Some Old and New Problems in Combinatorial Geometry

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Abstract. Let $x_1, x_2, \ldots, x_n$ be n distinct points in a metric space. Usually we will restrict ourselves to the plane. Denote by $D(x_1, \ldots, x_n)$ the number of distinct distances determined by $x_1, \ldots, x_n$. Assume that the points are in r-dimensional space. Denote by

$$f_r(n) = \min_{x_1, \ldots, x_n} D(x_1, \ldots, x_n).$$

I conjectured more than 40 years ago that $f_2(n) > c_1 n/(\log n)^{\frac{1}{2}}$. The lattice points show that this if true is best possible. In this paper we discuss problems related to the conjecture and other questions related to this parameter.

I wrote many papers with this or similar titles, and will try to avoid repetitions as much as possible.

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(1) $$f_2(n) > c_1 n/(\log n)^{1/2}.$$

I offer five hundred dollars for a proof or disproof of (1). The lattice points show that

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(1) if true is best possible. Denote by $d(x_i)$ the number of distinct distances from x_i. Probably for every choice of distinct points $x_1,\dots,x_n$ in the plane, we have

(2) $$\max_{1\le i\le n} d(x_i) > c_2 n/(\log n)^{1/2}$$

and perhaps even

(3) $$\sum_{i=1}^{n} d(x_i) > cn^2/(\log n)^{1/2}.$$

In 1946 I proved $f_2(n) > n^{1/2}$ and this was improved by L. Moser to $cn^{2/3}$, and in fact both Moser and I proved that $\max_i d(x_i) > \sqrt{n}$ respectively $cn^{2/3}$. A few years ago Fan Chung achieved a breakthrough. She proved $f_2(n) > cn^{5/7}$, but she did not prove max $d(x_i) > cn^{5/7}$. $f_2(n) > n^{3/4}$ seems to be the best current result, due to Trotter and Szemerédi (unpublished).

It is not impossible that for every choice of $x_1,\dots,x_n$ we in fact have

$$\max_i d(x_i) \ge (1+o(1))f_2(n)$$

or perhaps even

$$\max_i d(x_i) \ge f_2(n) - C$$

for some absolute constant C. Perhaps the last conjecture is too optimistic.

Let $x_1,\dots,x_n$ be a set of distinct points (in the plane) which implements $f_2(n)$, i.e., the number of distinct distances $D(x_1,\dots,x_n)$ determined by $x_1,\dots,x_n$ is $f_2(n)$. Consider all these n-tuples. Is it true that for $n > 5$ there are always two such sets which are dissimilar? i.e., there is no similarity transformation which carries one into the other. For $n = 3$ and $n = 5$ the equilateral triangle and the regular pentagon are the only sets which implement $f_2(3) = 1$ and $f_2(5) = 2$. Denote by $h(n)$ the largest integer so that any two sets $A_1(n)$ and $A_2(n)$ which implement $f_2(n)$ contain two sets of size $h(n)$ which are similar. The conjecture stated above is that, for $n > 5$, $h(n) < n$. Is it true that $h(n) \to \infty$ as $n \to \infty$? At present I cannot exclude the possibility that, for $n > n_0$, $h(n) = 2$, i.e., there are two sets $x_1,\dots,x_n$ and $y_1,\dots,y_n$ both of which implement $f_2(n)$ but no triangle (x_i,x_j,x_ℓ) is similar to any of the triangles (y_i,y_j,y_ℓ). I think this is unlikely since I expect that for $n > n_0$ all these sets must contain equilateral triangles.

Another somewhat related problem asks: Let $A(n)$ implement $f_2(n)$. For which k must it contain a subset which implements $f_2(k)$? I think that for $k = 3$ and $k = 4$ this must hold, but for $k = 5$ and $n > n_0$, it fails in the following strong sense. No set $A(n)$ which implements $f_2(n)$ can contain a regular pentagon. I have no guess what happens for $k > 5$. More generally one could ask the following problem: Consider all the sets $x_1,\dots,x_k$ which can occur as subsets of a set A which implements $f_2(n)$. What are the possible values of $D(x_1,\ldots,x_k)$, and, in particular, for which n and k is the value of $D(x_1,\ldots,x_k)$ uniquely determined.

By the way, I suspect that $A(n)$ must have lattice structure. Perhaps for $n > n_0$ it must be a subset of a triangular lattice. Again this conjecture could be completely wrongheaded. A much weaker conjecture would be that if $x_1,\ldots,x_n$ implements $f_2(n)$ then the points can be covered by $cn^{1/2}$ lines. I could not even prove that there is a line which contains $c_1 n^{1/2}$ of the x_i.

Many years ago I conjectured and Szemerédi proved that if $D(x_1,\ldots,x_n) = o(n)$ then there is a line which contains unboundedly many of the x_i's. In fact he showed that there is such a line which is a perpendicular bisector of two of our x_i's.

It is easy to see that if $D(x_1,\ldots,x_n) = o(n)$ then for every fixed k there is a subset $x_{i_1},\ldots,x_{i_k}$ for which

$$D(x_{i_1},\ldots,x_{i_k}) \leq 1 + \binom{k-1}{2}. \tag{4}$$

In fact (4) follows already from the weaker assumption that there is an x_i for which $d(x_i) = o(n)$. (4) follows trivially from the fact that there is a circle whose center is x_i and which must contain $\geq k$ of our points. $d(x_i) = o(n)$ for one i only of course does not imply that there are three of our points on a line. It is not clear to me that for how many i's must $d(x_i) = o(n)$ hold to imply that three of our points are on a line.

Perhaps (4) is best possible. In other words for every fixed k and $n > n_0(k)$ there is a set $x_1,\ldots,x_k$ for which $D(x_1,\ldots,x_k) = o(n)$ and for every $3 \leq \ell \leq k$ and every choice of ℓ points $x_{i_1},\ldots,x_{i_\ell}$ we have

$$D(x_{i_1},\ldots,x_{i_\ell}) \geq 1 + \binom{\ell-1}{2}.$$

I cannot even prove this for $\ell = 3$, in fact it may fail for $\ell = 3$ but hold for $\ell = 4$. It may be of interest to find out what happens if ℓ can tend to infinity with n. I would expect that if $D(x_1,\ldots,x_n) < \dfrac{cn}{(\log n)^{1/2}}$, then our set must contain equilateral (or at least isosceles) triangles and four points with $D(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) = 2$. I cannot even prove this with 3 instead of 4. I conjectured long ago that if $x_1,\ldots,x_n$ is such that any set of four points $(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})$ determines at least five distinct distances then

$$D(x_1,x_2,\ldots,x_n) > cn^2.$$

If we only assume that $D(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) \geq 4$ for every choice of $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}$, then I expect that this implies

$$D(x_1,\ldots,x_n)/n \to \infty\ ,$$

but I know that $D(x_1,\ldots,x_n) < n^{1+o(1)}$ is possible. Clearly many related questions can be asked and we leave their formulation to the interested reader.

The following question occurred to me a few weeks ago: Let S_1 and S_2 be two sets of distinct points $x_1,\ldots,x_n$; $y_1,\ldots,y_n$. The sets S_1 and S_2 do not have to be disjoint. Denote by $d(S_1;S_2)$ the number of distinct distances $d(x_i, y_j)$. Is it true that

$$(5) \qquad \min_{S_1,S_2} \ d(S_1,S_2)/f_2(n) \to 0$$

as n tends to infinity? (5) is perhaps of interest for the following reason: By a well known remark of Lenz, (5) certainly holds in 4-dimensions, since there min $d(S_1,S_2) = 1$ for every n. For 2 or 3 dimensions (5) is open and quite possibly the answer is negative.

Here is one final problem of this type: Let $x_1,\ldots,x_n$ be n points in the plane, no four on a circle and every circle whose center is one of the x_i contains at most two of our points. Clearly for every x_i we then have

$$d(x_i) \geq \frac{n-1}{2}.$$

Is it true that there is an absolute constant c so that

$$(6) \qquad \max_{1\leq i\leq n} \ d(x_i) > (1+c)\frac{n}{2} \quad ?$$

I offer 25 dollars for a solution.

We need the assumption that no four of our points are on a circle since otherwise the regular polygon gives a counterexample. Perhaps in fact

$$\sum_{i=1}^{n} d(x_i) > (1+c)\frac{n^2}{2}$$

also holds. It might be of some interest to try to deduce (6) from as weak an assumption as possible. It should certainly hold if we only assume that no k of our points are on a circle where k is independent of n, perhaps this assumption can be weakened further. We also assume that not too many of our points are on a line.

Let S be a set of n points in the plane no three on a line, no four on a circle. Denote by h(n) the largest integer for which such a set determines at least h(n) distinct distances. Pach just told me that $h(2^n) \leq 3^n$. The projection of the n-dimensional cube shows this. Perhaps $h(n)/n \to \infty$, but as far as I know this is still open.

Pach and I now ask: Suppose the n points further satisfy that they do not contain a parallelogram, or that no two lines determined by our n points are parallel. Is it then true that our n points determine $> cn^2$ distances?

To end this paper I discuss some decomposition problems, which are of a set theoretical character. Assume $c = \aleph_1$. Can one decompose E_n, the n-dimensional Euclidean space, as the union of $\aleph_0$ sets S_n so that for every n all the distances in S_n are distinct? Kakutani and I proved that for $n = 1$ the answer is affirmative (but it becomes negative if $c > \aleph_1$). Davies proved that the answer is affirmative for $n = 2$ and Kunen proved it for all n. A few years ago I asked whether such a decomposition is possible for Hilbert spaces. Pósa proved that the answer is negative in the following very strong sense. There is a set S in a Hilbert space of power $\aleph_1$ so that every subset S_1 of S with power $\aleph_1$ contains an equilateral triangle. $c = \aleph_1$ was not needed here. If $c = \aleph_1$ is assumed, Pósa shows that every subset of power $\aleph_1$ of his set contains an infinite dimensional equilateral simplex. (This was just proved by Kunen in a surprisingly simple way without using $c = \aleph_1$.)

Drop now the assumption $c = \aleph_1$. Can one decompose E_n into countably many sets S_i so that none of the S_i contain an isosceles triangle? For $n = 1$ the answer is well known to be affirmative, but as far as I know it is open for $n > 1$. Clearly many related questions can be asked.

Here is a Pizier type problem formulated by Nesetril, Rödl and myself: Let S be an infinite set in the plane (or more generally in a metric space). Assume that there is an $\varepsilon > 0$ so that for every n and every choice of n points $x_1, x_2, \dots, x_n$ of S there is a subset $x_{i_1}, \dots x_{i_m}$ of these n points with $m > \varepsilon n$ so that all the distances among these m points are distinct. Is it then true that S is the union of a finite number of sets S_i,

$$S = \bigcup_{i=1}^{t} S_i ,$$

where all the distances in S_i are distinct? The condition is clearly necessary. Is it also sufficient? Nesetril, Rödl and I have a paper in preparation about problems of this type. The same problem could be asked about decomposition into sets not containing any isosceles triples.

One could ask: Let $|S| = \aleph_2$, $S \subseteq E_n$ (we now assume $c > \aleph_1$). Assume that every subset S_1 of S of power $\aleph_1$ is the union of denumerably many sets $S_1^{(i)}$, $\cup_i S_1^{(i)} = S_1$ so that for every i all the distances in $S_1^{(i)}$ are distinct. Does S then have such a decomposition? Kunen just tells me that the answer is negative since his proof gives that every $S_1 \subseteq E_n$, $|S| = \aleph_1$ has such a decomposition. Assume now $c \geq \aleph_3$, $|S| = \aleph_3$, every subset $S_1 \subseteq S$, $|S_1| = \aleph_2$ has a decomposition into $\aleph_0$ sets $S_1^{(i)}$ so that all the distances in $S_1^{(i)}$ are distinct. Is it then true that S has such a decomposition?

The following sharpening of Kunen's result perhaps holds: One can decompose E_n into countably many sets S_i so that all the distances in S_i are distinct and every distance can occur in only a finite number of the S_i's. (Pósa just proved this for $n \leq 2$.)

Another interesting new type of decomposition problem was raised by Pach. Let S be a collection of sets in E^n. Does there exist a constant k(S) such that any k-fold covering of E^n by members of S can be split into two coverings? (A system of sets is said to form a k-fold covering of E^n, if every point of the space is contained in at least k sets.) Mani and Pach showed that if S is the family of all unit balls then the answer is in the affirmative only if $n = 2$. In this case $k(S) \leq 48$, but this bound is probably far from being sharp. Perhaps the most interesting unsolved case in the plane is, when S is the family of all strips, i.e., all regions bounded by two parallel lines. For half planes and, in general, for half-spaces the answer is positive and follows from Helly's Theorem.

One final problem: Let there be given n points in the plane no four on a line. Determine or estimate the largest h(n) so that one can always find h(n) of them, no three of which are on a line. Trivially $h(n) \geq \sqrt{2n}$. How far is this from being best possible? More generally one can ask: Let $x_1,\ldots,x_n$ be n points no k on a line. Let $\ell < k$. Determine (or estimate) the largest $h(n;\ell,k)$ so that one can always find $h(n;\ell,k)$ of them no ℓ on a line.

References

P. Erdös, Problems and results in combinatorial geometry. Discrete geometry and convexity, Annals of the New York Academy of Sciences, Vol. 940, 1-10. This paper contains many references but since the Annals may not be available everywhere I refer to two further papers of mine:

On some problems of elementary and combinatorial geometry, Annali di Mat., Ser 4, 103, 99-108. Some combinatorial problems in geometry, Lecture Notes in Math, Springer Verlag. Conference held in Haifa, Israel 1978, 46-53.

For a very rich source of problems, see W. O. J. Moser, Problems on extremal properties of a finite set of points, Annals of the New York Academy of Sciences, Vol. 440, 52-69. This paper has very many references. See also a forthcoming booklet of W. Moser and J. Pach, Research problems in discrete geometry. Mimeographed, 1986.

P. Erdös and S. Kakutani, On non-denumerable graphs, Bull. Amer. Math. Soc., 49(1943), 457-461. The results of Kunen and Pósa are unpublished.

R. O. Davies, Partitioning the plane into denumerably many sets without repeated distances, Proc. Cambridge Phil. Soc. 72 (1972), 179-183, Kunen's paper will soon appear. See also a forthcoming paper of Komjáth and myself.