On the Difference between Consecutive Ramsey Numbers

S.A. Burr[1]
City College, C.U.N.Y

P. Erdos
Hungarian Academy of Sciences

R.J. Faudree, R.H. Schelp
Memphis State University

Abstract. It is shown that the classical Ramsey numbers $r(m,n)$ satisfy

$$r(m,n) \geq r(m,n-1) + 2m - 3,$$

and, for $1 \leq k \leq n-2$,

$$r(m,n) \geq r(m,n-k) + r(m,k+1) - 1.$$

Consequences of the first result for some generalized Ramsey numbers will be considered.

If m and n are integers ≥ 2, define the (classical) Ramsey number $r(m,n)$ to be at least integer t such that if the edges of the complete graph K_t are colored red and blue, either a red K_m or a blue K_n must occur. These numbers have been extensively studied; see [2] for a survey. Various inequalities for $r(m,n)$ are known; for example,

$$\sqrt{2} \cdot n \cdot 2^{n/2} / e \leq r(n,n) \leq \binom{2n-2}{n-1}. \tag{1}$$

However, very little is known about the differences involving $r(m,n)$, such as $r(m,n) - r(m,n-1)$ or $r(m,n) - r(m-1,n-1)$. It seems very difficult to estimate these differences, but we have been able to establish the following.

Theorem 1. $r(m,n) \geq r(m,n-1) + 2m - 3$ *for* $m, n \geq 2$.

Corollary. $r(m,n) \geq r(m-1,n-1) + 2m + 2n - 8$ *for* $m, n \leq 2$.

The case $m = 3$ of Theorem 1 was proved by Graver and Yackel; see Corollary 4 on page 149 of [3]. We also note that Theorem 1 strengthens the trivial result

$$r(m,n) \geq r(m,n-1) + m - 1,$$

which was noted in [2]. In turn, this is a special case of the following.

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Theorem 2. *If* $1 \leq k \leq n-2$, *then*

$$r(m,n) \geq r(m,n-k) + r(m,k+1) - 1.$$

This theorem is nearly trivial, so we prove it first. Before giving the proof, we make the following definition. Call a coloring of K_t (m,n)-*good* if no red K_m or blue K_n occurs.

Proof of Theorem 2: Set $r_1 = (m,n-k)$, $r_2 = (m,k+1)$. Take a K_{r_1-1} with a $(m,n-k)$-good coloring, and a disjoint K_{r_2-1} with a $(m,k+1)$-good coloring. Join these two complete graphs entirely by blue edges, producing an edge-colored $K_{r_1+r_2-2}$. It is clear that this complete graph contains no red K_m, and the largest blue complete graph that occurs has no more than $(n-k-1)+(k+1-1) = n-1$ vertices. Therefore, $r(m,n) > r_1 + r_2 - 2$, completeing the proof.

We now turn to the less-trivial Theorem 1.

Proof of Theorem 1: We begin with a $(m,n-1)$-good colored $G = K_{r-1}$, where $r = r(m,n-1)$. G must contain a red K_{m-1}, since otherwise we could add a new vertex and join it to all of G with red edges, yielding a $(m,n-1)$-good coloring of $K_{r(m,n-1)}$. We actually use only the fact that G contains a red K_{m-2}. Denote the vertices of this K_{m-2} by $u_1, \ldots, u_{m-2}$. As a first step, adjoin $m-2$ more vertices, denoting them by $v_1, \ldots, v_{m-2}$. For each i, join v_i to u_i with a blue edge; for each other vertex x in G, join v_i to x with the same color as u_i is joined to x. Thus, u_iv_j is red for each $i \neq j$. Likewise, color v_iv_j red for each $i \neq j$. So far, we have colored a graph $H = K_{r+m-3}$, in effect by duplicating the u_i.

In H, no red K_m occurs, since a u and v could not both be used in any such K_m, and therefore and red K_m found could be converted to one that used only the u_i, contradicting the assumption that the original coloring was $(m,n-1)$-good. On the other hand, blue K_{n-1} does occur; but any such must use exactly one pair (u_i, v_j), and no other u or v.

We now adjoin $m-1$ more vertices, labeling them $x_1, \ldots, x_{m-1}$, and we must describe the coloring of all the edges involving the x_i. First, color x_ix_j red for all $i \neq j$, and color x_iy blue for all vertices y that are not a u_j, v_j, or an x_j. It remains to color the edges u_ix_j and v_ix_j. Color u_ix_j red if $i \geq j$; otherwise blue. On the other hand, color v_ix_j red if $i < j$; otherwise blue.

To finish the proof we must show that this 2-colored K_{r+2m-4} contains no red K_m and no blue K_n. Suppose first that, on the contrary, there exists a red K_m. Since H contains no such subgraph, this red K_m must use some vertices x_i, and hence only these and some of the u's and v's. Let x_k and x_l be the x's of minimum and maximum index respectively in this red K_m; thus there are no more than $l-k+1$ such x's. Furthermore, the u_i that could occur must satisfy $i \geq l$, and the v_j that could occur must satisfy $j < k$. Therefore, we can use at most $m-1-l$ u's and $k-1$ v's, so that the x's, u's, and v's amount to at most $m-1$ vertices, a contradiction.

Now suppose that there exists a blue K_n, which clearly must use exactly one x, say x_i. Therefore, this K_n must use a blue K_{n-1} from H. But as noted above, this K_{n-1} must use a pair (u_j, v_j). However, this is impossible, since either $x_i u_j$ or $x_i v_j$ must be red. This completes the proof.

It is clear that Theorem 1 is far short of what must be true. For instance, in view of (1), the value of $r(n,n) - r(n-1, n-1)$ must be exponentially large in n on the average, and it seems almost certain that this difference has an exponential lower bound as well.

However, Theorem 1 is strong enough to have consequences for generalized Ramsey numbers. (If G and H are graphs, $r(G,H)$ is defined like $r(m,n)$, but with G and H in place of K_m and K_n respectively.) For instance define $K^*_{k,l}$ to be a K_k with vertex-disjoint stars having a total of l edges emanating from the vertices of the K_k. There is not a unique way to adjoin the l edges of K_k, but we will take $K^*_{k,l}$ to be an arbitrary but fixed member of this family of possiblities. Of course, one of the possibilities is that all l edges are adjacent to just one vertex of the K_k.

We have the following consequence of Theorem 1.

Theorem 3. *For $m, n \geq 3$ and $m + n \geq 8$,*

$$r(K^*_{m,m-3}, K^*_{n,n-3}) = r(m,n).$$

Proof: The proof will be by induction on $m+n$ with the $m = n = 4$ and the $\{m,n\} = \{3,5\}$ cases left to the reader. Suppose the result fails and begin with a $(K^*_{m,n-3}, K^*_{n,n-3})$-good coloring of K_r with $r = r(m,n)$. We can assume that there is a red K_m. Some vertex v of the K_m is adjacent in red to at most $m-4$ vertices not in K_m, for otherwise there would be a red $K^*_{m,m-3}$. Therefore, v has a blue neighborhood N with at least $r(m,n) - 2m - 4 > r(m, n-1)$ vertices. If $n \geq 4$, then by induction N contains either a red $K^*_{m,m-3}$ or a blue $K^*_{n-1,n-4}$. In the first case we are done, and in the second case the vertex v and some additional blue adjacencies of v in N along with the blue $K^*_{n-1,n-4}$ gives the desired result. If $n = 3$, we use the fact that $r(m,3) \geq 4m - 7$, which follows by induction from $r(m,3) \geq r(m-8,3) + r(9,3) - 1$ and known bounds on $r(m,3)$ for $3 \leq m \leq 10$ [2]. Therefore N has at least $2m-3$ vertices. A blue edge in N gives a blue $K^*_{n,n-3}$, and otherwise there is a red K_{2m-3} in N, which completes the proof.

Let $\widehat{K}_{k,l}$ be the graph obtained from a K_k by adjoining a vertex adjacent to l vertices of K_k. Thus, in particular, $\widehat{K}_{k,k} = K_{k+1}$. Another consequence of Theorem 1 is the following.

Theorem 4. *For $m, n \geq 3$ and $m + n \geq 8$,*

$$r(\widehat{K}_{m,p}, \widehat{K}_{n,q}) = r(m,n)$$

$$\text{with } p = \left\lceil \frac{m}{n-1} \right\rceil \text{ and } q = \left\lceil \frac{n}{m-1} \right\rceil.$$

Proof: The three cases when $m+n = 8$ can be verified directly using $r(3,3) = 6$, $r(3,4) = 9$, $r(3,5) = 13$, and $r(4,4) = 18$, so we will proceed by induction on $m+n$.

We first verify the weaker result $r(m,n) = r(K_m, \widehat{K}_{n,q})$. Start with a $(K_m, \widehat{K}_{n,q})$-good coloring of a K_r with $r = r(m,n)$. There is a blue K_n, and since $r - n \geq r(m-1, n)$, there is a red K_{m-1} (even in the case when $m = 3$) that is vertex disjoint from the blue K_n. Each vertex of the K_n is adjacent in blue to at least one vertex of the K_{m-1}, so some vertex of the K_{m-1} is adjacent in blue to at least $\lceil n/(m-1) \rceil$ vertices of the K_n. This proves the weaker result.

This weaker result and the same strategy will yield a proof of Theorem 4.

In Theorem 4 the full strength of Theorem 1 was not needed in fact, only $r(m,n) \geq r(m, n-1) + m$. With this in mind, let $\widehat{\mathcal{K}}_{m,l}$ be the family of graphs obtained from K_m by adjoining $m-3$ independent vertices such that each is adjacent to l vertices of K_m. (if $\mathcal{G}$ and $\mathcal{H}$ are families of graphs, then $r(\mathcal{G}, \mathcal{H})$ only requires the existence of some graph in $\mathcal{G}$ or some graph in $\mathcal{H}$.) The same strategy used in the proof of Theorem 4 (induction with a weaker one-sided intermediate statement $r(\widehat{\mathcal{K}}_{m,p}, K_n) = r(m,n)$ proved), along with the full strength of Theorem 1, gives the following.

Theorem 5. *For $m, n \geq 3$ and $m+n \geq 8$,*

$$r(\widehat{\mathcal{K}}_{m,p}, \widehat{\mathcal{K}}_{n,q}) = r(m,n)$$

$$\text{with } p = \left\lceil \frac{m}{n-1} \right\rceil \text{ and } q = \left\lceil \frac{n}{m-1} \right\rceil.$$

References

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