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ON ADMISSIBLE CONSTELLATIONS OF CONSECUTIVE PRIMES

PAUL ERDÖS and HANS RIESEL

Németvölgyi Ut 72C
Budapest (XII)
Hungary

Department of Numerical Analysis and Computing Science
Royal Institute of Technology
S-100 44 Stockholm, Sweden

Dedicated to Peter Naur on the occasion of his 60th birthday.

Abstract.

Admissible constellations of primes are patterns which, like the twin primes, no simple divisibility relation would prevent from being repeated indefinitely in the series of primes. All admissible constellations, formed of *consecutive primes*, beginning with a prime < 1000, are established, and some properties of such constellations in general are conjectured.

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1. The twin prime hypothesis.

The so-called twin prime hypothesis deals with the number of occurrences of pairs $(p, p+2)$, with both integers primes. The hypothesis has been generalized by Hardy and Littlewood [1] to arbitrary constellations of integers $(x+a_1, x+a_2, \ldots, x+a_l)$.

2. Admissible constellations.

Let us start with a formal definition: $a_1, a_2, \ldots, a_k$ *is an admissible sequence of integers if the* a_i*'s do not form a complete set of residues* mod p *for any prime* p.—Clearly only the primes $p \le k$ have to be considered. A sequence which is not admissible is called *inadmissible*.

Beginning with the admissible sequence $\{a_i\}$ we now search for constellations of integers $\{x+a_i\}$, *with all its members prime*. We shall call such a sequence an *admissible constellation*.—According to our definition, the sequence $(0, 2, 4)$ leading to the constellation $(x, x+2, x+4)$ and being represented by the primes 3, 5, 7, is inadmissible, since precisely one of three consecutive even or odd integers is divisible by 3. This implies that 3, 5, 7 and −7, −5, −3 are the only instances for which all members of the constellation are primes. On the other hand, the sequence $(0, 2, 6, 8)$, leading to the quadruplet

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$(x, x+2, x+6, x+8)$, which is represented by the primes 11, 13, 17, 19 or by 101, 103, 107, 109, is admissible, since there is no prime p, for which all residue classes mod p can be filled by the members of the constellation. The generalized Hardy-Littlewood conjecture gives an asymptotic formula (see [4]) for the number of occurrences of any particular admissible constellation with all its members primes and $\leq N$, as $N \to \infty$. In particular, the hypothesis implies that their number tends to infinity, as $N \to \infty$.

3. Admissible constellations of consecutive primes.

In a search for repetitions of the pattern of small primes, the second author was led to consider admissible constellations of consecutive primes [5]. This study has now been carried a lot further, and some problems, which pose themselves, have been illuminated by computer runs.

A pattern of consecutive primes *starting from the very beginning of the prime series*, can never repeat, because (2, 3) or (3, 5, 7) never repeats. If, however, we start by some prime p_m, say, there always exists an admissible constellation $(p_m, p_{m+1}, \ldots, p_n)$ of maximal length. The first maximal admissible constellations are

$$(2), \quad (3,5), \quad (5,7,11,13,17), \quad (7,11,13,17,19,23).$$

If we continue, the constellations grow in length quite fast, so we have to introduce a shorter notation. Let $p_m(u)p_n$ denote $(p_m, p_{m+1}, \ldots, p_n)$, where $u = n-m+1$ is the number of primes in the constellation. With this notation, the sequence of maximal admissible constellations continues as follows:

$$\begin{array}{llllll} 11(15)67, & 13(21)101, & 17(20)101, & 19(35)181, & 23(42)229, & 29(56)313, \\ 31(74)433, & 37(73)433, & 41(78)463, & 43(77)463, & 47(105)653, \ldots & \end{array}$$

Here we make an interesting observation! Some of the constellations (viz. 17(20)101, 37(73)433 and 43(77)463) are *shorter* than their immediate predecessors. In order to explain why, let us see what limits the length of an admissible constellation of consecutive primes. Why cannot the constellation 31(74)433 be extended further? Because the residues of the 75 primes of the constellation 31(75)437 would fill *all* residue classes modulo some prime, which in this case turns out to be $p = 37$. This fact makes the extended constellation inadmissible. We shall say that the prime $p = 37$ *kills* the constellation. Usually, at least for small values of p_m, it is the prime p_m which ultimately kills the constellation, and since p_m is absent from the constellation beginning with p_{m+1}, it no longer interferes, and thus does not prevent the next constellation from growing longer.—But in some cases, where a maximal admissible constellation is *shorter* than its predecessor, the killing is due to some prime p_{m+s}, with $s > 0$, rather than p_m, and so there is a good chance that the prime p_{m+s} will kill also the next constellation, beginning with p_{m+1}. (This is, however, not certain! It could happen that it will not, viz. if the residue class p_m mod p_{m+s} is represented only once in the constellation $(p_m, \ldots, p_n)$. In such a case, it is the presence of p_m which is due to the killing by p_{m+s} of the constellation beginning with p_m, and thus, in this peculiar case, it is unlikely that p_{m+s} would kill the next constellation, lacking p_m, too! We shall give some explicit examples of this case below.)—As a result, a maximal constellation like 37(73)433 is a subset of the preceding one, and can be extended on the low side to give 31(74)433. We shall call an admissible constellation, which cannot be extended on either side, a *truly maximal* admissible constellation (of consecutive primes).

4. Truly maximal admissible constellations below 1000.

A computer run has exposed 46 truly maximal admissible constellations $p_m(u)p_n$ of consecutive primes with $p_m < 1000$. We give these in Table 1. For each prime p_m, the killing prime p_{m+s}, as well as the value of $s = 0, 1, 2, \ldots$, is also given.

The constellation 13(21)101 is unique, within the range of the table, by the property that its possible extension is killed *simultaneously* by the two primes 13 and 17.

Also, two maximal admissible constellations were found, which could serve as examples of the peculiar case, discussed in the preceding subsection. The maximal admissible constellation 193(550)4339 cannot be extended, because an extension to 193(551)4349 would be killed by $p_{m+2} = 199$. If the first prime 193, however, is removed, then the remaining constellation can be extended to 197(600)4789, whose extension is killed by $p = 223$.—This situation occurs once more within the range of the table, viz. for 367(1283)11177, killed by $p_{m+3} = 383$, while the next maximal admissible constellation is 373(1380)12143, which is killed by $p_{m+1} = 379$. Usually, a *truly* maximal admissible constellation starts with the prime immediately following the killing prime of the preceding one. See Table 1 for the verification of this statement! In the peculiar cases, however, the next truly maximal admissible constellation starts at an earlier prime, as can be seen from the two examples just given.

5. Maximal constellations immediately after 1000k.

In Table 2 we give the maximal admissible constellations $p_m(u)p_n$ for the first prime p_m, larger then $1000k$, for $k = 1, 2, \ldots, 10$. These values are of interest in connexion with the problems and conjectures discussed in the following subsection.

6. Some problems and conjectures.

We now mention some problems and conjectures in connexion with the tables we have computed:

1. Are there infinitely many maximal admissible constellations, for which $s > 0$?

2. Are there infinitely many maximal admissible constellations, for which $s = 0$? (This seems unlikely.)

3. If s is given, how often is p_{m+s} the killing prime?

4. How fast does p_n grow with p_m, i.e. how long should we expect a maximal admissible constellation, beginning with p_m, to be? Is the length of the constellation, measured in its number of terms, asymptotically $C \cdot p_m \log p_m$? Is $C = \frac{1}{2}$?

Carl Pomerance has shown [3] that for every m

$$p_n > \frac{p_m \log p_m \log_2 p_m \log_4 p_m}{(\log_3 p_m)^2}, \tag{1}$$

where $\log p_m$ stands for $\log_e p_m$ and $\log_2 p_m$ stands for $\log_e \log_e p_m$ etc.—It would be nice to have an upper bound for p_m. A result of Linnik [2] gives $p_n < p_m^c$ for some absolute constant c. Surely $p_n < p_m^{1+}$ for every > 0 and most likely $p_n < p_m(\log m)^c$ for some $c > 1$. Both these conjectures, especially the second, seem quite hopeless to prove at present.

5. Let $1 = a_1 < a_2 < \ldots < a_k$ be admissible. What can be said about $f(k) = \min a_k$? It is well known that

$$(1 + o(1))\frac{f(k)}{\log f(k)} < k < (2 + o(1))\frac{f(k)}{\log f(k)}. \tag{2}$$

The lower bound follows from the prime number theorem and the upper bound is a result proved by Selberg and, later, also by Montgomery and Vaughan.—From (2) one can get

$$\frac{1}{2}k \log k(1 - o(1)) < f(k) < k \log k(1 + o(1)), \tag{3}$$

and even the stronger

$$\frac{1}{2}k \log k < f(k) < k \log k \left(1 + O\left(\frac{\log\log k}{(\log k)^2}\right)\right). \tag{4}$$

6. A more fundamental problem, related to the others is: How long does a given prime modulus p "last," before all its residue classes are used up by $(p_m, p_{m+1}, \ldots, p_n)$?

7. Tables.

Table 1. The 46 truly maximal admissible constellations below 1000.

$p_m(u)p_n$	p_{m+s}	s	$p_m(u)p_n$	p_{m+s}	s
2(1)2	2	0	197(600)4783	223	3
3(2)5	3	0	227(714)5807	241	4
5(5)17	5	0	251(763)6271	257	1
7(6)23	7	0	263(901)7541	271	2
11(15)67	11	0	277(948)7963	277	0
13(21)101	13, 17	0, 1	281(970)8209	281	0
19(35)181	19	0	283(1034)8779	293	1
23(42)229	23	0	307(1217)10457	331	4
29(56)313	29	0	337(1288)11177	383	8
31(74)433	37	1	373(1380)12143	379	1
41(78)463	43	1	383(1388)12241	419	5
47(105)653	53	1	421(1472)13037	487	11
59(110)701	59	0	491(2048)18757	509	3
61(152)1009	67	1	521(2193)20261	599	11
71(195)1307	79	2	601(2261)21067	619	4
83(216)1493	97	2	631(2381)22277	659	5
101(272)1951	103	1	661(2544)23957	719	7
107(300)2179	113	2	727(2658)25247	797	10
127(411)3083	131	1	809(3194)30911	857	8
137(449)3433	149	2	859(3323)32359	887	5
151(459)3533	157	1	907(3429)33469	929	3
163(506)3919	167	1	937(3798)37361	953	3
173(554)4339	199	6	967(4260)42299	1051	14

Table 2. 10 maximal admissible constellations between 1000 and 10007.

$p_m(u)p_n$	p_{m+s}	s	$p_m(u)p_n$	p_{m+s}	s
1009(4254)42299	1051	8	6007(32645)394393	6007	0
2003(9060)97369	2053	6	7001(38879)476981	7237	24
3001(14619)164377	3011	1	8009(47459)591649	8123	14
4001(20589)238943	4019	4	9001(51662)649427	9161	18
5003(26467)314219	5119	15	10007(57411)728017	10321	36

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