Cycles in graphs without proper subgraphs of minimum degree 3

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1. Introduction.

Let G(n,m) denote the set of graphs with n vertices and m edges. It is well-known that each $G \epsilon G(n, 2n-2)$ contains a subgraph of minimum degree 3 but there exists a $G \epsilon G(n, 2n-3)$ with no subgraphs of minimum degree 3 (see [1] p. xvii).

It was proved in [2] that each $G \epsilon G(n, 2n-1)$ contains a *proper* subgraph of minimum degree 3, but there exists $G \epsilon G(n, 2n-2)$ without this property. In fact, a stronger result was proved in [2], namely that $G \epsilon G(n, 2n-1)$ must contain a subgraph of minimum degree 3 with at most $n - c\sqrt{n}$ vertices for some $c > 0$. It was conjectured in [2] that each $G \epsilon G(n, 2n-1)$ contains a subgraph of minimum degree 3 with at most cn vertices for some absolute constant $c < 1$.

In this paper we study cycle lengths of graphs which have no proper subgraphs of minimum degree 3. For ease of reference, let $G^*(n, m)$ denote the set of graphs with n vertices, m edges and with the property that no proper subgraph has minimum degree 3. The results mentioned so far show that $G \epsilon G^*(n, m)$ implies $m \leq 2n-2$, and if $G \epsilon G^*(n, 2n-2)$ then G has miminum degree 3. Throughout the paper we investigate the cycle structure of graphs G, with $G \epsilon G^*(n, 2n-2)$. In fact we give the following conjecture.

CONJECTURE: If $G \epsilon G^*(n, 2n-2)$, then G contains all cycles of length at most k where k tends to infinity with n.

Our results are all related to this conjecture. We have several examples to demonstrate the role of $2n-2$ in this conjecture. For example for each n there exists graphs G, $G \epsilon G^*(n, 2n-3)$, such that G has no triangle (Examples 1 and 2). It is also true that there are $G \epsilon G^*(n, 2n-3)$ such that G has no cycles of length 5 or more (Example 3). For every r, we construct a graph $G \epsilon G^*(n, 2n - c(r))$ such that G has no cycles of length less than or equal to r (Theorem 4). In fact, the minimum value of $c(r)$ is determined precisely for $r = 3, 4$.

On one hand, our conjecture says that the graphs in $G^*(n, 2n-2)$ contain small cycles. We prove that these graphs contain C_3, C_4 and C_5 (Theorem 2.) On the other hand, our conjecture says that the graphs in $G^*(n, 2n-2)$ contain long cycles. Our main result is that $G \epsilon G^*(n, 2n-2)$ contains a cycle of length at least $\lfloor log n \rfloor$ (Theorem 5.). However,

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graphs in $G^*(n, 2n-2)$ does not always contain very long cycles (as large as $c\sqrt{n}$ for some $c > 0$, Example 7).

2. Properties of Graphs without proper subgraphs of minimum degree 3.

In this section we give a lemma and a theorem which we shall use frequently in sections 3 and 4. We first introduce some terminology.

Consider an ordering $x_1, x_2, \ldots, x_n$ of the vertex set of a graph. An edge $x_i x_j, i > j$ of the graph is called a *forward* edge on x_i and a *backward* edge on x_j. The forward (backward) degree of x_i is the number of forward (backward) edges incident to x_i. We shall let $d^+(x_i)$, $d^-(x_i)$ denote the forward and backward degree of x_i, respectively.

For any graph G we formally define an ordering of the vertices of G as follows: x_1 is a vertex of minimum degree in G. If $x_1, x_2, \ldots, x_t$ are already defined and $t < |V(G)|$, then let x_{t+1} be a vertex of minimum degree in $G - \{x_1, x_2, \ldots, x_t\}$. If G has no proper subgraph of minimum degree 3, then $d^+(x_i) \leqq 2$ for $2 \leqq i \leq |V(G)|$. Since we shall use this ordering often, we formulate this statement as lemma.

LEMMA 1. *Let G have n vertices and contain no proper subgraph of minimum degree 3. Then, the vertices of G can be ordered so that $d^+(x_1)$ is the minimum degree of G and $d^+(x_i) \leqq 2$ for $i \geqq 2$.*

THEOREM 1. *If $G \in G^*(n, 2n-2)$, then the vertices of G can be ordered so that $d^+(x_1) = 3$, $d^+(x_i) = 2$ for $2 \leqq i \leqq n-2$, and $d^+(x_{n-1}) = 1$. Moreover $d^-(x_i) \geqq 1$ for $2 \leqq i \leqq n$.*

PROOF: In the ordering of the vertices described in Lemma 1 observe that

$$2n - 2 = |E(G)| = \sum_{i=1}^{n-1} d^+(x_i) \leq d(x_1) + 2(n-3) + 1 \leq 2n - 2.$$

Since $d(x_1) \leq 3$ (otherwise G has at least $2n$ edges), $d^+(x_i) \leqq 2$ for $i = 2, 3, \ldots, (n-2)$ and $d^+(x_{n-1}) \leqq 1$, all the inequalities are equalities. Thus, $d^+(x_1) = 3$, $d^+(x_i) = 2$ for $2 \leqq i \leqq n-2$, and $d^+(x_{n-1}) = 1$. Since $d(x_i) \geqq d(x_1) = 3$ and $d^+(x_i) \leq 2$ for $1 < i \leqq n$, $d^-(x_i) \geqq 1$ follows. ■

COROLLARY 1. *If $G \in G^*(n, 2n-2)$ then G has minimum degree 3.*

3. Small Cycles in $G^*(n, 2n-2)$.

THEOREM 2. *If $G \in G^*(n, 2n-2)$ then for $n \geqq 5$, G contains a C_3 and a C_5. If $G \in G^*(n, 2n-3)$ and $n \geqq 6$, then G contains C_4.*

PROOF: For $G \in G^*(n, 2n-2)$ consider the ordering of vertices given in Theorem 1. Clearly,

x_{n-2}, x_{n-1} and x_n determine a C_3. Without loss of generality we may assume that x_{n-3} is adjacent to x_{n-1} and x_n.

Assume that i is the largest index such that x_i is adjacent to x_j for some j, $i < j < n-1$. There exists such an index since $i = 1$ is a suitable choice. If x_i is adjacent to x_{n-1} or to x_n, say to x_n, then select any $k > i$ such that $k \neq j, k \neq n, k \neq n-1$. This gives the C_5, $x_i x_n x_k x_{n-1} x_j x_i$ in G.

If x_i is not adjacent to either x_{n-1} or to x_n, then (since $d^+(x_i) = 2$) x_i is adjacent to some x_k, with $i < k, j \neq k, n \neq k, n-1 \neq k$. But then $x_i x_k x_{n-1} x_n x_j x_i$ is a C_5 in G.

To see that $G \epsilon G^*(n, 2n-3)$ contains a C_4, observe that Theorem 1 almost holds in that we can order the vertices of G as $x_1, x_2, \ldots, x_n$ so that at most one of the equalities $d^+(x_i) = 2$ for $2 \leqq i \leqq n-2$, $d^+(x_1) = 3$, and $d^+(x_{n-1}) = 1$ fails to hold. Moreover if equality does not hold for some i then $d^+(x_i)$ is just one less than the value shown above. If each of the equalities $d^+(x_{n-1}) = 1, d^+(x_{n-2}) = d^+(x_{n-3}) = 2$ hold then the subgraph of G induced by $X = \{x_{n-3}, x_{n-2}, x_{n-1}, x_n\}$ has five edges and there is a C_4 in G. Therefore, we assume that there is no C_4 in the subgraph induced X. Also, by a suitable permutation of the vertices in X, we may assume that $x_{n-3}x_{n-2}, x_{n-3}x_{n-1}, x_{n-3}x_n$ and $x_{n-2}x_{n-1}$ are edges in X. But $d^+(x_{n-4}) = 2$ and the only way to avoid a C_4 in G is to assume x_{n-4} to be adjacent to x_{n-3} and to x_n. Since $n \geqq 6$, x_{n-5} exists and $d^+(x_{n-5}) \geqq 2$. Thus, there exists a C_4 in G containing x_{n-5} and three vertices of $\{x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}\}$.■

With more work it is possible to show that $G \epsilon G^*(2n-2)$ always contains C_6 for $n \geqq 6$. The following constructions show that Theorem 2 is sharp.

EXAMPLE 1: Let $n \geq 6$ be even. Consider the graph on n vertices defined as follows. Let $x_1 x_2 \ldots x_{n-2}$ be a cycle of length $n-2$. Let y and w be two new vertices with y adjacent to all x_i of even index and w adjacent to all x_i of odd index. Finally place an edge between y and w. The graph obtained contains no triangles, (in fact, is bipartite) has no proper subgraph of minimum degree 3, and has $2n-3$ edges. ■

EXAMPLE 2: Let $n = 2k+1 \geqq 9$ and consider a cycle of length k with vertices $x_1 x_2 \ldots x_k$. For $i = 1, 2, \ldots, k-1$ place new vertices y_i in the graph with each y_i adjacent to x_i. Finally, let v and w be two additional vertices of the graph such that each are adjacent to $y_1, y_2, \ldots, y_{k-1}$ and x_k. The resulting graph has $2n-3$ edges, no triangle and no proper subgraph of minimum degree 3. ■

EXAMPLE 3: Consider the graph obtained from $K_{2,n-2}$ by placing an edge between the two vertices of the two-vertex color class. This graph has no cycles of length 5 or more, has $2n-3$ vertices, and contains no proper subgraph of minimum degree 3. ■

EXAMPLE 4: Assume that $n-2$ is divisible by 4, $n \geqq 10$, and consider a cycle of length $n-2$ with vertices $x_1, x_2, x_3, \ldots, x_{n-2}$. Let y and w be two new vertices. Join vertex y to

x_i for $i \equiv 1$ or $i \equiv 2 (\mathrm{mod}\ 4)$ and join w to x_i for $i \equiv 0$ or $i \equiv 3$ (mod 4). This graph has no C_4, has $2n-4$ vertices, and has no proper subgraphs of minimum degree 3. It is easy to modify this example for $n \equiv 0,1,3$ (mod 4). ■

Based on these examples, we conclude that Theorem 2 is sharp: there exists $G \in G^*(n, 2n-3)$ without C_3 (Example 1 and 2); there exists $G \in G^*(n, 2n-3)$ without C_5 (Example 1 and 3); there exist $G \in G^*(n, 2n-4)$ without C_4 (Example 4).

Up to now we've only considered the existence of C_k (for k = 3,4,5) in $G \in G^*(n, 2n-2)$. We continue by looking for the minimum m that $G \in G^*(n,m)$ contains a cycle of length less than r. Theorem 2 and Examples 1 and 2 show that $m = 2n-2$ when $r = 4$. The upper bound for m in cases r = 5 and r = 6 are given in the next result.

THEOREM 3. *Let $g(G)$ denote the girth of G. If $n \geqq 6$ and $G \in G^*(n, 2n-4)$, then $g(G) \leqq 4$. If $n \geqq 8$ and $G \in G^*(n, 2n-6)$ then $g(G) \leqq 5$.*

PROOF: Assume $G \in G^*(n, 2n-4)$ and apply Lemma 1. Clearly $d^+(x_1) \leqq 3$, otherwise G has at least $2n$ edges. If $n \geqq 6$ the subgraph H induced by $x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}$ in G has at least $(2n-4) - 3 - 2(n-6) = 5$ edges. We may assume that H is a cycle of length 5, otherwise H contains C_3 or C_4 and $g(G) \leqq 4$ follows. Therefore $d^+(x_1) = 3, d^+(x_i) = 2$ for $i = 2, 3, \ldots, n-5$. But x_{n-5} is adjacent to two vertices of the five-cycle H giving a C_3 or C_4.

To prove the second part of the Theorem, assume $G \in G^*(n, 2n-6)$ and apply Lemma 1. Again, $d^+(x_1) \leqq 3$. Since $n \geqq 8$, we consider the subgraph H induced by $\{x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6}\}$ in G. Thus, H contains at least $(2n-6) - 3 - 2(n-8) = 7$ edges. Let C be a cycle of H with minimum length, so that C is a cycle without a diagonal. If $|C| = 7$ then $H = C$ and $d^+(x_1) = 3, d^+(x_i) = 2$ for $i = 2, 3, ..., n-7$. In particular, x_{n-7} is adjacent to at least two verices of C giving a cycle of length at most 5. If $|C| = 6$, then without loss of generality assume $x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}, x_n$ is a 6 - cycle and x_{n-6} is adjacent to x_{n-5}. If x_{n-6} is adjacent to any vertex x_i for $n-4 \leqq i \leqq n$ then we have a C_3, C_4 or C_5. Therefore, H has 7 edges and again $d^+(x_1) = 3, d^+(x_i) = 2$ for $2 \leq i \leqq n-7$. In particular $d^+(x_{n-7}) = 2$, and it is easy to check that the only case when $x_{n-7}, x_{n-6}, \ldots, x_n$ does not induce a cycle of length at most 5 in G occurs if x_{n-7} is adjacent to x_{n-6} and x_{n-2} (see Figure 4). It is easy to see that $d^+(x_{n-8}) = 2$ implies the existence of a cycle of length at most 5. Thus $|C| \leqq 5$ completing proof of the theorem. ■

To show that the first part of Theorem 3 is best possible we give the following example.

EXAMPLE 5: Assume n is divisible by 5 and $n \geqq 10$. Let $x_1 x_3 x_5 x_2 x_4 x_1$ be a five-cycle and $y_1 y_2 \cdots y_{n-5}\ y_1$ is a $n-5$ cycle. Vertex x_i is adjacent to y_j if and only if $j \equiv i$ (mod 5) (for all $i, 1 \leqq i \leqq 5$). This graph has $2n-5$ edges, has no proper subgraph of minimum degree 3 and contains no C_3 or C_4.

We do not know examples of $G \epsilon G^*(n, 2n-7)$ with $g(G) \geqq 6$ for infinitely many n. However, it is possible to find $G \epsilon G^*(n, 2n-8)$ with $g(G) = 6$ for infinitely many n.

The next theorem shows that graphs in $G^*(n, 2n-c)$ do not always contain small cycles.

THEOREM 4. *For every postive integer r there exists $c = c(r)$ and a graph $G \epsilon G^*(n, 2n - c(r))$ such that $g(G) > r$.*

PROOF: Let k be a natural number and let $C_1, C_2, \ldots, C_k$ be vertex disjoint cycles of length $t = 2 \cdot 5^{r+1} - 1$. We shall define the graph G_k by adding edges to the graph $C_1 \cup C_2 \cup \cdots \cup C_k$. Assume that the vertices of C_i are $x_1^i, x_2^i, \ldots, x_t^i$ (indexed in the natural order of the cycle). The definition of G_k is recursive. Set $G_1 = C_1$. If $G_1, G_2, \ldots, G_{k-1}$ are already defined we shall define G_k by adding edges xy to $G_{k-1} \cup C_k$ such that $x \epsilon C_k, y \epsilon C_{k-1}$. The definition will preserve the following properties (for each i, $1 \leqq i \leqq k$):

(i) each cycle of G_i is longer than r

(ii) the maximum degree of G_i is at most 5, and

(iii) $d_{G_i}(x_1^i) = 4$, $d_{G_i}(x_t^i) = 2$, $d_{G_i}(x_j^i) = 3$ for $2 \leqq j \leqq t-1$ and $i \geqq 2$.

Note that properties (i), (ii) and (iii) trivally hold for $i = 1$, since $G_1 = C_1$.

To define G_k we add edges $e_o = x_1^k y_o$, $e_1 = x_1^k y_1$, $e_2 = x_2^k y_2$, $e_3 = x_3^k y_3, \ldots, e_{t-1} = x_{t-1}^k y_{t-1}$ to $G_{k-1} \cup C_k$, such that $y_j \epsilon V(C_{k-1})$ for $j = 0, 1, \ldots, t-1$ and G_k satisfies properties (i), (ii), and (iii) for $i = k$. Observe that (iii) holds independent of the choice of each y_j, so that we need only select each y_j such that (i) and (ii) hold. The edge e_o can be defined arbitrarily. Assume that $e_o, e_1, \ldots, e_s$ are defined for $0 \leqq s < t-1$ in such a way that properties (i) and (ii) hold for $G' = G_{k-1} \cup C_k \cup \{e_o, e_1, \ldots, e_s\}$. We define e_{s+1} as follows. Let W denote the set of vertices in C_{k-1} which can be reached by a path of length at most r from x_{s+1}^k in the graph G'. Since (ii) holds for G', $|W| < 5^{r+1}$ and therefore $|V(C_{k-1}) - W| > t - 5^{r+1} = 5^{r+1} - 1$. Let T be a subset of $V(C_{k-1}) - W$ such that $|T| = 5^{r+1}$. By definition, for any $y \epsilon T$ the graph $G' \cup e_{s+1}$ statisfies (i) with $e_{s+1} = x_{s+1}^k y$. Using property (iii) for $i = k-1$

$$\sum_{y \epsilon T} d_{G_{k-1}}(y) \leqq 3|T| + 1 = 3.5^{r+1} + 1.$$

Since G' is obtained from G_{k-1} by adding $s+1$ edges,

$$\sum_{y \epsilon T} d_{G'}(y) \leqq \sum_{y \epsilon T} d_{G_{k-1}}(y) + s + 1 \leq 3 \cdot 5^{r+1} + 1 + t - 1 = 5^{r+2} - 1.$$

Thus there exists an $y_{s+1} \epsilon T$ with $d_G{}'(y_{s+1}) < 5$. Thus with $e_{s+1} = x_{s+1}^k y_{s+1}$, the graph

$G'' = G' \cup e_{s+1}$ satisfies properties (i), (ii) and (iii). Therefore G_k is defined.

It is clear that $|V(G_k)| = kt$ and $|E(G_k)| = 2kt - t$. The proof is completed by showing that G_k has no proper subgraph of minimum degree 3. Assume to the contrary that G^* is such a proper subgraph. Since $d_{G_k}(x_t^k) = 2$, $x_t^k \notin V(G^*)$. However, $d_{G_k - x_t^k}(x_{t-1}^k) = 2$ implies $x_{t-1}^k \notin V(G^*)$. Repeating this argument we get that $x_j^k \notin V(G^*)$ for $1 \leqq j \leqq t$. But $d_{G_k} - C_k\ (x^{k-1}) = 2$ and by observations just like those made above, none of the vertices of C_{k-1} belong to G^*. Continuing in this way we see that G^* is the empty graph, a contradiction. Hence $G_k \epsilon G^*(tk, 2tk - t)$ for all k with $t = 2 \cdot 5^{r+1} - 1$, showing that $c(r) = 2 \cdot 5^{r+1} - 1$ is a suitable choice. ■

4. Long cycles in $G^*(n, 2n-2)$.

In this section we prove one of the main results of the paper, that is $G \epsilon G^*(n, 2n-2)$ contains a long cycle. Note that $G \epsilon G^*(n, 2n-3)$ does not necessarily contain even a path of length 4 (see Example 3 in Section 2).

THEOREM 5: If $G \epsilon G^*(n, 2n-2)$, then G contains a cycle of length at least $\lfloor log n \rfloor$.

PROOF: Consider the ordering of G of Theorem 1. Since $d^-(x_i) > 0$ for $i = 2, \ldots, n$, we can find a spanning tree T recursively in G as follows. Place x_1 in T. If $x_1, x_2, \ldots, x_t$ are in T and $t < n$, then choose any edge $x_i x_{t+1}$ of G such that $1 \leqq i \leqq t$. Redefine T by adding vertex x_{t+1} and the edge $x_i x_{t+1}$ to the old T. By definition of the tree, $d^-(x_i) = 1$ in T for $2 \leqq i \leqq n$ and $d^+(x_i) \leqq 2$ in T for $2 \leqq i \leqq n$. Since $d^+(x_1) = 3$ in G, T is a tree of maximum degree $\leqq 3$. Therefore, T contains a path P of length at least $\lfloor log n \rfloor$ starting with x_i.

Let $x_1 = x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ denote the vertices of P in the natural order defined by P i.e $x_{i_j} x_{i_{j+1}}$ is an edge of P for $1 \leqq j \leq k-1$. Notice that $i_1 < i_2 < \cdots < i_k$ follows from the definition of T since $d^-(x_i) < 2$ in T for $2 \leqq i \leqq n$. We call a path $P = (x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ in G a *forward path* if $i_1 < i_2, \cdots < i_k$. Note that the definition depends on the order $x_1, x_2, \ldots x_n$ defined by Theorem 1. The discussion up to this point insures that G has a forward path of length at least $\lfloor log n \rfloor$.

Let $P = (x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ be a forward path of G with a maximum length. Since $d^-(x_i) \geqq 1$, and $d^+(x_i) \geqq 1$ in G for $2 \leqq i \leqq n-1$, it follows that $i_1 = 1, i_k = n$. Let t be any positive integer such that $1 \leqq t < k$ and $i_t \neq n-1$. Since $d^+(x_i) \geqq 2$ in G for $1 \leq i \leqq n-2$, we can find a forward path P_t in G starting at x_{i_t} and ending at some vertex $x_{i_{t'}}$ of P such that

(a) P_t and P are edge disjoint

(b) $V(P_t) \cap V(P) = \{x_{i_t}, x_{i_{t'}}\}$

Note that (a) implies $t+1 < t'$. We claim for $t < s$ that the paths P_t and P_s are vertex disjoint if $t' < s$ or if $s = t'-1$. The case $t' < s$ is obvious since both P_t and P_s are forward paths. Assume $s = t' - 1$. But $t + 1 < t'$ and $s + 1 < s'$ implies that P_t and P_s do not have common endpoints. Assume that x is the last common vertex of P_s and P_t we find on P_s by traveling along P_s from its starting point x_{i_s}. It is easy to see that the following edge sequence is a forward path: starting from $x_{i_1} = x_1$, travel along P to x_{i_s} ; continue on P_s to x; from x travel to $x_{i_{t'}} = x_{i_{s+1}}$ along P_t: finally from $x_{i_{t'}}$ to $x_{i_k} = x_n$ travel along P. This path is longer that P. This contradiction proves the claim.

Choose a subset $Q_1, Q_2, \dots Q_r$ of the paths $P_1, P_2, \dots, P_t$ as follows. Set $Q_1 = P_1$. If $Q_1, Q_2, \dots, Q_s$ are defined and the endpoint $x_{i_{s'}}$ of Q_s is not x_n then $Q_{s+1} = P_{s'-1}$. If the endpoint $x_{i_{s'}}$ of Q_s is x_n then set $r = s$.

It is now easy to construct a cycle using all the vertices of $P \cup Q_1 \cup \dots \cup Q_r$. But P has at least $\lfloor log n \rfloor$ vertices so that the cycle C has length at least $\lfloor log n \rfloor$.■

Finally, to see that $G \epsilon G^*(n, 2n-2)$ does not contain necessarily a very long cycle, (larger that $c\sqrt{n}$) consider the following example.

EXAMPLE 6: Let k be an integer, $k \geqq 4$. Let C be a k -cycle with vertices $x_1, x_2, \dots, x_k$. Select a new vertex w and connect w to each x_i with vertex-disjoint paths of length $k-1$. (The only common vertex of these paths is w). Select another new vertex y and let y be adjacent to all vertices except those of $\{x_2, x_3, \dots, x_k\}$. Let G_k be the graph just defined.

The graph G_k has $k(k-1)+2 = n$ vertices. Since $d(w) = k+1$, $d(y) = n-k$, $d(x_1) = 4$ and all the other vertices are of degree 3, G_k has

$$\frac{k + 1 + n - k + 4 + (n-3)3}{2} = 2n - 2$$

edges. It is easy to check that G_k has no proper subgraph of minimum degree 3. It is also easy to see that the longest path of $G_k - y$ is smaller that $5k$. Therefore the longest path of G_k is smaller that $10k \leqq 10\sqrt{n+1}$.

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