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On the representing number of intersecting families

By

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1. Introduction. One of the best-known results in extremal set theory is the Theorem of Erdös-Ko-Rado [3]:

Suppose $n \geqq 2k$, and let $\mathfrak{M}$ be a family of k-subsets of an n-set M such that any two members of $\mathfrak{M}$ intersect non-trivially, then $|\mathfrak{M}| \leqq \binom{n-1}{k-1}$. Furthermore, the bound can be attained, and the extremal families are precisely the families $\mathfrak{M}_a = \{X \ni a: a \in M\}$ for $k \geqq 3$. Many proofs of this result have been given, in addition to the original proof see e.g. [4, 9, 10]. Since all the members of an extremal familiy $\mathfrak{M}$ have an element in common, we say that $\mathfrak{M}$ has *representing number* 1.

What if we do not allow the sets of $\mathfrak{M}$ to have an overall nontrivial intersection? How large can then $\mathfrak{M}$ be? The answer to this question has been given by Hilton-Milner [8] with a further proof appearing e.g. in [6]: *Let $\mathfrak{M}$ be an intersecting family of k-subsets of an n-set M such that $\bigcap_{X \in \mathfrak{M}} X = \emptyset$, then $|\mathfrak{M}| \leqq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ for $n > 2k$.* Again the extremal families are characterized. Since the members of $\mathfrak{M}$ are allowed to contain one of two points, but not a single one we say that $\mathfrak{M}$ has representing number 2.

In this paper we estimate the cardinality of an intersecting family with an arbitrary representing number r, $1 \leqq r \leqq k$. We first give the relevant definitions. All sets will be assumed to be finite. The collection of all k-subsets of a set M will be denoted by $\binom{M}{k}$. We say that a family $\mathfrak{M}$ is *intersecting* if any two members of $\mathfrak{M}$ have a non-trivial intersection.

Definition. Let $\mathfrak{M}$ be a family of sets, and R a single set. R is said to *represent* $\mathfrak{M}$ or be a *representing set* for $\mathfrak{M}$ if $R \cap X \neq \emptyset$ for all $X \in \mathfrak{M}$. $\mathfrak{M}$ has *representing number* r if r is the cardinality of a smallest set representing $\mathfrak{M}$.

Since an intersecting family $\mathfrak{M}$ is represented by every one of its members we note that the representing number r of such a family satisfies $r \leqq \min(|X|: X \in \mathfrak{M})$. In particular, if $\mathfrak{M} \subseteq \binom{M}{k}$ then $1 \leqq r \leqq k$.

Theorem. *Let n, r, k be natural numbers with $1 \leqq r \leqq k \leqq n$. Denote by $g(n; r, k)$ the maximal cardinality of an intersecting family $\mathfrak{M} \subseteq \binom{M}{k}$ of an n-set M with representing number r. Then there are constants $c_{r,k}$, $C_{r,k}$ only depending on r and k, such that*

$$c_{r,k} n^{k-r} \leqq g(n; r, k) \leqq C_{r,k} n^{k-r}.$$

Sections 2 and 3 are devoted to a proof of this result with a few additional comments appearing in Section 4.

2. Proof of the upper bound. This section establishes the existence of the constant $C_{r,k}$ as spelled out in the statement of the theorem. We divide the proof into a series of lemmas. First we need a definition.

Definition. Let $\mathfrak{A}$ be a family of sets and let $u \in \mathbb{N}$, $u > 1$. A *$\Delta(u)$-system* of $\mathfrak{A}$ is a subfamily $\mathfrak{B} \subseteq \mathfrak{A}$ such that

(i) $|\mathfrak{B}| = u$,
(ii) any two members of $\mathfrak{B}$ have the same intersection C. C is called the *stem* of $\mathfrak{B}$.

The following lemma appeared in [2]. The easy proof goes by induction on a.

Lemma 1. *Let $a, b \in \mathbb{N}$, $b > 1$. Then there exists a smallest number $f(a, b) \in \mathbb{N}$ such that any family of sets $\mathfrak{A}$ with $|\mathfrak{A}| > f(a, b)$ and $(X \in \mathfrak{A} \Rightarrow |X| \leqq a)$ possesses a $\Delta(b)$-system. Furthermore, $f(a, b) \leqq a!\,(b-1)^a$.*

Lemma 2. *Let $\mathfrak{A}$ be a family of sets with $X \in \mathfrak{A} \Rightarrow |X| \leqq k$. Let, further, $\mathfrak{B}$ be a family of sets such that every $X \in \mathfrak{B}$ is a representing set of $\mathfrak{A}$ and satisfies $|X| \leqq b$. If $|\mathfrak{B}| > f(b, k+1)$, then there exists a representing set Y of $\mathfrak{A}$ with $|Y| \leqq b - 1$ and $Y \subseteq Z$ for some $Z \in \mathfrak{B}$.*

Proof. Let $\{Y_1, \ldots, Y_{k+1}\}$ be a $\Delta(k+1)$-system of $\mathfrak{B}$ with $|Y_i| \leqq b$ for all i and stem Y (guaranteed by Lemma 1). Then $|Y| \leqq b - 1$, $Y \subseteq Y_i \in \mathfrak{B}$. We claim that Y represents $\mathfrak{A}$. If, on the contrary, there existed $X \in \mathfrak{A}$ with $X \cap Y = \emptyset$ then X would have to intersect all the disjoint set $Y_1 - Y, Y_2 - Y, \ldots, Y_{k+1} - Y$, in contradiction to $|X| \leqq k$. □

To facilitate the induction used in the proof of the theorem we introduce the following function.

Definition. Let $n, r, k \in \mathbb{N}$. For $\ell \in \mathbb{N}$, $\ell \leqq k$ define the functions $h_\ell: \mathbb{Q} \to \mathbb{Q}$

$$h_k(x) = x$$

$$h_\ell(x) = \frac{1}{\binom{n-r}{k-r}}(x - f(k, k+1)) - \sum_{i=\ell+1}^{k-1} f(i, k+1) \quad \text{for} \quad \ell < k.$$

The following facts are immediately verified from the definition.

Lemma 3. i) $h_{\ell+1}\left(x - h_\ell(x)\binom{n-r}{k-r}\right) = f(\ell+1, k+1)$ *for all* x,

ii) *if* $x > \binom{n-r}{k-r}\sum_{i=r}^{k-1} f(i, k+1) + f(k, k+1)$ *then* $h_{r-1}(x) > 0$.

We come to the crux of the proof.

Lemma 4. *Let* n, k, r *and* $M, \mathfrak{M}$ *be given as in the statement of the theorem. For a subfamily* $\mathfrak{M}' \subseteq \mathfrak{M}$ *and* $\ell \leqq k$ *let*

$$\mathfrak{M}'_\ell = \{X \subseteq M : X \text{ represents } \mathfrak{M},\ |X| \leqq \ell \text{ and there exists } Y \in \mathfrak{M}' \text{ with } X \subseteq Y\}.$$

Then $|\mathfrak{M}'_\ell| \geqq h_\ell(|\mathfrak{M}'|)$.

Proof. We use downward induction on ℓ. For $\ell = k$ we have $\mathfrak{M}'_k \supseteq \mathfrak{M}'$ and thus $|\mathfrak{M}'_k| \geqq h_k(|\mathfrak{M}'|) = |\mathfrak{M}'|$. Suppose we already know that $|\mathfrak{M}'_{\ell+1}| \geqq h_{\ell+1}(|\mathfrak{M}'|)$ holds for all subfamilies $\mathfrak{M}' \subseteq \mathfrak{M}$. We determine step by step distinct sets $X_1, X_2, \ldots, X_\alpha \in \mathfrak{M}'_\ell$ with $\alpha = \max(0, \lceil h_\ell(|\mathfrak{M}'|)\rceil)$. Let $\alpha > 0$ and $1 \leqq \beta \leqq \alpha$. Suppose we have already found sets $X_1, X_2, \ldots, X_{\beta-1} \in \mathfrak{M}'_\ell$. Set

$$\mathfrak{M}'' = \{X \in \mathfrak{M}' : X \supseteq X_i \text{ for some } i, 1 \leqq i \leqq \beta - 1\}$$

$$\tilde{\mathfrak{M}} = \mathfrak{M}' - \mathfrak{M}''.$$

Then $\tilde{\mathfrak{M}} \subseteq \mathfrak{M}$ and hence $|\tilde{\mathfrak{M}}_{\ell+1}| \geqq h_{\ell+1}(|\tilde{\mathfrak{M}}|)$ by the induction hypothesis. As every X_i represents $\mathfrak{M}$ we have $|X_i| \geqq r$ by the assumption on $\mathfrak{M}$, and thus

$$|\{X \subseteq M : X \supseteq X_i\}| \leqq \binom{n-r}{k-r} \quad (i = 1, \ldots, \beta - 1).$$

From this we infer

$$\begin{aligned}
|\tilde{\mathfrak{M}}| &= |\mathfrak{M}'| - |\mathfrak{M}''| \\
&\geqq |\mathfrak{M}'| - (\beta - 1)\binom{n-r}{k-r} \\
&\geqq |\mathfrak{M}'| - (\alpha - 1)\binom{n-r}{k-r} \\
&> |\mathfrak{M}'| - h_\ell(|\mathfrak{M}'|)\binom{n-r}{k-r}.
\end{aligned}$$

Since $h_{\ell+1}$ is strictly increasing we conclude from Lemma 3 (i)

$$|\tilde{\mathfrak{M}}_{\ell+1}| \geqq h_{\ell+1}(|\tilde{\mathfrak{M}}|) > f(\ell+1, k+1).$$

Now Lemma 2 applied to $\mathfrak{A} = \mathfrak{M}$, $\mathfrak{B} = \tilde{\mathfrak{M}}_{\ell+1}$ implies the existence of a set X_β with $|X_\beta| \leqq \ell$ representing $\mathfrak{M}$ and of $Y \in \tilde{\mathfrak{M}}_{\ell+1}$ with $X_\beta \subseteq Y$. Y is, in turn, contained in a set $Z \in \tilde{\mathfrak{M}}$, $Y \subseteq Z$, by the definition of $\tilde{\mathfrak{M}}_{\ell+1}$. In summary, $X_\beta \subseteq Z \in \tilde{\mathfrak{M}} \subseteq \mathfrak{M}'$. Hence $X_\beta \in \mathfrak{M}'_\ell$ and X_β must be distinct from all sets $X_1, \ldots, X_{\beta-1}$ since $X_\beta = X_i$ would imply $Z \in \mathfrak{M}'' = \mathfrak{M}' - \tilde{\mathfrak{M}}$, whereas $Z \in \tilde{\mathfrak{M}}$. □

Proof of the upper bound. Suppose, on the contrary, there is no such constant $C_{r,k}$. Then there are n, M and a family $\mathfrak{M}$ satisfying the assumptions of the theorem with

$$(*)\qquad |\mathfrak{M}| > \binom{n-r}{k-r}\sum_{i=r}^{k-1} f(i, k+1) + f(k, k+1).$$

Applying Lemma 4 with $\mathfrak{M}' = \mathfrak{M}$ and $\ell = r-1$, we conclude $|\mathfrak{M}_{r-1}| \geqq h_{r-1}(|\mathfrak{M}|)$ and thus $|\mathfrak{M}_{r-1}| > 0$ by Lemma 3 (ii). But this contradicts the fact that $\mathfrak{M}$ cannot be represented by a set of cardinality less than r, and the proof is complete. □

From the inequality (*) and Lemma 1 we obtain the following estimate of $C_{r,k}$.

Corollary. *For given n, r, k and M, $\mathfrak{M}$ as in the statement of the theorem we have*

$$|\mathfrak{M}| \leqq \left(\sum_{i=r}^{k} i!\, k^i\right) n^{k-r}.$$

3. Proof of the lower bound. Let r and k be given. The Erdös-Ko-Rado Theorem states $g(n; 1, k) = \binom{n-1}{k-1}$ for $n \geqq 2k$, hence $c_{1,k}$ exists. For $r > 1$ we use a generalization of the construction in [1] which includes the optimal family of the Hilton-Milner Theorem [8] for $r = 2$ and the one given by Frankl [5] for $r = 3$ as special cases.

Assume $n \geqq k + (k-1) + \cdots + (k-r+2) + 1$. Choose pairwise disjoint sets S_i $(i = 0, \ldots, r-2)$ with $|S_i| = k - i$, a subset $T \subseteq S_0$ with $|T| = r-1$ and an element $x \notin \bigcup S_i$. Denote by $\mathfrak{M}_i$ the family

$$\mathfrak{M}_i = \{X : X \supseteqq S_i, |X \cap S_j| = 1 \text{ for } 1 \leqq j < i, |X \cap T| = 1\}$$
$$(i = 1, \ldots, r-2),$$

and by $\mathfrak{M}_x$ the family

$$\mathfrak{M}_x = \{X : |X| = k, x \in X, X \cap S_i \neq \emptyset \text{ for all } i\} \cup \{X : |X| = k, x \cup T \subseteqq X\}.$$

The family $\mathfrak{M} = \bigcup_{i=1}^{r-2} \mathfrak{M}_i \cup \mathfrak{M}_x \cup \{S_0\}$ is intersecting, has $T \cup x$ as representing set, and it is readily seen that no smaller set can represent $\mathfrak{M}$. Since the second part of $\mathfrak{M}_x$ contains already $\binom{n-r}{k-r}$ sets, the existence of $c_{r,k}$ is established.

4. Families with representing number k. As mentioned before, the precise value of $g(n; 1, k)$ and $g(n; 2, k)$ is known whereas the family $\mathfrak{M}$ of the previous section was shown to be optimal in [5] for $r = 3$ and $n \geqq n_0(k)$. Let us go to the other end and consider $g(n; k, k)$.

The theorem says in this case that $g(n; k, k)$ is independent of n for $n \geqq n_0(k)$, so we denote it shortly by $g(k)$.

The corollary in Sect. 2 gives $g(k) \leqq k!\, k^k$, and it was shown in [1] that, in fact, $g(k) \leqq k^k$. To gain further insight into $g(k)$ we observe that any maximal family

$\mathfrak{M} \subseteq \binom{M}{k}$ with representing number k must include all representing sets of $\mathfrak{M}$ of size k. This, in turn, immediately yields the following alternate characterization.

Proposition. *Let* $\mathfrak{M} \subseteq \binom{M}{k}$ *be an intersecting family. Then the following conditions are equivalent:*

i) $\mathfrak{M}$ *is maximal with representing number* k.
ii) $\mathfrak{M}$ *is maximal with respect to the condition that to every* $X \in \mathfrak{M}$, $x \in X$ *there exists* $Y \in \mathfrak{M}$ *with* $X \cap Y = \{x\}$.

The construction of Erdös and Lovász in [1] yields $g(k) \geqq k! \sum_{i=1}^{k} \frac{1}{i!}$, and thus $g(k) \geqq (e-1)\,k!$ for $k \to \infty$. For small k, we have $g(1) = 1$, $g(2) = 3$. Using the preceding proposition it can be easily shown that $g(3) = 10$ and, with a little more work, $g(4) = 41$ which was also found in [7]. Hence for these values, the construction in [1] is optimal, and it is quite plausible that optimality always holds.

Two interesting questions come to mind: First, improve the bounds on $g(k)$, and, secondly, estimate the threshold value $n_0(k)$.

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