Annals of Discrete Mathematics 41 (1989) 117–130

Bandwidth versus Bandsize

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Dedicated to the memory of G. A. Dirac

The bandwidth (bandsize) of a graph G is the minimum, over all bijections $\mu : V(G) \to \{1, 2, \ldots, |V(G)|\}$, of the *greatest* difference (respectively the *number* of distinct differences) $|\mu(v) - \mu(w)|$ for $vw \in E(G)$.

We show that a graph on n vertices with bandsize k has bandwidth between k and $cn^{1-\frac{1}{k}}$, and that this is best possible. In the process we obtain best possible asymptotic bounds on the bandwidth of circulant graphs.

The bandwidth and bandsize of random graphs are also compared, the former turning out to be $n - c_1 \log n$ and the latter at least $n - c_2(\log n)^2$.

1 Introduction

The problem of bandwidth minimization was motivated by the needs of matrix manipulations in structural engineering [1, 10, 19]: there it is desirable to store the matrices in such a way that their non-zero entries are all as close to the main diagonal as possible. Simultaneous row/column permutations are applied to transform a given (large, sparse, symmetric) matrix to a form in which all non-zeros are in a narrow band of sub- and super-diagonals surrounding the main diagonal. Given a symmetric matrix $M = (m_{i,j})$ $(i, j = 1, \ldots, n)$, one may consider the graph G on n vertices in which i is adjacent to j just if $m_{i,j} \neq 0$. We now give the precise statement of the bandwidth minimization problem in terms of graphs; its correspondence to the above problem for symmetric matrices is self-evident (via the translation just given).

Let G be a graph with n vertices. A *numbering* of G is a bijection $\mu : V(G) \to \{1, \ldots, n\}$; the numbers $|\mu(u) - \mu(v)|$ for $uv \in E(G)$ are called the *edge-differences* of the numbering μ. The *width* of a numbering μ is its largest edge-difference. The *bandwidth* of a graph G, $\mathrm{bw}(G)$, is the smallest width of any numbering of G.

A matrix in which all non-zeros are in a narrow band is convenient for storage and computation. For some applications it may be enough to have all the non-zero entries concentrated in a small number of sub- and super-diagonals. (Such matrices also seem to arise in certain applications, e.g., in queueing network analysis of job line production models, cf. [8] Figures 1 and 2.) The corresponding graph-theoretic analogue is the following: The *size* of a numbering μ is the number of distinct edge-differences of μ; the *bandsize* of a graph G, $\mathrm{bs}(G)$, is the smallest size of any numbering of G. Of course, it follows from the definitions that

$$\mathrm{bs}(G) \leq \mathrm{bw}(G).$$

The actual motivation for the study of bandsize (as opposed to the possible application described above) originated from an investigation of spanning subtrees of the k-dimensional hypercube Q_k. J. Malkevitch studied spanning subtrees of Q_k, and derived a number of their properties; M. Rosenfeld observed that such trees must admit a numbering with k edge-differences (in fact, with edge-differences $1, 2, 4, \ldots, 2^{k-1}$), and was led to ask if there was a bound to the number of edge-differences required by numberings of trees of maximum degree k. The notion of bandsize is formally introduced in [12], where it is shown that the bandsize of the complete binary tree of height n, T_n, is between $\frac{n}{7}$ and $\frac{4n}{5} + 2$. Since the maximum degree in T_n is three, this answers Rosenfeld's question in the negative.

Throughout the paper, we reserve the symbol "lg" for logarithms base 2, and "ln" for logarithms base e.

The tree T_n has $v = 2^{n+1} - 1$ vertices; thus its bandsize is roughly $c \lg v$ (for $\frac{1}{7} < c < \frac{4}{5}$). On the other hand, it can be shown that the bandwidth of T_n is as high as $\frac{v}{2 \lg v}$, (cf. [4] for a tree similar to T_n). When the bandsize is so much smaller than the bandwidth, storing the matrix in the form we suggest, with few non-zero diagonals, would seem particularly attractive.

In this paper we take up the comparison between bandwidth and bandsize. The largest bandwidth among all graphs of fixed bandsize is asymptotically determined in the next section: a graph with n vertices and bandsize k can have bandwidth as large as $O(n^{1-\frac{1}{k}})$, but no more. Our method also yields best asymptotic bounds for the bandwidth of circulant graphs. In the last section we compare the bandwidth and bandsize of random graphs: it turns out that their values are quite close, $n - c_1 \log n$ for bandwidth, and at least $n - c_2(\log n)^2$ for bandsize.

The bandwidth problem, i.e., the problem of deciding for a given graph G and integer k, whether there exists a numbering of G of width at most k, is well known to be NP-complete, even in the case of trees, [16, 9]. On the other hand, if k is fixed, the problem of deciding if $\mathrm{bw}(G) \leq k$ can be solved in polynomial time, [17, 11]. In contrast to this, the problem of deciding if $\mathrm{bs}(G) \leq k$ is NP-complete for every fixed $k \geq 2$, [18].

A variety of other numbering problems have been studied recently, [6]. For instance the minsum (or optimal linear arrangement) problem [6] may be stated as follows: The *sum* of a numbering μ is the sum of all its edge-differences; the *minsum* of a graph G is the smallest sum of any numbering of G. The reader may find it amusing to note that the *largest* size of a numbering of G bears an obvious relation to graceful graphs; this notion, in some sense dual to the notion of bandsize of a graph, may be called the *gracesize* of G, $gs(G)$. Because of the famous graceful graph conjecture, it could be interesting to prove non-trivial lower bounds on the gracesize of trees. (In this terminology, the graceful graph conjecture asserts that the gracesize of any tree is equal to its number of edges.)

2 The extremal case

2.1 General remarks

In this section, and the next, we prove the following theorem:

Theorem 1. *Let k be fixed. A graph G with n vertices and bandsize k has bandwidth only $O(n^{1-\frac{1}{k}})$. Moreover, this bound is best possible.*

Let $I = \{i_1, i_2, \ldots, i_l\}$ be a set of integers, $0 < i_1 < i_2 < \ldots < i_l < n$. The *linear* graph $L_n(I)$ (or $L_n(i_1, i_2, \ldots, i_l)$) has the vertex set $Z_n = \{0, 1, \ldots, n-1\}$ and the edge-set $\{uv : |u - v| \in I\}$. The *circulant* graph $C_n(I)$ (or $C_n(i_1, i_2, \ldots, i_l)$) also has the vertex-set Z_n and its edge-set is $\{uv : |u - v| \equiv i \pmod{n} \text{ for some } i \in I\}$. A graph is linear if it is isomorphic to some linear graph, and is a circulant if it is isomorphic to some circulant graph. Note that each $L_n(I)$ is a spanning subgraph of $C_n(I)$.

Let G be a graph with n vertices. It follows from the definitions that the bandwidth of G is the smallest integer l such that G is isomorphic to a spanning subgraph of $L_n(1,2,\dots,l)$ (the isomophism taking a vertex numbered x to the vertex $x-1$ of $L_n(1,2,\dots,l)$), and the bandsize of G is the smallest integer l such that G is isomophic to a spanning subgraph of some $L_n(i_1,i_2,\dots,i_l)$.

Our proof of Theorem 1 (or rather of Theorem 2, below) depends heavily on the use of the natural "circular" extension of these notions: The *circular bandwidth* of the graph G (still with n vertices), $\text{cbw}(G)$, is the smallest l such that G is isomorphic to a spanning subgraph of $C_n(1,2,\dots,l)$; the *circular bandsize* of G, $\text{cbs}(G)$, is the smallest l such that G is isomorphic to a spanning subgraph of some $C_n(i_1,i_2,\dots,i_l)$. (Circular bandwidth shall play a central role in our proof; circular bandsize is introduced only for symmetry.) We shall also need the notion of "circular length"; formally the (circular) *n-norm* of a positive integer i, $\|i\|_n$, is the unique $|b|$, $-\frac{n}{2} \le b \le \frac{n}{2}$, such that $i = an + b$. (Thus for i in Z_n, the n-norm of i is the distance from 0 to i in the graph $C_n(1)$.) If we call the *norm* of a numbering the largest norm of its edge-differences, then the circular bandwidth of a graph is the smallest norm of its numberings. (If we call the *normsize* of a numbering the number of distinct norms of its edge-differences, then the circular bandsize of a graph is the smallest normsize of its numberings.)

The usefulness of circular bandwidth is due to the fact that

$$\text{cbw}(G) \le \text{bw}(G) \le 2\text{cbw}(G). \tag{1}$$

The first inequality follows directly from the definitions. To prove the second inequality it is enough to show that

$$\text{bw}(C_n(1,2,\dots,l)) \le 2l:$$

Number all vertices $x = 0,1,\dots,\lceil\frac{n}{2}\rceil - 1$ by $2x+1$, and all vertices $y = \lceil\frac{n}{2}\rceil,\dots,n-1$ by $2(n-y)$. It is straightforward to verify that this is a numbering of $C_n(1,2,\dots,l)$ and that the edge-difference of any edge uv with $v \equiv u+1 \pmod n$ is either 1 or 2; hence any edge-difference is at most $2l$.

In the next subsection we shall prove the following result:

Theorem 2. $\text{bw}(C_n(i_1,i_2,\dots,i_k)) \le 4n^{1-\frac{1}{k}}$.

Proof of Theorem 1 (from Theorem 2). A graph G with n vertices and bandsize k is (up to isomorphism) a spanning subgraph of $L_n(I)$ for som I with k elements; hence a spanning subgraph of $C_n(I)$. By Theorem 2, its bandwidth is only $O(n^{1-\frac{1}{k}})$.

This bound is best possible, as for $n = m^k$, the graph $G = L_n(1, m, m^2, \ldots, m^{k-1})$ has bandwidth at least

$$\frac{2}{k+1} \cdot m^{k-1} - 1 = \Omega(n^{1-\frac{1}{k}})$$

(for k fixed and $m \to \infty$).

To see this, note that any two vertices of G can be joined by a path of at most $\frac{k+1}{2} \cdot m$ edges (at most m steps of type m^{k-1}, at most $\frac{m}{2}$ steps of each of the types $m^{k-2}, \ldots, m, 1$). Thus, whatever the numbering of G, the shortest path joining the vertices numbered 1 and n must have some edge-difference at least

$$\frac{n-1}{\frac{k+1}{2} m} \geq \frac{2}{k+1} \cdot m^{k-1} - 1.$$

□

Note that $\frac{4n^{1-\frac{1}{k}}}{k}$ is maximized for $k = \ln n$; hence

Corollary 1. *The ratio of bandwidth to bandsize for a graph on n vertices cannot exceed* $(\frac{1}{\ln n}) \cdot 4n^{1-\frac{1}{\ln n}}$.

2.2 The bandwidth of circulants

Here we prove Theorem 2; we restrict our attention to *connected* circulants (as the bandwidth of a graph is the maximum bandwidth of its components).

In fact we shall show that (for any fixed k)

$$\max_{i_1, \ldots, i_k} \mathrm{bw}(C_n(i_1, i_2, \ldots, i_k)) = \Theta(n^{1-\frac{1}{k}}). \tag{2}$$

The lower bound,

$$\max_{i_1, \ldots, i_k} \mathrm{bw}(C_n(i_1, i_2, \ldots, i_k)) = \Omega(n^{1-\frac{1}{k}}),$$

follows the same way it did for linear graphs. The remainder of this subsection contains the proof of

$$\mathrm{cbw}(C_n(i_1, i_2, \ldots, i_k)) \leq 2n^{1-\frac{1}{k}}, \tag{3}$$

which implies (2) and Theorem 2 because of (1). In other words, we seek an isomorphism of any $C_n(i_1, i_2, \ldots, i_k)$ onto a subgraph of $C_n(1, 2, \ldots, l)$ with $l \leq 2n^{1-\frac{1}{k}}$.

Lemma 1. *Let n, k, and $i_1, i_2, \ldots, i_k$ be given positive integers. Then there exists an integer m with $0 < m < n$ such that $\|m \cdot i_j\|_n \leq n^{1-\frac{1}{k}}$ for all $j = 1, 2, \ldots, k$.*

Proof. Note that the n-norm satisfies the triangle inequality. Let $S(n,k)$ denote the torus $[0,n)^k$ with entries taken modulo n. Let

$$A = \{(m \cdot i_1, m \cdot i_2, \ldots, m \cdot i_k) \in S(n,k) : 0 \leq m < n\},$$

and for each m let

$$B_m = \{(y_1, y_2, \ldots, y_k) \in S(n,k) : \|y_j - m \cdot i_j\|_n \leq \frac{1}{2} \cdot n^{1-\frac{1}{k}} \text{ for all } j\}.$$

The (k-dimensional Euclidean) volume of each B_m is

$$(2 \cdot \frac{1}{2} \cdot n^{1-\frac{1}{k}})^k = n^{k-1}.$$

Since the combined volume of the B_m's is n^k (the volume of $S(n,k)$), and since they are all closed sets, there is a point $x = (x_1, x_2, \ldots, x_k)$ in some $B_{m'} \cap B_{m''}$, $m' \neq m''$. We then take $m = |m' - m''|$ so that $0 < m < n$ and we have

$$\|m \cdot i\|_n = \|m' \cdot i - m'' \cdot i\|_n \leq \|m' \cdot i - x_i\|_n + \|x_i - m'' \cdot i\|_n \leq n^{1-\frac{1}{k}}$$

for all $i = i_j$, $j = 1, 2, \ldots, k$, as required. □

We are grateful to Miklós Simonovits for pointing out that Lemma 1 may also be derived from Dirichlet's theorem on simultaneous diophantine approximation, [2, p.159]. Because of its relation to [14], this may allow us to find the "multiplier" m figuring in Lemma 1 efficiently (cf. [14. p.524–525]).

Also note that Lemma 1 is sufficient to imply that

$$\text{cbw}(C_n(i_1, i_2, \ldots, i_k)) \leq n^{1-\frac{1}{k}}$$

when n is prime. In fact, a long as the m from Lemma 1 is relatively prime to n, the mapping taking x to $m \cdot x \bmod n$ is a bijection $Z_n \to Z_n$ and hence an isomorpism of $C_n(i_1, i_2, \ldots, i_k)$ onto a spanning subgraph of $C_n(m \cdot i_1, m \cdot i_2, \ldots, m \cdot i_k)$; and therefore onto a spanning subgraph of $C_n(1, 2, \ldots, l)$ with

$$l = \lfloor n^{1-\frac{1}{k}} \rfloor.$$

If m and n are not relatively prime, then the above mapping is not a bijection. Nevertheless there always exists a bijection accomplishing our aims. This is not hard to see by arguing that the above mapping $x \to m \cdot x$ takes exactly g points of Z_n onto each of the points $g, 2g, \ldots, ng = 0$; thus it can be made bijective by local perturbations. To define such a bijection explicitely we can use the following facts:

Lemma 2. *Let* $g = \gcd(m,n)$ *and* $d = \frac{n}{g}$.

(a) *Each* $x \in Z_n$ *can be uniquely written as* $x = dq + r$ *with* $0 \leq q < g$ *and* $0 \leq r < d$.

(b) *Each* $x \in Z_n$ *can be uniquely written as* $x = mu + v$ *with* $0 \leq u < d$ *and* $0 \leq v < g$.

Proof. Each x can be written as $x = dq + r$ with $0 \leq r < d$; since $x \in Z_n$ and $dg = 0$ in Z_n, q may be assumed to satisfy $0 \leq q < g$. The uniqueness in (a) follows from the fact that $gd = n = |Z_n|$. Each x can also be written as $x = mu + v$ with $0 \leq v < m$; evidently $u < d$ because $md = \frac{m}{g} \cdot n \geq n$. Since $\alpha \cdot m \equiv g \pmod{n}$ for some α, v may be assumed to satisfy $0 \leq v < g$. The uniqueness in (b) follows by the same argument as in (a). □

Let $F : Z_n \to Z_n$ be defined as follows : if $x = dq + r$ with $0 \leq q < g$ and $0 \leq r < d$, then $F(x) = mr + q$. According to Lemma 2, F is well defined and a bijection. It remains to verify that F takes $C_n(i_1, i_2, \ldots, i_k)$ onto a spanning subgraph of $C_n(1, 2, \ldots, l)$ with $l \leq 2n^{1-\frac{1}{k}}$.

Lemma 3. *If* $\|x - x'\|_n = i$, *then* $\|F(x) - F(x')\|_n \leq \|m \cdot i\|_n + \|g\|_n$.

Proof. Let $x = dq + r$, $0 \leq q < g$, $0 \leq r < d$, and $x' = dq + r + i = d(q + f) + r'$ where $0 \leq r' < d$.

Case 1: $0 \leq q + f < g$.

Then $F(x) = mr + q$ and $F(x') = mr' + q + f$. Hence

$$\begin{aligned} \|F(x) - F(x')\|_n &= \|m \cdot (r - r') - f\|_n = \|m \cdot (df - i) - f\|_n \\ &= \|m \cdot i + f\|_n \leq \|m \cdot i\|_n + \|g\|_n. \end{aligned}$$

(Note that $f \leq g$ because $dg \equiv 0 \pmod{n}$.)

Case 2: $0 \leq q + f - g < g$.

Then $x' = d(q + f - g) + r'$ and $F(x') = m \cdot r' + q + f - g$. Hence

$$\begin{aligned} \|F(x) - F(x')\|_n &= \|m \cdot (r - r') - f + g\|_n = \|m \cdot (d \cdot f - i) - f + g\|_n \\ &= \|m \cdot i + f - g\|_n \leq \|m \cdot i\|_n + \|g\|_n. \end{aligned}$$

□

Lemma 4. *If* $\gcd(n, i_1, i_2, \ldots, i_k) = 1$, *and if* $m < n$ *has each* $\|m \cdot i_j\|_n \leq n^{1-\frac{1}{k}}$ $(j = 1, 2, \ldots, k)$, *then* $g = \gcd(m, n) \leq n^{1-\frac{1}{k}}$.

Proof. If all $\|m \cdot i_j\|_n = 0$ then each $m \cdot i_j$ is a common multiple of m and n, hence also a multiple of $\frac{m \cdot n}{g}$. Therefore $\frac{n}{g}$ is a divisor of all i_j as well as of n; according to our assumption $\frac{n}{g} = 1$, contrary to $g \leq m \leq n$. On the other hand, note that g divides $m \cdot i_j - a \cdot n$ for each $j = 1, 2, \ldots, k$, and every integer a. Assume that $\|m \cdot i_j\|_n \neq 0$; since $\|m \cdot i_j\|_n$ is either $m \cdot i_j - a \cdot n$ or $a \cdot n - m \cdot i_j$ for some integer a, g divides $\|m \cdot i_j\|_n$ and hence $g \leq \|m \cdot i_j\|_n \leq n^{1-\frac{1}{k}}$ as claimed. □

Proof of (3) (and thus of (2) and Theorems 2 and 1). Lemmas 1, 3 and 4 imply that the mapping F given above is an isomorphism of $C_n(i_1, i_2, \ldots, i_k)$ onto a spanning subgraph of $C_n(1, 2, \ldots, l)$, where $l \leq 2n^{1-\frac{1}{k}}$, provided $\gcd(n, i_1, i_2, \ldots, i_k) = 1$. Since this condition is always satisfied for connected circulants $C_n(i_1, i_2, \ldots, i_k)$, (3) has been proved. □

Remark. Let $d_s(G)$ denote the number of vertices of G of distance s from a fixed vertex. We have studied, jointly with Martin Farber, the behaviour of $d_s(C_n(i_1, i_2, \ldots, i_k))$ and $d_s(L_n(i_1, i_2, \ldots, i_k))$ for fixed k and believe that there exists a constant c depending only on k such that

$$d_s(L_n(i_1, i_2, \ldots, i_k)) \leq cn^{1-\frac{1}{k}} \tag{4}$$

for all n, s, and $i_1, i_2, \ldots, i_k$. This would then offer another proof of Theorem 1: indeed any breadth first numbering (or so-called "level algorithm" in the terminology of [20]) of $L_n(i_1, i_2, \ldots, i_k)$ would have width at most $2c \cdot n^{1-\frac{1}{k}}$. When $k = 1$ (4) is obvious (the constant c is 1 or 2 depending on the choice of the starting vertex); we were also able to prove (4) for $k = 2$ and $k = 3$. It turns out that to complete the proof of Theorem 1 this way, it would suffice to show that, for fixed k,

$$\mathrm{bw}(C_n(i_1, i_2, \ldots, i_k)) = O(\mathrm{bw}(L_n(i_1, i_2, \ldots, i_k))).$$

3 The random case

Our object in this section is to compare the bandwidth and bandsize of random graphs. For $0 < p < 1$ and n a positive integer, let $G_{n,p}$ be a graph with vertex set $V_n = \{1, 2, \ldots, n\}$ and edges defined randomly as follows: for each pair $\{i, j\}$ of vertices with $i \neq j$ the edge ij is included with probability p and excluded with probability $q = 1 - p$; the $\binom{n}{2}$ choices are made independently. This construction has been studied extensively, especially by Erdös and Rényi [7]; see also Bollobás [3]. We wish to compute bounds on $\mathrm{bw}(G_{n,p})$ and $\mathrm{bs}(G_{n,p})$ for almost all $G_{n,p}$ when n is large.

For simplicity we concentrate below on the case $p = \frac{1}{2}$; the results are easily generalized. Note that when $p = \frac{1}{2}$ the graph $G_{n,p}$, which we denote below simply by G_n, has the special property that every graph on the labelled set V_n is equally likely to occur.

The following theorem is proved in [13].

Theorem 3. *With probability approaching* 1 *as* $n \to \infty$,

$$n - (2 + \sqrt{2} + o(1)) \cdot \lg n < \mathrm{bw}(G_n) < n - (2 + \sqrt{2} - o(1)) \cdot \lg n.$$

As we found, a number of recent papers studied the bandwidth of random graphs, [5, 13, 15, 20, 21]. Theorem 3 is stronger than similar results in [20, 21], and weaker than the most general version of [13]. We have stated it in this way for simplicity, and also because this was the form of the result we had before discovering [13]; we had the constant $2 + \sqrt{2}$ in the lower bound, but not in the upper bound. (In [15] the authors study the average bandwidth of trees, which turns out to be between $c_1\sqrt{n}$ and $c_2\sqrt{n}\log n$.)

Theorem 4. *With probability approaching* 1 *as* $n \to \infty$,

$$\mathrm{bs}(G_n) \geq n - \left(\frac{3}{\ln 2} + o(1)\right) \cdot (\ln n)^2.$$

Proof. Fix $c > \frac{3}{\ln 2}$ and $t = \lceil c(\ln n)^2 \rceil$; we show that

$$\Pr(G_n \text{ has a numbering of size less than } n - t) \to 0.$$

It suffices to show that

$$\sum_{0<d_1<\ldots<d_t<n} \Pr(G_n \text{ has a numbering omitting lengths } n - d_1, \ldots, n - d_t)$$

tends to 0.

For values of $d_1, d_2, \ldots, d_t$ which are large relative to n, the constraints are severe and the probability that a given numbering satisfies them will be much less that $n!$. When the d_i's are small, however, it becomes necessary to count only partial numberings, namely bijections from subgraphs of G_n to the union of an initial and final segment of $1, 2, \ldots, n$. To determine the optimum size of these segments we compare the d_i's to a certain exponential sequence.

Let $0 < d_1 < d_2 < \ldots < d_t < n$ be a sequence of integers; set $r = 1 + \frac{1}{b \ln n}$ where $c > b > \frac{3}{\ln 2}$. Note that for large n

$$\begin{aligned}
\log_r n &= \frac{\ln n}{\ln\left(1+\frac{1}{b\ln n}\right)} \\
&\le \frac{\ln n}{\frac{1}{b\ln n} - \frac{1}{2}\left(\frac{1}{b\ln n}\right)^2} \\
&= \frac{2b^2(\ln n)^3}{2b\ln n - 1} \\
&< c(\ln n)^2 - 1 \le t-1,
\end{aligned}$$

hence $d_t < n < r^{t-1}$. It follows that there is a least integer j such that $d_j < r^{j-1}$; then $d_i \ge r^{i-1}$ for $i < j$ and hence

$$\begin{aligned}
\sum_{i=1}^{j} d_i &\ge \sum_{i=1}^{j-1} d_i \\
&\ge \sum_{i=0}^{j-2} r^i \\
&= (b \ln n)\cdot(r^{j-1}-1).
\end{aligned}$$

Note that $d_j \ge j$ and that $j \to \infty$ as $n \to \infty$.

We now consider partial numberings of G_n into the set $\{1,2,\ldots,d_j\} \cup \{n-d_j+1, n-d_j+2, \ldots, n\}$ where edge-differences $n-d_1, n-d_2, \ldots, n-d_j$ are avoided; since each difference $n-d_i$ forbids exactly d_i edges within the domain of the partial numbering, altogether there are $d_1 + d_2 + \cdots + d_j$ forbidden pairs. For a given value of j there are fewer than n^j choices for $d_1, d_2, \ldots, d_j$ and fewer than n^{2s} partial numberings, where $s = r^{j-1}$. Hence for fixed j the probability that G_n has a partial numbering avoiding forbidden pairs which determine j is less than

$$\begin{aligned}
n^j \cdot n^{2s} \cdot 2^{-(b\ln n)\cdot(s-1)} &< n^{3s} \cdot 2^{-(b \lg n)\cdot(\ln 2)\cdot(s-1)} \\
&= n^{(-(b-\frac{3}{\ln 2})s+b)\ln 2} \\
&< n^{-1} \text{ (say)},
\end{aligned}$$

since $s = r^{j-1}$ is forced like j to increase without bound as $n \to \infty$.

However, there are fewer than $c(\ln n)^2$ choices for j so

$$\Pr(\mathrm{bs}(G_n) < n - c(\ln n)^2) < c(\ln n)^2 \cdot n^{-1} \to 0,$$

and Theorem 4 is proved. □

Corollary 2. *With probability approaching* 1 *as* $n \to \infty$,

$$n - c_2(\log n)^2 \le \mathrm{bs}(G_n) \le \mathrm{bw}(G_n) \le n - c_1 \log n.$$

We have determined that the "correct" value of bandwidth of G_n is $n - c\log n$; in fact even the constant c is determined by the result of Kuang and McDiarmid, as stated in Theorem 3. We do not know at present whether the correct value of the bandsize is more like $n - c\log n$ or $n - c(\log n)^2$.

Acknowledgement

We are grateful to Pauline van den Driessche, Martin Farber, Dale Olesky, Moshe Rosenfeld, and Miklós Simonovits for useful conversations concerning bandsize.

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