Unsolved Problems

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In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

Has Every Latin Square of Order n a Partial Latin Transversal of Size $n - 1$?

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The notion of a transversal of a latin square easily generalizes to more general arrays. We present some of the known combinatorial results in this area and indicate the open questions.

They concern $m \times n$ rectangular arrays of mn cells ($m \le n$), each cell containing a symbol. A **transversal** of an array is a set of m cells, no two in the same row or same column. A **latin transversal** is one whose symbols are distinct. A row or column is **latin** if its symbols are distinct. If $m = n$ and each symbol occurs exactly n times, we call the array an **equi-n-square**. If each row and column of an equi-n-square is latin, it is a **latin square**. The definitive work [**1**] on the subject, published in 1974, contains 639 references and 73 problems. The earliest result is due to Euler.

Marshall Hall [**2**, and see the earlier work of Lowell Paige, **3**] examined transversals of $n - 1$ by n arrays obtained from an abelian group A of order n, as follows. Let $A = \{a_1, \ldots, a_n\}$. Corresponding to each element a in A is the sequence $\{a + a_1, \ldots, a + a_n\}$, recording translation by a. Let $\{b_1, \ldots, b_{n-1}\}$ be a sequence of $n - 1$ not necessarily distinct elements of A, and form the $n - 1$ by n array of the translations of A by $b_1, \ldots, b_{n-1}$. Each row is latin, but there may be duplications in the columns. Hall proved that such an array has a latin transversal.

In [**5**] Stein introduced several types of arrays, including the equi-n-square, and established the existence of transversals with many distinct elements. For example, the equi-n-square has a transversal with at least

$$n\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots \pm \frac{1}{n!}\right) \approx (1 - 1/e)n \approx 0.63n$$

distinct symbols. He also gave equi-n-squares ($n \ge 2$) without a latin transversal, and showed that n by n arrays in which each symbol appears exactly q times have transversals with at least $n - q/2$ symbols.

More recently, Shor [4] showed that every latin square has a partial latin transversal of length at least

$$n - 5.53(\ln n)^2$$

and observed that his method will yield better results, but not as good as $n - \log_2 n$ (log to base 2).

There are several outstanding conjectures. The most noteworthy are:

Conjecture 1. *An equi-n-square has a transversal with at least $n-1$ distinct symbols.*

The special case of the title has been associated with Herb Ryser's name, while [**1**, p. 103] attributes it to Richard Brualdi. The answer is affirmative for latin squares which are the Cayley tables of abelian groups.

Conjecture 2. *An $n-1$ by n array in which each symbol appears at most q times $(q \leq n)$ has a latin transversal.*

The case $q = n$ would imply that a row-latin $n-1$ by n array has a latin transversal, and would also imply Conjecture 1. For $q = 1$, conjecture 2 is trivally true, and the case $q = 2$ follows from $q = 3$, which can be proved as follows.

THEOREM 1. *Let $2 \leq k < m \leq n$ and assume that every m by n array in which each symbol appears at most 3 times has a latin transversal that misses k proscribed cells. Then every such $m+1$ by $n+1$ array has a latin transversal that misses k proscribed cells.*

Proof. Consider an $m+1$ by $n+1$ array in which each symbol appears at most 3 times and in which k cells are proscribed. There are two cases: (1) some row or column contains at least 2 proscribed cells; (2) no row or column contains more than 1 proscribed cell.

Case 1. Consider a row (or column) with at least 2 proscribed cells. Since $k < m$, there is a cell c in that row which is not proscribed. Denote the symbol in that cell by t. Delete the row and column containing c, producing an m by n array. This array has at most $k-2$ proscribed cells and at most 2 cells with the symbol t, so it has a latin transversal that misses the proscribed cells and does not have the symbol t in it. Adjunction of cell c gives a latin transversal for the original $m+1$ by $n+1$ array.

Case 2. Consider a cell that is on a row of one proscribed cell and on a column of another proscribed cell. Delete the row and column of this cell and argue as in Case 1.

An exhaustive search showed that every 4 by 5 array in which each symbol occurs at most 3 times has a latin transversal avoiding any 2 proscribed elements. Hence every m by $m+1$ array has this property for $m \geq 4$. A similar analysis showed that for $n \geq 6$, every n by n array in which each symbol appears at most 3 times possesses a latin transversal avoiding 2 proscribed cells.

Theorem 2 is proved much as Theorem 1.

THEOREM 2. *If every m by n array in which each symbol occurs at most 3 times has a latin transversal that passes through any given cell, then every $m+1$ by $n+1$ array in which each symbol appears at most 3 times has the same property.*

Note that there is no latin transversal through the top left cell of a 4 by 4 array which contains the configuration of FIG. 1.

Our final theorem is an unpublished result of Erdős and Joel Spencer.

a			
	a	b	b
	b	a	c
	c		

FIG. 1.

THEOREM 3. *Let k be fixed. An $n \times n$ array in which each symbol appears at most k times has a latin transversal if n is sufficiently large.*

Their proof will also work if $k < (\ln n)^{1-\varepsilon}$ and they believe that it may be possible to make it do so for $k < (\ln n)^{c}$ for every fixed c, but beyond this there may be serious difficulties. [J. Dénes [6] made a conjecture equivalent to Conjecture 1, and some other relevant remarks.—Ed.]

REFERENCES

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