r)}(n) = 2\}$$

and

$$K(n) = \min \{r : a^{(r)}(n) = 1\}.$$

The existence of both $k(n)$ and $K(n)$ is easily established, and in [3] it was shown that

$$0 < \limsup_{n \to \infty} \frac{k(n)}{\log \log \log n} < \infty. \tag{1.9}$$

It was noted in [3] that the summatory function of $k(n)$ is very difficult to estimate. On the other hand, we shall establish in Th. 2 a sharp asymptotic formula for the summatory function of $K(n)$, which implies that $K(n)$ has a positive mean value. The upper bound in (1.9) remains true if $k(n)$ is replaced by $K(n)$. This will follow trivially from our upper bound for $a^{(r)}(n)$ in Th. 1, but we are unable to determine whether the lim sup in question for $K(n)$ is positive or (which seems to us to be more likely true) is equal to zero. We also think that

$$\limsup_{n \to \infty} K(n) = \infty, \tag{1.10}$$

but we are unable to prove (1.10) at present.

## 2. *Statement of results*

Before we formulate our results we note that there are many $n$ for which $a(a(n))$ must be fairly large. Namely, let

$$n = (p_1 p_2 \cdots p_k)^2,$$

where $p_j$ is the $j$-th prime number. Then

$$a(n)=P^k(2)=2^k,$$
$$a(a(n))=P(k)\gg\exp(Ck^{1/2})\quad (C>0), \tag{2.1}$$

which follows from (1.1). But the prime number theorem gives

$$\log n=2\sum_{p\le p_k}\log p=(2+o(1))p_k=(2+o(1))k\log k\quad (k\to\infty),$$

hence $k\gg\log n/\log_2 n$, and (2.1) implies that

$$a(a(n))=a^{(2)}(n)\gg\exp(C_1(\log n/\log\log n)^{1/2})\qquad (C_1>0) \tag{2.2}$$

holds for infinitely many integers $n$. Lower bounds for $a^{(r)}(n)$ for $r\ge3$ are difficult to obtain, since very little is known about the structure of prime factors of $P(k)$. The situation with upper bounds for $a^{(r)}(n)$ is better, and we shall prove

THEOREM 1. There is a constant $B>0$ such that

$$a^{(2)}(n)=a(a(n))\ll\exp\left\{\frac{B(\log n)^{7/8}}{(\log\log n)^{19/16}}\right\}, \tag{2.3}$$

and if $c_r$ is the constant defined by

$$\log a^{(r)}(n)\ll_r(\log n)^{c_r}, \tag{2.4}$$

then for $r\ge3$

$$c_r\le\frac{1}{2}c_{r-1}+\frac{3}{8}c_{r-2}\qquad \left(c_1=1,\ c_2=\frac{7}{8}\right). \tag{2.5}$$

As in § 1, let $K(n)$ for a given $n$ be the smallest $r$ such that $a^{(r)}(n)=1$. Then we have

THEOREM 2. There is a constant $E>1$ such that for any given $\varepsilon>0$

$$\sum_{n\le x}K(n)=Ex+O(x^{1/2+\varepsilon}). \tag{2.6}$$

While the asymptotic formula (2.6) is sharp, the bounds for the constants $c_r$ (defined by (2.4)) can probably be improved. In the proof of Th. 1 we shall use the upper bound

$$\omega(P(n))\ll\frac{n^{1/2}}{\log n}, \tag{2.7}$$

which is the immediate consequence of (1.1) and $\omega(n)\ll\log n/\log_2 n$, where as usual $\omega(n)$ denotes the number of distinct prime factors of $n$ (and $\Omega(n)$

is the number of all prime factors of $n$). Better bounds than (2.7) would lead to better results in Th. 1, and in particular we conjecture that

$$\omega(P(n)) \ll \log^C n \tag{2.8}$$

for some suitable $C>0$, which would give $c_2 \leq 3/4$ in Th. 1. If true, (2.8) seems unattainable by present methods.

## 3. *Upper bound estimates for iterates*

In this section we shall prove Th. 1. The crucial element in the proof is the upper bound for $\omega(a(n))$, contained in the following

*Lemma* 1.

$$\omega(a(n)) \ll (\log n)^{3/4} (\log\log n)^{-11/8}. \tag{3.1}$$

*Proof.* Let $n = p_{j_1}^{\alpha_1} \dots p_{j_r}^{\alpha_r}$ $(r = \omega(n))$ be the canonical decomposition of $n$. Then

$$a(n) = P(\alpha_1) \dots P(\alpha_r),$$

and in bounding $\omega(a(n))$ we can suppose that the $\alpha_j$'s are distinct, since $\omega(m^k) = \omega(m)$. If $S \geq 2$ is a parameter which will be determined later, then using (2.7) we obtain

$$\begin{aligned}
\omega(a(n)) &= \omega\Big(\prod_{\alpha_i \leq S} P(\alpha_i) \prod_{\alpha_i > S} P(\alpha_i)\Big) \\
&\leq \sum_{\alpha_i \leq S} \omega(P(\alpha_i)) + \sum_{\alpha_i > S} \omega(P(\alpha_i)) \\
&\ll \frac{S^{1/2}}{\log S} \sum_{\alpha_i \leq S} 1 + \sum_{\alpha_i > S} \frac{\alpha_i^{1/2}}{\log \alpha_i} \\
&\ll \frac{S^{3/2}}{\log S} + \frac{1}{\log S} \Big(\sum_{i=1}^{r} \alpha_i\Big)^{1/2} \Big(\sum_{\alpha_i > S} 1\Big)^{1/2} \\
&\ll \frac{S^{3/2}}{\log S} + \frac{1}{\log S} (\log n)^{1/2} \Big(\sum_{\alpha_i > S} 1\Big)^{1/2},
\end{aligned}$$

since

$$\sum_{i=1}^{r} \alpha_i = \Omega(n) \leq \frac{\log n}{\log 2}.$$

To estimate $R = R(S, n) = \sum_{\alpha_i > S} 1$, note that from $n = \prod_{i=1}^{r} p_{j_i}^{\alpha_i} \geq \prod_{\alpha_i > S} p_{j_i}^{\alpha_i}$

we have ($p_R$ is the $R$-th prime)

$$\log n \geq S \sum_{p \leq p_R} \log p \gg SR \log R.$$

Hence for $\log^A n \ll S \ll \log^B n \quad (0<A<B<1)$ we have

$$R = R(S, n) = \sum_{\alpha_i > S} 1 \ll \frac{\log n}{S \log\log n}. \tag{3.2}$$

Therefore we obtain

$$\omega(a(n)) \ll \frac{S^{3/2}}{\log S} + \frac{S^{-1/2} \log n}{\log S (\log\log n)^{1/2}}$$

$$\ll (\log n)^{3/4} (\log\log n)^{-11/8}$$

on taking

$$S = (\log n)^{1/2} (\log\log n)^{-1/4}.$$

This ends the proof of Lemma 1, but we remark that our method can be used to bound $\omega(f(n))$ for a fairly wide class of prime-independent, integer-valued multiplicative functions. In particular, we obtain

$$\omega(d(n)) \ll \left(\frac{\log n}{\log_2 n \log_3 n}\right)^{1/2}. \tag{3.3}$$

Namely,

$$\omega(d(n)) = \omega\left(\prod_{\alpha_i \leq S} (\alpha_i + 1) \prod_{\alpha_i > S} (\alpha_i + 1)\right)$$

$$\leq \omega\left(\prod_{\alpha_i \leq S} (\alpha_i + 1)\right) + \sum_{\alpha_i > S} \omega(\alpha_i + 1)$$

$$\ll \sum_{p \leq S+1} 1 + \sum_{\alpha_i > S} \log \alpha_i / \log_2 \alpha_i \ll S/\log S + \sum_{\alpha_i > S} \log_2 n / \log_3 n$$

$$\ll \frac{S}{\log S} + \frac{\log n}{S \log_3 n} \ll \left(\frac{\log n}{\log_2 n \log_3 n}\right)^{1/2}$$

on taking

$$S = \left(\frac{\log n \log_2 n}{\log_3 n}\right)^{1/2},$$

where we used again (3.2). Note that the order of $d(d(n))$ is closely related to the order of $\omega(d(n))$, since trivially

$$d(n) = \sum_{\delta | n} 1 \geq \sum_{\delta | n} \mu^2(\delta) = 2^{\omega(n)},$$

hence $\omega(n)\ll\log d(n)$, $\omega(d(n))\ll\log d(d(n))$, and on the other hand

$$\log d(n)=\sum_{i=1}^{r}\log(\alpha_i+1)\ll r\log\log n=\omega(n)\log\log n,$$

which then yields

$$\omega(d(n))\ll\log d(d(n))\ll\omega(d(n))\log\log n. \tag{3.4}$$

We also remark that (3.3) remains valid if $d(n)=d_2(n)$ is replaced by $d_k(n)$ ($k\geq 2$ fixed), which represents the number of ways $n$ can be written as a product of $k$ factors. This is a multiplicative function satisfying

$$d_k(p^\alpha)=\frac{(\alpha+1)\ldots(\alpha+k-1)}{(k-1)!}.$$

Having at our disposal Lemma 1, it is a fairly simple matter to prove (2.3) and (2.5). Namely, from (1.1) we have by the Cauchy-Schwarz inequality

$$a(n)=P(\alpha_1)\ldots P(\alpha_r)<\exp\left\{C\sum_{i=1}^{r}\alpha_i^{1/2}\right\}<\exp\left(C(\omega(n)\,\Omega(n))^{1/2}\right). \tag{3.5}$$

It follows that

$$a(a(n))<\exp\left(C(\omega(a(n))\Omega(a(n))^{1/2}\right)$$
$$\ll\exp\left(C_1\{(\log n)^{3/4}(\log_2 n)^{-11/8}(\log n/\log_2 n)\}^{1/2}\right)=$$
$$=\exp\left(C_1(\log n)^{7/8}(\log_2 n)^{-19/16}\right),$$

since using (1.2) we have

$$\Omega(a(n))\leq\frac{\log a(n)}{\log 2}<\frac{\log n}{\log\log n}.$$

To prove (2.5) we use induction (trivially $c_1=1$ and $c_2\leq 7/8$ by (2.3)) and (3.5) with $n$ replaced by $a^{(r-1)}(n)$. This gives by Lemma 1, for $r\geq 3$,

$$a^{(r)}(n)=a(a^{(r-1)}(n))<\exp\left(C\{\omega(a(a^{(r-2)}(n)))\Omega(a^{(r-1)}(n))\}^{1/2}\right)$$
$$<\exp\left(C_1\{(\log a^{(r-2)}(n))^{3/4}\log a^{(r-1)}(n)\}^{1/2}\right)$$
$$<\exp\left(C_2(\log n)^{(3c_{r-2}+4c_{r-1})/8}\right),$$

where $C_2>0$ possibly depends on $r$. Hence (2.4) holds with $c_r$ satisfying (2.5). We could have also obtained (2.4) with certain negative powers of $\log_2 n$ multiplying $(\log n)^{c_r}$, but this did not seem of great importance.

## 4. *Proof of the asymptotic formula for iterates*

In this section we shall prove the asymptotic formula (2.6) of Th. 2. The proof will use the following

*Lemma* 2. For $j \geq 1$ and $k \geq 2$ we have uniformly

$$\sum_{n \leq x, a^{(j)}(n)=k} 1 = d_{j,k}\, x + O\,(x^{1/2} \log^2 x)$$

with suitable constants $d_{j,k} \geq 0$. Moreover for some suitable constants $C_1$, $C_2$, $C_3 > 0$

$$d_{j,k} \leq C_1 \exp\,(-C_2 \log\,(C_3 2^j k) \log\log\,(C_3 2^j k)). \tag{4.2}$$

*Proof.* For $j=1$ (4.1) and (4.2) reduce to a result proved by A. Ivić [7], so that we can suppose $j \geq 2$. Note that in the terminology of Ivić-Tenenbaum [9] the function $f(n) = a^{(j)}(n)$ ($j \geq 1$ fixed) is an $s$-function. This means that $f(n) = f(s(n))$, where $s = s(n)$ is the squarefull part of $n$ ($s$ is squarefull if $p^2 \mid s$ whenever $p$ is a prime such that $p \mid s$). Hence from [9] we have (4.1) with

$$0 \leq d_{j,k} = 6\pi^{-2} \sum_{s=1, a^{(j)}(s)=k}^{\infty} (s \prod_{p \mid s} (1+1/p))^{-1}, \tag{4.3}$$

where summation is over all squarefull $s$ (empty sum being understood as zero). Let $s_1$ be the smallest squarefull $s$ for which $a^{(j)}(s) = k$ (if no such $s_1$ exists, then $d_{j,k} = 0$). Using multiplicativity and the properties of the partition function we have $a(n) \leq n^{1/2}$ for $n \geqslant n_0$, which combined with (1.2) gives, for $j \geq 2$,

$$k = a^{(j)}(s_1) \leq (a(s_1))\, 2^{1-j} \leq C_0 \exp\,(C_1 2^{-j} \log s_1 / \log\log s_1).$$

$$s_1 \geq \exp\,(C_2 \log\,(2^j k / C_0) \log\log\,(2^j k) C_0)),$$

and

$$d_{j,k} \leq 6\pi^{-2} \sum_{s \geq s_1} s^{-1} \ll s_1^{-1/2} \leq C_1 \exp\,(-C_2 \log\,(C_3 2^j k) \log\log\,(C_3 2^j k))$$

as asserted. Here we used the fact that the elementary formula

$$\sum_{s \leq x_1} 1 = \frac{\zeta\,(3/2)}{\zeta\,(3)}\, x^{1/2} + O\,(x^{1/3})$$

gives by partial summation

$$\sum_{s \geq y} s^{-1} \ll y^{-1/2}.$$

A more careful argument, based on Th. 1, would give

$$d_{j,k} \leq C_1 \exp\,(-C_2 (\log\,(k+1))^{C_3^j}) \quad (C_3 = \sqrt{8/7}\ ;\ C_1, C_2 > 0,\ k \geq 2,\ j \geq j_0). \tag{4.4}$$

We pass now to the proof of Th. 2. We shall use only the weak bound $K(n) \leq \log n$ ($n \geq n_0$), although $K(n) \ll \log_3 n$ follows easily from Th. 1. We can write

$$\sum_{n \leq x} K\,(n) = \sum_{1 \leq k \leq \log x} \Big( \sum_{n \leq x, K(n)=k} 1 \Big). \tag{4.5}$$

If $K(n)=1$, this means that $n$ is squarefree. Hence

$$\sum_{n\leq x, K(n)=1} 1=\sum_{n\leq x} \mu^2(n)=6\pi^{-2}x+O(x^{1/2}).$$

Since $a(n)=1$ is equivalent to $n$ being squarefree, this means that if $K(n)=k$ $(k\geq 2)$, then we must have $a^{(k-1)}(n)=r$, $r$ squarefree and $r>1$. Hence for $k\geq 2$ Lemma 2 gives

$$\sum_{n\leq x, K(n)=k} 1= \sum_{n\leq x, a^{(k-1)}(n)=r,\ 1<r=\mathit{squarefree}} 1$$

$$=\sum_{2\leq r\leq x^{\varepsilon}} \mu^2(r)\,(d_{k-1,r}\,x+O(x^{1/2}\log^2 x))$$

$$=\left(\sum_{r=2}^{\infty} \mu^2(r)\,d_{k-1,r}\right)x+O(x^{1/2+\varepsilon}),$$

with the error term uniform in $k$ for any fixed $\varepsilon>0$. Thus we obtain

$$\sum_{2\leq k\leq \log x}\ \sum_{n\leq x, K(n)=k} 1=x \sum_{2\leq k\leq \log x}\ \sum_{r=2}^{\infty} \mu^2(r)\,d_{k-1,r}+O(x^{1/2+\varepsilon})$$

$$=x\sum_{k=2}^{\infty}\ \sum_{r=2}^{\infty} \mu^2(r)\,d_{k-1,r}+O(x^{1/2+\varepsilon})+O\left(x \sum_{k>\log x}\ \sum_{r=2}^{\infty} \mu^2(r)\,d_{k-1,r}\right).$$

Using (4.2) it is seen that the last double sum is majorized by

$$\sum_{k>\log x}\ \sum_{r=2}^{\infty} \exp\left(-C_2\log(C_4 2^k r)\log\log(C_4 2^k r)\right)$$

$$\ll \exp\left(-C_5 2^{\log x}\right) \sum_{r=2}^{\infty} \exp\left(-C_2 \log r \log\log(C_4 r)\right)\ll x^{-A}$$

for any fixed $A>0$. Inserting the preceding estimates in (4.5) we obtain Th. 2 with

$$E=6\pi^{-2}+\sum_{k=2}^{\infty}\ \sum_{r=2}^{\infty} \mu^2(r)\,d_{k-1,r}.$$

Trivially $E>1$, since $K(n)=1$ for squarefree $n$ (which have density $6\pi^{-2}$) and $K(n)\geq 2$ for other $n$.

We conclude by making two remarks. If $0\leq F(n)\ll \exp(Cn)$, then our arguments would give an asymptotic formula analogous to (2.6) for the sum of $F(K(n))$. The second remark concerns the constant $B_r$ in the asymptotic formula (1.8). It is easy to see that $\lim_{r\to\infty} B_r=1$, but it is also possible to show that $B_r$ converges very quickly to 1. Namely we have

$$x^{-1}\sum_{n\leq x} a^{(r)}(n)=x^{-1}\sum_{k\leq x^{\varepsilon}}\ \sum_{n\leq x, a^{(r)}(n)=k} k,$$

hence using (4.1), (4.2) and letting $x \to \infty$ we obtain

$$B_r = \sum_{k=1}^{\infty} k\, d_{r,k}.$$

Similarly from

$$x^{-1} \sum_{n \le x} 1 = x^{-1} \sum_{k \le x\varepsilon} \sum_{n \le x, a^{(r)}(n)=k} 1$$

we obtain

$$1 = \sum_{k=1}^{\infty} d_{r,k}.$$

Using then (4.4) we obtain

$$0 \le B_r - 1 = \sum_{k=2}^{\infty} (k-1)\, d_{r,k} \ll \exp\{-C_2 (\log 3)^{C^r}\} \tag{4.6}$$

for $r \ge r_0$ and some $C>1$, $C_2>0$. Presumably a lower bound for $B_r - 1$ analogous to (4.6) also holds for infinitely many $r$, but this does not seem easy to show. Perhaps even $B_r = 1$ for $r \ge r_1$ might be true; this would follow from (1.10).

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