Graphs with Unavoidable Subgraphs with Large Degrees

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ABSTRACT

Let $\mathcal{G}(n, m)$ denote the class of simple graphs on n vertices and m edges and let $G \in \mathcal{G}(n, m)$. There are many results in graph theory giving conditions under which G contains certain types of subgraphs, such as cycles of given lengths, complete graphs, etc. For example, Turan's theorem gives a sufficient condition for G to contain a K_{k+1} in terms of the number of edges in G. In this paper we prove that, for $m = \alpha n^2$, $\alpha > (k-1)/2k$, G contains a K_{k+1}, each vertex of which has degree at least $f(\alpha)n$ and determine the best possible $f(\alpha)$. For $m = \lfloor n^2/4 \rfloor + 1$ we establish that G contains cycles whose vertices have certain minimum degrees. Further, for $m = \alpha n^2$, $\alpha > 0$ we establish that G contains a subgraph H with $\delta(H) \geq f(\alpha, n)$ and determine the best possible value of $f(\alpha, n)$.

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1. INTRODUCTION

All graphs considered in this paper are finite and loopless, and have no multiple edges. For the most part, our notation and terminology follows that of Bondy and Murty [2]. Thus a graph G has vertex set $V(G)$, edge set $E(G)$, $\nu(G)$ vertices, $\epsilon(G)$ edges, and minimum degree $\delta(G)$. K_n denotes the complete graph on n vertices and C_l a cycle of length l.

When the number of vertices and edges of a graph are suitably restricted much can be said about the structure of the graph. Indeed, the graph theory literature contains many results concerned with the structure of extremal graphs containing (or not containing) certain prescribed subgraphs. We refer the reader to the book by Bollobás [1] for an excellent presentation of such results. One of the best known results of this type is that of Turan [7, 8]:

Turan's Theorem. Let $T_{q,n}$ denote the complete q-partite graph on n vertices in which all parts are as equal in size as possible. If G is a simple graph on n vertices containing no K_{q+1}, then

$$\epsilon(G) \leq \epsilon(T_{q,n}) = \frac{q-1}{2q}n^2 - \frac{r(q-r)}{2q},$$

where $n \equiv r(\mathrm{mod}\ q)$, with equality possible if and only if $G \cong T_{q,n}$.

Let $\mathcal{G}(n, m)$ denote the class of graphs on n vertices and m edges. Let $G \in \mathcal{G}(n, m)$. When $m > [(k-1)/2k]n^2$, k a positive integer, Turan's theorem asserts that G contains K_{k+1}. In this paper we prove (Theorem 3) that G contains K_{k+1}, each vertex of which has degree (in G) at least $f(\alpha)n$, where

$$f(\alpha) = \begin{cases} \dfrac{k}{k+1}\left(1 - \sqrt{1 - \dfrac{2k+2}{k}\alpha}\right), & \text{if } \dfrac{k-1}{2k} < \alpha \leq \dfrac{k(k+3)}{2(k+2)^2} \\ 1 - \dfrac{2}{k}(1 - \sqrt{2k\alpha - k + 1}), & \text{otherwise}. \end{cases}$$

Moreover, we establish that this result is best possible.

For the particular case $m = \lfloor n^2/4 \rfloor + 1$ our result asserts that G contains a triangle, each vertex of which has degree greater than $n/3$. We establish (Theorem 4) that, when $m = \lfloor n^2/4 \rfloor + 1$, G contains a C_r for each r, $3 \leq r \leq \lfloor n/6 \rfloor + 2$, each vertex of which has degree greater than $n/3$ and that this result is best possible.

We also prove (Theorem 5) that every member of $\mathcal{G}(n, \alpha n^2)$ contains a subgraph H with

$$\delta(H) \geq \left\lceil \left(1 - \sqrt{\left(1 - \frac{1}{2n}\right)^2 - 2\alpha}\right)n - \frac{1}{2} \right\rceil$$

and that this result is best possible.

Some related unsolved problems are discussed.

2. PRELIMINARIES

Consider a graph H on n or fewer vertices. For sufficiently large m, it is easily seen that all graphs in $\mathcal{G}(n, m)$ would contain a subgraph isomorphic to H. The problem of determining this maximum value of m such that $\mathcal{G}(n, m)$ contains at least one graph G, which has no subgraph isomorphic to H, has attracted considerable interest in the literature and Turan's Theorem mentioned in the introduction is one such example. We shall denote by $e(n, H)$ this maximum value of m for a given n and H. We restate below two well-known results (see [1]) on the value of $e(n, H)$ in the form of

Theorem 1.

(a) When H is the complete graph K_{k+1},

$$e(n, K_{k+1}) = \frac{k-1}{2k} n^2 - \frac{r(k-r)}{2k}, \tag{2.1}$$

where

$$n \equiv r (\mathrm{mod}\ k).$$

(b) When H is a cycle C_l of length l,

$$e(n, C_l) \le \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor, & \text{if } l < \dfrac{n+3}{2} \\ \dbinom{l-1}{2} + \dbinom{n-l+2}{2}, & \text{otherwise.} \end{cases}$$

The bound is achievable if l is odd or $l \ge (n+3)/2$. ∎

In this paper we investigate the minimum degree on the vertices of the H-isomorphic subgraph of G from $\mathcal{G}(n, m)$ when m is greater than $e(n, H)$.

In establishing that our results are best possible, the following constructions are useful. Given two vertex disjoint graphs J_1 and J_2, $J_1 \vee J_2$ will denote the graph whose vertex set is the union of vertex sets of J_1 and J_2, and the edges are all those in J_1 and J_2 together with all those edges with one end in vertex set of J_1 and the other in vertex set of J_2. The two graphs we construct are

$$H_1(a, k) = \overline{K}_{n-a} \vee T_{k,a}$$

and

$$H_2(a, k) = R_{n-a, n-2a} \vee T_{k,a}, \qquad a \le \frac{n}{2},$$

where $\overline{K}_{n-a}$ denotes the complement of K_{n-a}, $T_{k,a}$ the complete k-partite graph of order a with all parts as equal as possible, and $R_{n-a,n-2a}$ a graph on $n - a$ vertices and $\lceil \frac{1}{2}(n - a)(n - 2a) \rceil$ edges with minimum degree $n - 2a$. It may be noted that every K_{k+1} subgraph of $H_1(a,k)$ has exactly one vertex of $\overline{K}_{n-a}$ and of $H_2(a,k)$ has at least one vertex of $R_{n-a,n-2a}$. The number of edges in H_1 and H_2 are

$$h_1(a,k) = (n - a)a + \left(\frac{k - 1}{2k}\right)a^2 - \frac{r(k - r)}{2k}$$

and

$$h_2(a,k) = h_1(a,k) + \left\lceil \frac{1}{2}(n - a)(n - 2a) \right\rceil$$

respectively, where $r \equiv a (\text{mod } k)$.

3. MAIN RESULTS

The following general lemma is easily established by simple counting:

Lemma 1. Let G_1 denote the subgraph of $G \in \mathcal{G}(n,m)$ induced by the vertices of degree at least d and $n_1 = |V(G_1)|$. If G_1 does not contain a subgraph H, then

$$m \le g(n_1, H) = e(n_1, H) + \begin{cases} (n - n_1)(d - 1), & \text{if } n_1 \ge d - 1 \\ \left\lfloor \frac{1}{2}(n - n_1)(d + n_1 - 1) \right\rfloor, & \text{otherwise}. \end{cases} \quad \blacksquare \tag{3.1}$$

In obtaining the results mentioned in the introduction, we need to maximize $g(n_1, H)$ with respect to n_1 for the two subgraph structures considered.

The following lemma will be useful for this purpose:

Lemma 2. For $H = K_{k+1}$

$$\max_{n_1 \ge d-1} \{g(n_1)\} = \max\{g(n), g(d - 1)\} \tag{3.2}$$

and

$$\max_{n_1 \le d-1} \{g(n_1)\} = \begin{cases} g(d - 1), & \text{if } d - 1 \le \left\lfloor \dfrac{nk}{k + 2} \right\rfloor \\ g\left(\left\lfloor \dfrac{1}{2}(n - k + 1) \right\rfloor k\right), & \text{otherwise}, \end{cases} \tag{3.3}$$

where $g(n_1) = g(n_1, K_{k+1})$ [given in (3.1)].

Proof. From (2.1) and (3.1) we get

$$g(n_1+1)-g(n_1)=n_1-\left\lfloor\frac{n_1}{k}\right\rfloor + \begin{cases} -(d-1), & \text{if } n_1 \ge d-1 \\ \left\lfloor\frac{1}{2}(n-d+1)\right\rfloor - n_1 - \delta(n-d)\,\delta(n-n_1-1), \end{cases} \tag{3.4}$$

where $\delta(x)=0$ or 1 according to whether x is even or odd.

For $n_1 \ge d-1$, (3.4) is monotonically increasing in n_1. Hence $g(n_1)$ attains its maximum at one of its end points (i.e., at $d-1$ or n). This proves (3.2).

For $n_l \le d-1$,

$$\begin{aligned} g(n_1+1)-g(n_1) &= -\left\lfloor\frac{n_1}{k}\right\rfloor + \left\lfloor\frac{1}{2}(n-k+1)\right\rfloor - \delta(n-d)\delta(n-n_1-1) \\ &\ge -\left\lfloor\frac{n_1}{k}\right\rfloor + \left\lceil\frac{1}{2}(n-d-1)\right\rceil \\ &\ge \left\lceil\frac{n-d-1}{2}-\frac{n_1}{k}\right\rceil \\ &\ge 0, \end{aligned} \tag{3.5}$$

when

$$\frac{n_1}{k} < \frac{n-d-1}{2}+1=\frac{n-d+1}{2}.$$

In the range of 0 to $d-2$, (3.5) is satisfied if $d-1 \le \lfloor nk/(k+2)\rfloor$. For $d-1 > \lfloor nk/(k+2)\rfloor$, we note that $g(n_1+1)-g(n_1)$ changes sign from positive to negative when $n_1 = \lfloor (n-d+1)/2\rfloor k$ and hence $g(n_1)$ is maximized at this point. This completes the proof of the lemma. ∎

Theorem 2. Suppose all graphs in $\mathcal{G}(n,\alpha n^2)$, $\alpha > (k-1)/2k$, contain subgraphs isomorphic to H. Also let

$$e(n_1,H) \le e(n_1,K_{k+1}).$$

Then for any $G \in \mathcal{G}(n,\alpha n^2)$, G, contains a subgraph J isomorphic to H such that every vertex of J has degree at least $f(\alpha)\cdot n$, where

$$f(\alpha)=\begin{cases} \dfrac{k}{k+1}\left\{1-\sqrt{1-\dfrac{2k+2}{k}\alpha}\right\}, & \text{if } \alpha \le \dfrac{k(k+3)}{2(k+2)^2} \\ 1-\dfrac{2}{k}\left(1-\sqrt{2k\alpha-k+1}\right), & \text{otherwise}. \end{cases} \tag{3.6}$$

Proof. For a graph $G \in \mathcal{G}(n, \alpha n^2)$, let $d(G)$ be the smallest value such that the subgraph G_1 of G induced by the vertices of degree at least $d(G)$ has no subgraph isomorphic to H. Let $\min_G\{d(G)\} = d(G^*) = d$, and $|V(G_1^*)| = n_1$. We will show that $d > \lceil f(\alpha) \cdot n \rceil$, thus proving the theorem. Obviously

$$\begin{aligned} \alpha n^2 &\le g(n_1, H) \\ &\le g(n_1) = g(n_1, K_{k+1}) \\ &\le \begin{cases} g(d-1), & \text{if } d - 1 \le \left\lfloor \dfrac{nk}{k+2} \right\rfloor \\ g\left(\left\lfloor \dfrac{n-d+1}{2} \right\rfloor k\right), & \text{otherwise}. \end{cases} \qquad \text{(By Lemma 2.)} \end{aligned}$$

Now, by observing that

$$\left\lfloor \frac{n-d+1}{2} \right\rfloor = \frac{n-d+1-\delta(n-d+1)}{2},$$

and simplifying we get

$$\begin{aligned} g\left(\left\lfloor \frac{n-d+1}{2} \right\rfloor k\right) &= \left\lfloor \frac{1}{2}\left\{k\left(\frac{n-d+1}{2}\right)^2 + n(d-1) \right.\right. \\ &\qquad \left.\left. - \delta^2(n-d+1) \cdot \frac{k}{4}\right\}\right\rfloor \\ &\le \frac{1}{2}\left\{k\left(\frac{n-d+1}{2}\right)^2 - n(d-1)\right\} \end{aligned}$$

and hence

$$\alpha n^2 \le \begin{cases} \dfrac{k-1}{2k}(d-1)^2 + (n-d+1)(d-1), & \text{if } d - 1 \le \left\lfloor \dfrac{nk}{k+2} \right\rfloor \\ \dfrac{1}{2}\left\{k\left(\dfrac{n-d+1}{2}\right)^2 + n(d-1)\right\}, & \text{otherwise}. \end{cases} \tag{3.7}$$

Observing that the right-hand side of (3.7) is a quadratic in $(d-1)$, (3.7) then implies that

$$\begin{aligned} d - 1 &\ge \begin{cases} a(\alpha) = \dfrac{nk}{k+1}\left(1 - \sqrt{1 - 2\dfrac{k+1}{k}\alpha}\right), & \text{if } d - 1 \le \dfrac{nk}{k+2} \\ b(\alpha) = n\left(1 - \dfrac{2}{k}(1 - \sqrt{2k\alpha - k + 1})\right), & \text{otherwise}. \end{cases} \\ &\ge \min(a(\alpha), b(\alpha)). \end{aligned} \tag{3.8}$$

In (3.8), we note that $a(\alpha) = b(\alpha)$ when $\alpha = [k(k+3)]/[2(k+2)^2]$ and $a(\alpha) - b(\alpha)$ is an increasing function in α. Hence

$$d - 1 \geq \begin{cases} a(\alpha), & \text{if } \alpha \leq \dfrac{k(k+3)}{2(k+2)^2} \\ b(\alpha), & \text{otherwise}. \end{cases} \tag{3.9}$$

Observe that the right-hand side of (3.9) is $f(\alpha)$, and hence if $d = f(\alpha)$, G_1^* would contain a subgraph isomorphic to H. Hence the theorem. ∎

For the particular case when $H = K_{k+1}$ we have

Theorem 3. Let $G \in \mathcal{G}(n, \alpha n^2)$, $\alpha > (k-1)/2k$. Then G contains a K_{k+1}, each vertex of which has degree at least $f(\alpha)n$, where $f(\alpha)$ is given by (3.6). Moreover, this result is the best possible.

Proof. The first part of the theorem follows from Theorem 2. The graphs $H_1(\lceil f(\alpha)n \rceil, k)$ when $\alpha \leq [k(k+3)]/[2(k+2)^2]$ and $H_2(n - \lceil f(\alpha)n \rceil, k)$ when $\alpha > [k(k+3)]/[2(k+2)^2]$, have at least αn^2 edges and every K_{k+1} contains at least one vertex with degree $\lceil f(\alpha)n \rceil$. Hence the result is the best possible. ∎

Remark 1. It follows from Theorem 3 that every graph in $\mathcal{G}(n, \lfloor n^2/4 \rfloor + 1)$ contains a triangle, each vertex of which has degree greater than $n/3$ and that this result is the best possible.

Remark 2. Erdös [4] proved that every graph in $\mathcal{G}(n, \lfloor n^2/4 \rfloor + 1)$ contains a subgraph

$$H = \overline{K}_{u_n} \vee (\overline{K}_{u_n} + e)$$

where $u_n = \lfloor c \log n \rfloor$ and c is a positive constant. It follows from the above that every vertex of H has degree greater than $n/3$ in G.

Next we consider the case when H is a cycle. We make use of the following result:

Lemma 3 [1, p. 150]. Let $G \in \mathcal{G}(n, m)$ with $m > n^2/4$. If G has circumference c, then G contains a C_r for each r, $3 \leq r \leq \max\{\lfloor (n+3)/2 \rfloor, c\}$. ∎

Theorem 4. Let $G \in \mathcal{G}(n, \lfloor n^2/4 \rfloor + 1)$. Then G contains a C_r for each r, $3 \leq r \leq \lfloor n/6 \rfloor + 2$, each vertex of which has degree at least $\lceil (n+1)/3 \rceil$. Moreover, this result is best possible.

Proof. That G contains cycles of the specified length follows from Theorem 1. Let d be the smallest integer such that the subgraph G_1 of G induced by the vertices of degree at least d does not contain cycles of every length from 3 to $\lfloor n/6 \rfloor + 2$. We may assume that $d \le \lceil (n+1)/3 \rceil$, as otherwise there is nothing to prove. Suppose G_1 has n_1 vertices and a maximum cycle of length $l - 1 \le \lfloor n/6 \rfloor + 1$.

We may suppose that $\epsilon(G_1) > \frac{1}{4}n_1^2$, as otherwise the result follows from the proof of Theorem 2. This, together with Theorem 1 and Lemmas 1 and 3, implies that

$$\left\lfloor \frac{n_1 + 3}{2} \right\rfloor \le l - 1 \le \left\lfloor \frac{n}{6} \right\rfloor + 1, \tag{3.10}$$

and

$$\left\lfloor \frac{n^2}{4} \right\rfloor + 1 \le \binom{l-1}{2} + \binom{n_1 - l + 2}{2} + \begin{cases} (n - n_1)(d - 1), & \text{if } n_1 \ge d - 1 \\ \dfrac{1}{2}(n - n_1)(d + n_1 - 1), & \text{otherwise}, \end{cases} \tag{3.11}$$

Let $g(n_1, d, l)$ denote the right-hand side of (3.11). We have

$$\begin{aligned} g(n_1, d, l) &\le g\left(n_1, \left\lceil \frac{n+1}{3} \right\rceil, l\right) \\ &\le g\left(\left\lceil \frac{n+1}{3} \right\rceil - 1, \left\lceil \frac{n+1}{3} \right\rceil, l\right) \\ &\le g\left(\left\lceil \frac{n+1}{3} \right\rceil - 1, \left\lceil \frac{n+1}{3} \right\rceil, \left\lfloor \frac{n}{6} + 1 \right\rfloor\right) \\ &\le \frac{n^2}{4}. \end{aligned}$$

This contradiction establishes that $l - 1 \ge \lfloor n/6 \rfloor + 1$. The following simple construction establishes that the result is best possible: Let $H_{m,c}$ denote a graph on m vertices having no more than $\lfloor m^2/4 \rfloor$ edges and containing no cycle of length c, $c \le \lfloor (m+3)/2 \rfloor$. For each r, $3 \le r \le \lfloor n/6 \rfloor + 2$, we can choose a $H_{\lceil \frac{n+1}{3} \rceil, r}$ graph such that the graph

$$G = \overline{K}_{\lfloor \frac{2n-1}{3} \rfloor} \vee H_{\lceil \frac{n+1}{3} \rceil, r}$$

has $\lfloor n^2/4 \rfloor + 1$ edges and all cycles of length r have at least one vertex of degree $\lceil (n+1)/3 \rceil$. ∎

Remark 3. Continuing from Theorem 4, Caccetta and Vijayan [3] have established that every $G \in \mathcal{G}(n, \lfloor n^2/4 \rfloor + 1)$ contains a cycle C_r for each r,

$3 \le r \le f(\beta, n)$, each vertex of which has degree greater than $\beta n, \beta \le \frac{1}{3}$, where

$$f(\beta, n) \ge$$

$$\begin{cases} \dfrac{1}{2}(n+3), & \text{if } \beta \le \dfrac{1}{4} \\ \beta n + \dfrac{1}{2} + \sqrt{\left(n - \beta n - \dfrac{1}{2}\right)^2 - \left(\dfrac{n^2}{2} - n\right)}, & \text{if } \dfrac{1}{4} < \beta < \left(1 - \dfrac{1}{2n}\right) - \sqrt{\dfrac{1}{2} - \dfrac{1}{n}} \\ \dfrac{1}{2}(\beta n + 1) + \sqrt{(1-\beta)(1-3\beta)n^2 + 4}, & \text{otherwise}. \end{cases}$$

Moreover, this result is best possible.

The more general problem for the class $\mathcal{G}(n, \alpha n^2)$ has been considered and the results will form part of a forthcoming paper.

We conclude our discussion on cycles with the following problem:

Problem 1. Let $f(n, r)$ denote the largest integer such that every $G \in \mathcal{G}(n, \lfloor n^2/4 \rfloor + 1)$ contains an r-cycle, with the sum of the degrees of its vertices at least $f(n, r)$. Determine $f(n, r)$. Theorem 4 asserts that $f(n, r) > nr/3$ for $3 \le r \le \lfloor n/6 \rfloor + 2$. Erdös and Laskar [5] proved that

$$(1 + c)n < f(n, 3) < \left(\frac{3}{2} - c\right)n,$$

where c is a positive constant. This result has recently been improved by Fan [6], who proves that for every $G \in \mathcal{G}(n, m)$

$$f(n, 3) \ge \begin{cases} \dfrac{5m}{n}, & \text{if } \dfrac{n^2}{4} < m < n^2(10 - \sqrt{32})/17 \\ 2n + \dfrac{4}{n}(\sqrt{m(4m - n^2)} - m), & \text{if } \dfrac{n^2(10 - \sqrt{32})}{17} \le m < \dfrac{n^2}{3} \\ (3\Delta - 2n + 4m)/\Delta, & \text{otherwise}. \end{cases}$$

Determining $f(n, 3)$ exactly seems difficult.

We next turn our attention to the following problem. If H is a subgraph of $G \in \mathcal{G}(n, m)$, what can be said about $\delta(H)$? The following lemma is useful in our investigations.

Lemma 4. Let G be a graph on n vertices containing no subgraph H with $\delta(H) \ge a$.
Then

$$\epsilon(G) \le (n-1)(a-1) + \frac{1}{2}a(a-1). \tag{3.12}$$

Proof. Define a sequence of graphs $G_0, G_1, \ldots, G_{n-a}$ as follows: $G_0 = G$, and for $1 \leq i \leq n - a$,

$$G_i = G_{i-1} - x_{i-1},$$

where x_{i-1} is a vertex of G_{i-1} having minimum degree. Note that $d_{G_i}(x_i) \leq a - 1$ for $0 \leq i \leq n - a - 1$. Let $A = \{x_0, x_1, \ldots, x_{n-a-1}\}$. Since there are at most $(n - a)(a - 1)$ edges of G incident to the vertices of A, and G_{n-a} has a vertices and at most $(a/2)$ edges, inequality (3.12) follows immediately. ∎

Theorem 5. Let $G \in \mathcal{G}(n, \alpha n^2)$, $\alpha > 0$. Then G contains a subgraph H with

$$\delta(H) \geq \left\lceil \left(1 - \sqrt{\left(1 - \frac{1}{2n}\right)^2 - 2\alpha}\right) n - \frac{1}{2} \right\rceil$$

and this result is best possible.

Proof. Suppose G has no subgraph H with

$$\delta(H) \geq a .$$

Then, by Lemma 4,

$$\alpha n^2 \leq (n - a)(a - 1) + \frac{1}{2} a(a - 1) . \tag{3.13}$$

The right-hand side of (3.13) is a monotonically increasing function of a. Further, equality in (3.13) is possible only if

$$a = \left(1 \pm \sqrt{\left(1 - \frac{1}{2n}\right)^2 - 2\alpha}\right) n + \frac{1}{2} .$$

Hence inequality (3.13) holds only if

$$a \geq \left(1 - \sqrt{\left(1 - \frac{1}{2n}\right)^2 - 2\alpha}\right) n + \frac{1}{2} .$$

This establishes the existence of a subgraph H with the desired degree property.

That this result is best possible follows from the following construction: Let

$$t = \left\lceil \left(1 - \sqrt{\left(1 - \frac{1}{2n}\right)^2 - 2\alpha}\right) n - \frac{1}{2} \right\rceil .$$

The graph G_1 is obtained from the graph $\overline{K}_{n-t-1} \vee K_{t+1}$ by deleting one edge from each of the $n - t - 1$ vertices in $\overline{K}_{n-t-1}$. Clearly

$$\epsilon(G_1) = (n - t - 1)t + \frac{1}{2}t(t + 1).$$

Since

$$t \geq \left(1 - \sqrt{\left(1 - \frac{1}{2n}\right)^2 - 2\alpha}\right)n - \frac{1}{2},$$

we have

$$\epsilon(G_1) \geq \alpha n^2.$$

Further,

$$\frac{1}{2}t(t + 1) < \alpha n^2.$$

Hence we can construct a graph $G_2 \in \mathcal{G}(n, \alpha n^2)$ from G_1 by deleting appropriate edges so that G_2 contains a subgraph $H \cong K_{t+1}$ and every vertex of G_2 not in H has degree at most t. ∎

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