MANY HEADS IN A SHORT BLOCK

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ABSTRACT. Let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s with $\mathbf{P}(X_i = +1) = \mathbf{P}(X_i = -1) = 1/2$. Further let $S_0 = 0, S_n = X_1 + X_2 + \ldots + X_n (n = 1, 2, \ldots)$ and $I(N, K) = \max_{0 \le n \le N-K}(S_{n+K} - S_n)$ $(K = 1, 2, \ldots, N; N = 1, 2, \ldots)$. Consider a sequence $\{K_N\}$ of positive integers and investigate the properties of the maximal increments $I(N, K_N)$. This problem was studied by many authors in case of different $\{K_N\}$'s. In the present paper we intend to summarize the results and prove a few new theorems. We are especially interested in the case $K_N = \log N + o(\log N)$. In section 1 we introduce a few notations and concepts and recall the known results in the case $K_N \le c \log N$. In section 2 a key-inequality will be proved. The main results are presented in section 3. Section 4 gives a survey of the case $\log N \ll K_N \le N$.

1. NOTIONS AND A FEW KNOWN RESULTS

In order to present the results of our paper in a pleasant form, it is worthwile to recall some definitions, see, e.g. Révész (1980,1982).

Let $\varsigma = \{Z_n\}$ be a sequence of r.v.'s. Then we formulate:

DEFINITION 1. The sequence $f_1(n)$ $(n = 1, 2, \ldots)$ belongs to the upper-upper class of ς $(f_1 \in \mathcal{UUC}(\varsigma))$ if $Z_n \le f_1(n)$ a.s. for all but finitely many n.

DEFINITION 2. The sequence $f_2(n)$ $(n = 1, 2, \ldots)$ belongs to the upper-lower class of ς $(f_2 \in \mathcal{ULC}(\varsigma))$ if $Z_n > f_2(n)$ a.s. i.o.

DEFINITION 3. The sequence $f_3(n)$ $(n = 1, 2, \ldots)$ belongs to the lower-upper class of ς $(f_3 \in \mathcal{LUC}(\varsigma))$ if $Z_n < f_3(n)$ a.s. i.o.

DEFINITION 4. The sequence $f_4(n)$ $(n = 1, 2, \ldots)$ belongs to the lower-lower class of ς $(f_4 \in \mathcal{LLC}(\varsigma))$ if $Z_n \ge f_4(n)$ a.s. for all but finitely many n.

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DEFINITION 5. If there exists a deterministic sequence $f(n)$ such that $\lim_{n\to\infty}(Z_n - f(n)) = 0$ then we say that ς is asymptotically deterministic (AD).

DEFINITION 6. If there exist $f_1(n) \in \mathcal{UUC}(\varsigma)$, $f_2(n) \in \mathcal{LLC}(\varsigma)$, and $K > 0$ such that $f_1 - f_2 \leq K$ then we say that ς is quasi asymptotically deterministic (QAD).

Utilizing these concepts we present some known results.

(i) (Erdős-Révész 1975) Let $\epsilon > 0$ and

$$K_N \leq [\log N - \log\log\log N + \log\log e - 2 - \epsilon] = \beta_1(N, \epsilon)$$

Then $\{I(N, K_N)\}$ is AD and $I(N, K_N) = K_N$ if $N \geq N_0 = N_0(\omega, \epsilon)$.

(Here and in what follows, log means logarithm with base 2; $[x]$ is the integral part of x.) This clearly means that with probability 1 for all N big enough the sequence $X_1, X_2, \ldots, X_N$ contains a run of length $\beta_1(N, \epsilon)$. A careful investigation of the number of such runs can be found in Deheuvels (1985).

(ii) (Erdős-Révész 1975) Let

$$\beta_1(N, \epsilon) < K_N \leq [\log N + \log\log N - \log\log\log N + \log\log e - 2 - \epsilon] = \beta_2(N, \epsilon).$$

Then $I(N, K_N)$ is QAD and $I(N, K_N) = K_N$ or $K_N - 2$ if N is big enough.

(iii) (Erdős-Révész 1975) Let

$$\beta_2(N, \epsilon) < K_N \leq [\log N + \log\log N + (1 + \epsilon)\log\log\log N] = \gamma_1(N, \epsilon).$$

Then $I(N, K_N)$ is QAD and $I(N, K_N) = K_N$ or $K_N - 2$ or $K_N - 4$ if N is big enough.

(iv) (Erdős-Révész 1975) In general, if

$$\begin{aligned} \gamma_T(N, \epsilon) &= [\log N + T\log\log N + (1+\epsilon)\log\log\log N] < K_N \\ \leq \beta_{T+2}(N, \epsilon) &= [\log N + (T+1)\log\log N - \log\log\log N - \\ &\quad -\log((T+1)!) + \log\log e - 2 - \epsilon], \end{aligned}$$

then $\{I(N, K_N)\}$ is QAD and $I(N, K_N) = K - 2T$ or $K - 2T - 2$, and if

$$\beta_{T+2}(N, \epsilon) < K_N \leq \gamma_{T+1}(N, \epsilon),$$

then $\{I(N, K_N)\}$ is QAD and $I(N, K_N) = K_N - 2T$ or $K_N - 2T - 2$ or $K_N - 2T - 4$.

(v) (Deheuvels-Devroye-Lynch 1986) Let $K_N = c\log N + o(\log\log N)$, $c > 1$. Then for any $\epsilon > 0$

$$\alpha[c\log N] + (1 + \epsilon)\rho\log\log N \in \mathcal{UUC}(I(N, K_N)),$$

$$\alpha[c\log N] + (1 - \epsilon)\rho\log\log N \in \mathcal{ULC}(I(N, K_N)),$$

$$\alpha[c\log N] - (1 - \epsilon)\rho\log\log N \in \mathcal{LUC}(I(N, K_N)),$$

$$\alpha[c\log N] - (1 + \epsilon)\rho\log\log N \in \mathcal{LLC}(I(N, K_N)),$$

where α is the unique solution of the equation

$$1/c = 1 - h\left(\frac{1+\alpha}{2}\right),$$

$$h(x) = -x\log x - (1-x)\log(1-x),$$

and

$$\rho = 1/\log\frac{1+\alpha}{1-\alpha}.$$

Comparing the above statements one can realize that $\{I(N, K_N)\}$ is QAD when K_N is "regular enough" and smaller than $\log N + T\log\log N$ with some fixed $T > 0$. However, when $K_N = (1+\epsilon)\log N$ then the actual value of $I(N, K_N)$ strongly depends on chance. In fact, the upper and lower bounds differ by $O(\log\log N)$. One of the main aims of this exposition is to fill the gap between the cases $K_N = \log N + T\log\log N$ and $K_N = (1+\epsilon)\log N$.

2. AN INEQUALITY

In this section we prove

THEOREM 1. Let $0 < K = K_N < N$ and $0 < T = T_K < K/2$. Assume also $K_N \to \infty, N/K_N \to \infty$. Furthermore, let

$$p = p(K, T) = \left(1 - \frac{2T}{K-1}\right)2^{-K-1}\binom{K-1}{T}. \tag{1}$$

Then it holds:

(i) If $Np \to \infty$ and $N^2Kp^3 \to 0$ then there are constants C_1 and C_2 such that

$$C_1\exp(-Np) \le \mathbf{P}(I(N, K) < K - 2T) \le C_2\exp(-Np). \tag{2}$$

(ii) If $Np \to \infty$ and $Kp \to 0$ then for any $\epsilon > 0$ it holds:

$$\exp(-(1+\epsilon)Np) < \mathbf{P}(I(N, K) < K - 2T) < \exp(-(1-\epsilon)Np). \tag{3}$$

In order to make some of our arguments more transparent, we shall first give a proof of the special case $T = 0$.

Lemma 1. For any $M > 0$ it holds

$$(M+2)2^{-K-1} - (M+2)^2 2^{-2K-2} \le \mathbf{P}(I(M+K, K) = K) \le (M+2)2^{-K-1} \tag{4}$$

Proof: Let us first define some events:

$$A_0 = \{S_K = K\},$$

$$A_j = \{X_j = 0, S_{K+j} - S_j = K\}.$$

It clearly holds:

$$\mathbf{P}(A_0) = 2^{-K},$$

$$\mathbf{P}(A_j) = 2^{-K-1},$$

and, since $\{I(M+K,K)=K\}=\bigcup_{j=0}^{M}A_j$, we obtain from the inclusion-exclusion formula:

$$\sum_{0\le j\le M}\mathbf{P}(A_j)-\sum_{0\le j<r\le M}\mathbf{P}(A_jA_r)\le \mathbf{P}(I(M+K,K)=K)\le \sum_{0\le j\le M}\mathbf{P}(A_j),$$

and the assertion of our lemma follows from the fact that $\mathbf{P}(A_jA_r)=0$ if $|j-r|\le K$ and $=\mathbf{P}(A_j)\mathbf{P}(A_r)$ otherwise.

Now, in order to prove assertion (i) of theorem 1, let M be chosen in such a way that $MNp^2\to 0$ and $K/M\to 0$. Let

$$C_j=\bigcup_{r=j(M+K)}^{j(M+K)+M-1}A_r$$

$$D_j=\bigcup_{r=j(M+K)-M}^{(j+1)(M+K)-1}A_r$$

$$E=\bigcup_{r=0}^{N/(M+K)-1}C_j$$

$$\hat{E}=\bigcup_{r=0}^{N/(M+K)+1}C_j$$

$$F=\bigcup_{j=0}^{N/(M+K)-1}D_j.$$

Obviously,

$$\hat{E}^CF^C\subset\{I(N,K)<K\}\subset E^C,$$

and

$$\mathbf{P}(E^C)=\prod_{j=0}^{N/(M+K)-1}\mathbf{P}(C_j^C).$$

$$\mathbf{P}(\hat{E}^C)=\prod_{j=0}^{N/(M+K)+1}\mathbf{P}(C_j^C).$$

$$\mathbf{P}(F^C)=\prod_{j=0}^{N/(M+K)-1}\mathbf{P}(D_j^C).$$

Now, since it is also clear that $\mathbf{P}(E^CF^C)\ge\mathbf{P}(E^C)\mathbf{P}(F^C)$, we finally obtain

$$\prod_{j=0}^{N/(M+K)+1}\mathbf{P}(C_j^C)\mathbf{P}(D_j^C)\le\mathbf{P}(I(N,K)<K-2T)\le\prod_{j=0}^{N/(M+K)-1}\mathbf{P}(C_j^C).$$

Using lemma 1, this can be restated as

$$((1-Mp)(1-Kp))^{\lfloor \frac{N}{M+K} \rfloor} \le \mathbf{P}(I(N,K) < K-2T) \le (1-Mp+M^2p^2)^{\lfloor \frac{N}{M+K} \rfloor + 2}$$

By our choice of M, both products on the left and right-hand sides are of size $\exp(-Np+o(1))$, so (2) is proven in this special case.

Now, in order to prove theorem 1 in the general case, let us redefine the events A_j in a suitable way:

$$A_j = \begin{cases} \{S_K \ge K-2T\} & \text{if } j=0 \\ \{S_{K+r}-S_r < K-2T : 0 \le r < j, S_{K+j}-S_j \ge K-2T\} & \text{if } 0<j\le K \\ \{S_{K+r}-S_r < K-2T : j-K \le r < j, S_{K+j}-S_j \ge K-2T\} & \text{if } K<j \end{cases}$$

The probabilities of these events can be estimated in the following way:

<u>Lemma 2.</u> If $T < K/2$ then for $1 \le j \le K$ it holds

$$\left(1-\frac{2T}{K-1}\right)2^{-K-1}\binom{K-1}{T} \le \mathbf{P}(A_j) \le \left(1-\frac{2T}{K-1}+\sqrt{\frac{2}{j}}\right)2^{-K-1}\binom{K-1}{T}$$

Proof: Let

$$Y = \#\{r : 1 \le r \le j-1 : X_r \le X_{r+K}\}$$

$$Y_1 = \#\{r : 1 \le r \le j-1 : X_r = -1, X_{r+K} = +1\}$$

$$Y_2 = \#\{r : 1 \le r \le j-1 : X_r = +1, X_{r+K} = -1\}$$

Clearly

$$\begin{aligned} \mathbf{P}(A_j) &= \mathbf{P}(A_j | X_j = -1, X_{j+K} = +1, S_{K+j-1} - S_j = K-2T-1) \times \\ &\quad \times \mathbf{P}(X_j = -1)\mathbf{P}(X_{j+K} = +1) \end{aligned}$$

From the ballot theorem (cf. Takács 1967) it follows that

$$\begin{aligned} &\mathbf{P}(A_j | X_j = -1, X_{j+K} = +1, S_{j+K-1} - S_j = K-2T-1, Y_1, Y_2) \\ &= \frac{(1+Y_1-Y_2) \vee 0}{Y_1+Y_2+1} \end{aligned}$$

and

$$\begin{aligned} &\mathbf{P}(A_j | S_{j+K-1} - S_j = K-2T-1, Y) \\ &= \frac{1}{Y+1}\mathbf{E}(1+Y_1-Y_2 | S_{j+K-1} - S_j = K-2T-1, Y) \end{aligned}$$

Now, since the conditional distribution of Y_1 is hypergeometric with parameters $K-1, K-T-1$, and Y, we get the estimates

$$\begin{aligned} &\mathbf{E}((1+Y_1-Y_2) \vee 0 | S_{j+K-1} - S_j = K-2T-1, Y) \\ &\ge \mathbf{E}((1+Y_1-Y_2 | S_{j+K-1} - S_j = K-2T-1, Y) \\ &= 1+Y\left(1-\frac{2T}{K-1}\right) \end{aligned}$$

and

$$
\begin{aligned}
&\mathbf{E}((1+Y_1-Y_2)\vee 0|S_{j+K-1}-S_j=K-2T-1,Y)\\
&\quad\le\ \mathbf{E}(|Y_1-Y_2|\,|S_{j+K-1}-S_j=K-2T-1,Y)+1\\
&\quad\le\ \mathbf{E}((Y_1-Y_2)^2|S_{j+K-1}-S_j=K-2T-1,Y)^{1/2}+1\\
&\quad\le\ \sqrt{Y+1}+\frac{K-2T-1}{K-1}Y+1
\end{aligned}
$$

Finally, the conditional distribution of Y is binomial with parameters $j-1$ and $1/2$, so the assertion of our lemma is simply obtained by taking expectations in the last two estimates.

From this point, we can proceed in exactly the same manner as in the proof for the case $T=0$. A proof of the second assertion of theorem 1 is also obtained in a quite similar way; the only difference is that the final argument only yields somewhat coarser bounds on $\mathbf{P}(I(N,K)<K-2T)$.

3. STRONG THEOREMS

THEOREM 2. Let $K=K_N\sim\log N$ and $0<T=T_K<K/2$ be nondecreasing sequences of integers. Then

$$K-2T\in\mathcal{LLC}(I(N,K))\ \text{if}\ \sum_{n\in\mathbb{N}}\exp(-2^n p(2^n))<\infty \tag{5}$$

$$K-2T\in\mathcal{LUC}(I(N,K))\ \text{if}\ \sum_{n\in\mathbb{N}}\exp(-2^n p(2^n))=\infty \tag{6}$$

$$K-2T\in\mathcal{ULC}(I(N,K))\ \text{if}\ \sum_{n\in\mathbb{N}}2^n p(2^n)=\infty \tag{7}$$

$$K-2T\in\mathcal{UUC}(I(N,K))\ \text{if}\ \sum_{n\in\mathbb{N}}2^n p(2^n)<\infty \tag{8}$$

Here p is defined as in theorem 1, (1):

$$p=\left(1-\frac{2T}{K-1}\right)2^{-K-1}\binom{K-1}{T} \tag{9}$$

Proof. (5) and (8) are simple consequences of theorem 1 and the Borel-Cantelli lemma, while to prove (6) and (7) it is worthwile to utilize the Erdős-Rényi form (Erdős-Rényi 1970) of the Borel-Cantelli lemma. It is quite obvious that the conditions (5), (6), (7), and (8) can be replaced respectively by

$$(1-\frac{2T}{K-1})N2^{-K-1}\binom{K}{T}\ge\frac{1+\epsilon)\log\log N}{\log e}, \tag{5*}$$

$$(1-\frac{2T}{K-1})N2^{-K-1}\binom{K}{T}\le\frac{(1-\epsilon)\log\log N}{\log e}. \tag{6*}$$

$$N2^{-K-1}\binom{K}{T} \geq (\log N)^{-1-\epsilon}, \tag{7*}$$

$$N2^{-K-1}\binom{K}{T} \leq (\log N)^{-1-\epsilon}, \tag{8*}$$

for some $\epsilon > 0$ and every N big enough.

We shall investigate the case $K_N = [C\log N]$ with $C > 1$ in a little more detail proving the following

Consequence 1. Let $K = K_N = [C\log N]$ with $C > 1$. Then

$$C(1-2\beta)\log N + (1+\epsilon)2\rho\log\log N \in \mathcal{UUC}(I(N, K_N)),$$

$$C(1-2\beta)\log N + (1-\epsilon)2\rho\log\log N \in \mathcal{ULC}(I(N, K_N)),$$

$$C(1-2\beta)\log N - 2\rho\log\log N - 2\theta\log\log\log N$$
$$+2\theta\log(1-2\beta) + 2\theta\log\log e + 2\rho\log\pi + 3\theta + 1 + \epsilon \in \mathcal{LUC}(I(N, K_N))$$

$$C(1-2\beta)\log N - 2\rho\log\log N - 2\theta\log\log\log N$$
$$+2\theta\log(1-2\beta) + 2\theta\log\log e + 2\rho\log\pi + 3\theta + 1 - \epsilon \in \mathcal{LLC}(I(N, K_N))$$

where β is the solution of

$$(2\beta^\beta(1-\beta)^{1-\beta})^C = 2,$$

$$\rho = (2\log\frac{1-\beta}{\beta})^{-1},$$

$$\theta = 2\rho$$

and ϵ is an arbitrary positive number.

Remark. Observe that the $\mathcal{UUC}$ and $\mathcal{ULC}$ classes in the above consequence are exactly the same as the corresponding classes in (v) of section 1 while the herewith given results for the $\mathcal{LUC}$ and $\mathcal{LLC}$ classes are a bit sharper than the corresponding results in (v).

Proof of consequence 1. Let

$$T = \gamma K$$

then

$$Np(K, T) \approx \frac{(1-2\gamma)\sqrt{1-\gamma}}{2\sqrt{2\pi\gamma}}\left(\frac{2^{1/C}}{2\gamma^\gamma(1-\gamma)^{1-\gamma}}\right)^K$$

and a little calculation shows that the desired conditions are equivalent to (5*) to (8*), respectively.

Now, let $K = K_N = \log N + f(N)$ be a non-decreasing sequence of positive integers with $f(n) = o(\log N)$ and consider the equation

$$\binom{K}{T} = 2^{f(N)}$$

An easy calculation gives that the solution of this equation is

$$T \approx \frac{f(N)}{\log\log N - \log f(N)}$$

We prove

Consequence 2.

(i) Assume that

$$\lim_{N\to\infty} \frac{f(N)}{(\log N)^{\epsilon}} = 0 \text{ for any } \epsilon > 0.$$

Then $I(N, K_N)$ is QAD and there exists an $f_1 \in \mathcal{UUC}(I(N, K_N))$ and an $f_4 \in \mathcal{LLC}(I(N, K_N))$ such that $f_1 - f_4 \le 3$.

(ii) Assume that

$$f(N) \approx (\log N)^{\alpha} \quad (0 < \alpha < 1)$$

Then $I(N, K_N)$ is QAD and there exists an $f_1 \in \mathcal{UUC}(I(N, K_N))$ and an $f_4 \in \mathcal{LLC}(I(N, K_N))$ such that $f_1 - f_4 \le \frac{2}{1-\alpha} + 1$.

(iii) Assume that

$$\lim_{N\to\infty} \frac{f(N)}{(\log N)^{1-\epsilon}} = 0 \text{ for any } \epsilon > 0.$$

Then $I(N, K_N)$ is not QAD.

Proof. Observe that

$$\frac{K-T}{T} \approx \frac{\log N(\log\log N - \log f(N))}{f(N)},$$

consequently

(i) if $f(N) = o(\log N)^{\epsilon})$ and

$$2^{-f(N)}\binom{K}{T_1} = (\log N)^{-1-\epsilon}$$

then

$$2^{-f(N)}\binom{K}{T_1+3} \ge \frac{(1+\epsilon)\log\log N}{\log e}$$

what proves (i).

(ii) if $f(N) = (\log N)^{\alpha}$ and

$$2^{-f(N)}\binom{K}{T_2} = (\log N)^{-1-\epsilon}$$

then

$$2^{-f(N)}\binom{K}{T_1 + \frac{2}{1-\alpha} + 1} \ge \frac{(1+\epsilon)\log\log N}{\log e}$$

what proves (ii).

(iii) is trivial.

4. THE CASE $K_N \gg \log N$

Up to now we have studied the properties of $I(N, K_N)$ when $K_N \nearrow \infty, N/K_N \nearrow \infty$ and $K_N \leq C \log N$ with some $C > 0$. We have proved that $I(N, K_N)$ is QAD when $K_N \leq \log N + (\log N)^\alpha$ $(0 < \alpha < 1)$ and in the case $K_N = [C \log N]$ $(C > 1)$ the difference between the $\mathcal{UUC}$ and $\mathcal{LLC}$ is $O(\log \log N)$. We expect that this difference becomes greater as K_N becomes greater. It is really the case, however, we will see that the available results become less complete as K_N becomes greater with the exception that in the case $K_N = N$ the law of iterated logarithm gives the complete description of the four classes.

From now on the results can be more suitably presented using the natural logarithm instead of the logarithm of base 2, hence *log* will be meant in this sense. We present

THEOREM 3. (Deheuvels-Steinebach 1986) Let K_N be a sequence of positive integers with $K_N = [\tilde{K}_N]$ where $\tilde{K}_N/\log N$ is increasing and for some $p > 1$ $\tilde{K}_N(\log N)^{-p}$ is decreasing. Then for any $\epsilon > 0$ we have

$$\alpha_N K_N - t_N^{-1} \log K_N + (3/2 + \epsilon) t_N^{-1} \log \log N \in \mathcal{UUC}(I(N, K_N)),$$

$$\alpha_N K_N - t_N^{-1} \log K_N + (3/2 - \epsilon) t_N^{-1} \log \log N \in \mathcal{ULC}(I(N, K_N)),$$

$$\alpha_N K_N - t_N^{-1} \log K_N + (1/2 + \epsilon) t_N^{-1} \log \log N \in \mathcal{LUC}(I(N, K_N)),$$

$$\alpha_N K_N - t_N^{-1} \log K_N + (1/2 - \epsilon) t_N^{-1} \log \log N \in \mathcal{LLC}(I(N, K_N)),$$

where α_N is the unique positive solution of the equation

$$\exp(-\log N/K_N) = (1 + \alpha_N)^{(1+\alpha_N)/2}(1 - \alpha_N)^{(1-\alpha_N)/2}$$

and

$$t_N = \frac{1}{2} \log \frac{1 + \alpha_N}{1 - \alpha_N}$$

Note that $\alpha_N \approx (2K_N^{-1} \log N)^{1/2}$

In the case when $K_N \gg \log^3 N$ we have

THEOREM 4. (Ortega-Wschebor 1984, Révész 1982) Let K_N be a sequence of positive integers such that $K_N = [\tilde{K}_N]$ where $\tilde{K}_N$ is a sequence of positive real numbers with

(i) $\tilde{K}_N \nearrow \infty$

(ii) $\tilde{K}_N \leq N$ and $N/\tilde{K}_N$ is nondecreasing.

(iii) $\tilde{K}_N / \log^3 N \to \infty$.

(iv) $\dfrac{\log N \tilde{K}_N^{-1}}{\log \log N} \to \infty$.

Then

$$\phi_1(N)K_N^{1/2} \in \mathcal{UUC}(I(N, K_N))$$

and

$$\phi_2(N)K_N^{1/2} \in \mathcal{ULC}(I(N, K_N))$$

if $\phi_1(N)$ and $\phi_2(N)$ are increasing sequences with

$$\sum_{N=1}^{\infty} \phi_1^2(N)K_N^{-1}\exp(-\phi_1^2(N)/2) < \infty$$

and

$$\sum_{N=1}^{\infty} \phi_2(N)K_N^{-1}\exp(-\phi_1^2(N)/2) = \infty.$$

Further for any $\epsilon > 0$

$$K_N^{1/2}\Big(2\log NK_N^{-1} + \log\log NK_N^{-1} - 2\log\log\log N + \log(\frac{51^2}{\pi} + \epsilon)\Big)^{1/2}$$
$$\in \mathcal{LUC}(I(N, K_N))$$

and

$$K_N^{1/2}\Big(2\log NK_N^{-1} + \log\log NK_N^{-1} - 2\log\log\log N - \log(\pi(1+\epsilon))\Big)^{1/2}$$
$$\in \mathcal{LLC}(I(N, K_N)).$$

Now we turn to the case when K_N is so big that not even (iv) of Theorem 4 surely holds. We consider at first the case $K_N = CN(\log\log N)^{-1}$. The following constant will be essential in our proofs.

Lemma 3. There exists a constant $\frac{1}{2}\log\frac{4\pi}{\pi-2} \leq \Gamma \leq \log\frac{4\pi}{\pi-2}$ such that

$$\Gamma = \lim_{n\to\infty}\left(-\frac{1}{n}\log\left(\int_{E_n}\det\begin{pmatrix}\phi(y_i - y_{j-1})\\ 0 \leq i,j \leq n\end{pmatrix}dy_1\ldots dy_{n-1}\right)\right) \tag{10}$$

where $E_n = \{0 = y_0 < y_1 < \ldots < y_{n-1}\}$ and $\phi(s) = (2\pi)^{-1/2}\exp(-s^2/2)$.
Proof. Let $(W(t), t \geq 0)$ be a Wiener process and set $S(t) = W(t+1) - W(t)$ and $M_\lambda = \sup_{0\leq t\leq\lambda} S(t)$. Put $P_\lambda = \mathbf{P}(M_\lambda \leq 0)$. We shall prove that

$$\Gamma = \lim_{\lambda\to\infty}(-\frac{1}{\lambda}\log P_\lambda) \tag{11}$$

exists.

Knowing that (11) holds, (10) is straightforward from a result due to Shepp (1971, see, e.g., theorem 3.1 in Cressie 1980) by which precisely

$$P_n = \int_{E_n}\det\begin{pmatrix}\phi(y_i - y_{j-1})\\ 0 \leq i,j \leq n\end{pmatrix}dy_1\ldots dy_{n-1} \tag{12}$$

In order to show (11), we remark that P_λ is nonincreasing in $\lambda > 0$ with $P_0 = 1/2$. Futhermore, define a process

$$S_\lambda(t) = \begin{cases} S(t) & \text{if } 0 \le t \le \lambda \\ S(t+1) & \text{if } t > \lambda \end{cases}$$

We have $\rho(s,t) = \mathbf{E}(S(s)S(t)) = (1-|s-t|) \vee 0$ and $\rho_\lambda(s,t) = \mathbf{E}(S_\lambda(s)S_\lambda(t)) = (1-|s-t|) \vee 0$ if $0 \le s,t \le \lambda$ or $\lambda < s,t$ and $\rho_\lambda(s,t) = 0$ otherwise. Hence $\rho_\lambda(s,t) \le \rho(s,t)$ for all $s,t \ge 0$, and it follows from Slepian's lemma (Slepian 1962) that

$$\mathbf{P}(\sup_{0 \le t \le \lambda + \lambda'} S_\lambda(t) \le 0) \le \mathbf{P}(\sup_{0 \le t \le \lambda + \lambda'} S(t) \le 0),$$

and hence that for all $\lambda, \lambda' \ge 0$

$$P_{\lambda+\lambda'} \ge P_\lambda P_{\lambda'} \tag{13}$$

Observing that $S(t)$ and $S(t+h)$ are independent for $|h| \ge 1$, and using (13), we can prove that for any $\lambda, \alpha > 0$,

$$P_\alpha^{\lceil \lambda/\alpha \rceil} \le P_\lambda \le P_\alpha^{\lfloor \lambda/(\alpha+1) \rfloor} \tag{14}$$

The first inequality in (14) follows from (13) and the remark that $\alpha\lceil \lambda/\alpha \rceil \ge \lambda$. For the second inequality, observe that

$$\{M_\lambda \le 0\} \subset \bigcap_{i=1}^{\lfloor \lambda/(\alpha+1) \rfloor} \{ \sup_{(i-1)(\alpha+1) \le t \le i(\alpha+1)-\alpha} S(t) \le 0\}.$$

By (12), we can easily compute $P_1 = \frac{\pi-2}{4\pi}$ as follows:

$$\begin{aligned} P_1 &= \int\int_{0<s<t} \left(\frac{1}{2\pi}\left(\exp(-\frac{s^2+(s-t)^2}{2} - \exp(-\frac{t^2}{2})\right)\right) ds\,dt \\ &= \frac{1}{2\pi}\int_0^1 du \int_0^\infty \left(\exp(-\frac{t^2}{2}(u^2+(1-u)^2)) - \exp(-\frac{t^2}{2})\right) t\,dt \\ &= \frac{1}{2\pi}\int_0^1 du \int_0^\infty \left(\exp(-v(u^2+(1-u)^2)) - \exp(-v)\right) dv \\ &= \int_0^1 (\frac{1}{(u^2+(1-u)^2} - 1) du = \frac{\pi-2}{4\pi}. \end{aligned}$$

By (14), we see that, for any fixed $\alpha > 0$, we have

$$\frac{1}{\alpha}\log P_\alpha \le \liminf_{\lambda\to\infty}\frac{1}{\lambda}\log P_\lambda \le \limsup_{\lambda\to\infty}\frac{1}{\lambda}\log P_\lambda \le \frac{1}{\alpha+1}\log P_\alpha \tag{15}$$

This in turn implies that $|\frac{1}{\lambda}\log P_\lambda|$ is ultimately bounded as $\lambda \to \infty$. It follows that

$$\frac{1}{\alpha+1}\log P_\alpha - \frac{1}{\alpha}\log P_\alpha \le \frac{1}{\alpha+1}\sup_{\lambda\ge\alpha}|\frac{1}{\lambda}\log P_\lambda| \to 0$$

as $\alpha \to \infty$ which proves the existence of $\Gamma = \lim_{\lambda\to\infty}(-\frac{1}{\lambda}\log P_\lambda)$ together with the bounds, for any $\alpha > 0$,

$$-\frac{1}{\alpha+1}\log P_\alpha \le \Gamma \le -\frac{1}{\alpha}\log P_\alpha. \tag{16}$$

Taking $\alpha = 1$ in (16) completes the proof of lemma 3.
Remark. The exact value of Γ is not known at present.

Similar arguments as above enable us to prove the following
Lemma 4. For any fixed $a \in \mathbb{R}$ there exists a $0 < \Gamma(a) < \infty$ such that with the notation of lemma 3,

$$\Gamma(a) = \lim_{n\to\infty}\left(-\frac{1}{n}\log\left(\int_{E_n}\det\begin{pmatrix}\varphi(y_i - y_{j+1} + a)\\ 0 \le i, j \le n\end{pmatrix} dy_1 \dots dy_{n-1}\right)\right)$$

$$= \lim_{\lambda\to\infty}\left(-\frac{1}{\lambda}\log \mathbf{P}(\sup_{0\le t\le\lambda} S(t) \le a)\right).$$

Furthermore, $\Gamma(.)$ is strictly decreasing.

Starting from lemmas 3 and 4 and using the Komlós-Major-Tusnady approximation, one can easily prove the following result.
Lemma 5. Let $\Gamma = \Gamma(0)$ and $\Gamma(a)$ be as in lemmas 3 and 4. Assume that $1 \le K_N \le N$ is such that $K_N \to \infty$, $K_N/N \to 0$, and $K_N^{-1/2}\log N \to 0$ as $N \to \infty$. Then, for any fixed $a \in \mathbb{R}$,

$$\lim_{N\to\infty}\left(-\frac{K_N}{N}\log \mathbf{P}(I(N, K_N) \le a K_N^{1/2})\right) = \Gamma(a).$$

Furthermore, we have for any integer $m \ge 1$

$$\lim_{n\to\infty}\mathbf{P}(I((m+1)n, n) \le 0) = P_m$$

and in particular,

$$\lim_{n\to\infty}\mathbf{P}(I(2n, n) \le 0) = \frac{\pi - 2}{4\pi}.$$

We may now state our main result concerning the case where $K_N \approx CN/\log\log N$.
THEOREM 5. Let $K_N = CN(\log\log N)^{-1}$. Then

$$\liminf_{N\to\infty} I(N, K_N) = \begin{cases} +\infty, & \text{if } C < \Gamma \\ -\infty, & \text{if } C > \Gamma \end{cases}$$

with probability one.
Proof. First note, by lemma 5, that

$$\begin{aligned}\mathbf{P}(I(N, K_N) \le a K_N^{1/2}) &= \exp\left(-\frac{N}{K_N}\Gamma(a)(1 + o(1))\right)\\ &= \exp\left(-(1 + o(1))\frac{\Gamma(a)}{C}\log\log N\right).\end{aligned}$$

Suppose in the first place that $C < \Gamma$. It follows that there exists an $\alpha > 0$ such that $C \le \Gamma(\alpha) < \Gamma$. Next, if $N_k = \exp(k/\log k)$, it follows evidently that

$$\sum_{k\in\mathbb{N}}\mathbf{P}(I(N_k, K_{N_k}) \le \alpha K_{N_k}^{1/2}) < \infty$$

Since $K_{N_{k+1}} - K_{N_k} \approx (\log k)^{-1} K_{N_k} \approx (\log\log N_k)^{-1} K_{N_k}$, standard methods show that this implies that

$$\liminf_{N\to\infty} K_N^{-1/2} I(N, K_N) > 0 \text{ a.s.},$$

which in turn implies the first half of theorem 5.

For the second half, assume that $C > \Gamma$ and let $\beta < 0$ be such that $\Gamma < \Gamma(\beta) < C$. The same arguments as above show in this case that

$$\sum_{k \in \mathbb{N}} \mathbf{P}(I(N_k, N_{N_k}) \le \beta K_{N_k}^{1/2}) = \infty$$

In a similar way as before, this enables us to prove that in this case

$$\liminf_{N\to\infty} K_N^{-1/2} I(N, K_N) < 0 \text{ a.s.},$$

which implies the second half of theorem 5.

The study of the limiting behaviour of $I(N, K_N)$ when $K_N \approx \Gamma N/\log\log N$ looks a challenging problem. As a special case of this problem we propose the following question: Does there exist a sequence (K_N) for which $\liminf_{N\to\infty} I(N, K_N) = 0$ a.s.?

A result describing the upper classes of $I(N, K_N)$ when K_N is big follows:

THEOREM 6. (Csörgő-Révész 1979) Let K_N be a nondecreasing sequence of positive integers for which $K_N \le N$, N/K_N is non-decreasing and $K_N \log^{-2} N \to \infty$. Then

$$(1+\epsilon)(2K_N(\log NK_N^{-1} + \log\log N))^{1/2} \in \mathcal{UUC}(I(N, K_N)),$$

$$(1-\epsilon)(2K_N(\log NK_N^{-1} + \log\log N))^{1/2} \in \mathcal{ULC}(I(N, K_N)).$$

In the case when $K_N = [\alpha N]$ $(0 < \alpha \le 1)$ the lower classes of $I(N, K_N)$ can be described by

THEOREM 7. (Csáki-Révész 1979) Assume that $K_N = [\alpha N]$ with $0 < \alpha \le 1$. Then it holds

$$\liminf_{N\to\infty} (2N \log\log N)^{-1/2} I(N, K_N) = -c_\alpha \text{ a.s.}$$

where

$$c_\alpha = \left(\frac{(2r+1)\alpha - 1}{r(r+1)}\right)^{1/2} \quad \text{and } r = [1/\alpha]$$

We also mention that Strassen's law of the iterated logarithm implies that

$$\limsup_{N\to\infty} (2N \log\log N)^{-1/2} I(N, K_N) = \alpha^{1/2}$$

It seems worth wile to mention that some results on the lower classes of

$$I^*(N, K) = \max_{0 \le n \le N-K} \max_{0 \le j \le K} |S_{n+j} - S_n|$$

are available. In fact we have

THEOREM 8. (Csáki-Révész 1979) Let K_N be a non-decreasing sequence of positive integers satisfying the conditions of theorem 6. Then

$$(46+\epsilon)(2K_N\log(1+\frac{\pi^2}{16}\Delta_N))^{1/2}\in\mathcal{LUC}(I^*(N,K_N)),$$

$$(18^{-1}-\epsilon)(2K_N\log(1+\frac{\pi^2}{16}\Delta_N))^{1/2}\in\mathcal{LLC}(I^*(N,K_N)),$$

where $\Delta_N=[NK_N^{-1}](\log\log N)^{-1}$.

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