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k-Connectivity in Random Graphs

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1. INTRODUCTION

Motivated by applications of evolving random graphs as models for phase transitions in physical systems [1, 2, 3, 4], problems were posed [5] concerning threshold functions for the appearance of giant k-connected subgraphs in random graphs, random f-graphs (i.e. random graphs with maximum vertex degree f), and random lattice-graphs (i.e. random graphs restricted to be embeddable [6] in some lattice-graph). For more details about these classes of random graphs see [2] or [4].

We present here a solution to the problem for the first two classes of random graphs and for all $k = 1, 2, \ldots$. The problem concerning random lattice-graphs remains open.

2. RANDOM GRAPHS AND RANDOM f-GRAPHS

We employ *random graph* in the sense of Erdös and Rényi [7], that is a graph $R_{n,N}$ selected with equal probability from among the $\binom{\binom{n}{2}}{N}$ graphs on n (labelled) points and with N edges. It is the statistical properties as the random graph evolves (i.e. as N increases) in the asymptotic limit $n \to \infty$ that are of interest.

An *f-graph* is a graph with maximum degree $\leqslant f$. A random f-graph $R_{(f),n,N}$ is defined analogously to $R_{n,N}$ but is subject to the constraint that no vertex has degree $> f$. That is, $R_{(f),n,N}$ is a graph selected with equal probability from among the $A_f(n, N)$ f-graphs on n (labelled) points and with N edges. Since the number $A_f(n, N)$ is an unsolved problem (see e.g. [8]), for practical purposes we may adapt one of the operational formulations of $R_{n,N}$ as a stochastic process (see e.g. [7b]) to the degree restricted case. Random f-graphs are of interest as chemical models where degree restrictions are imposed by bonding considerations [2, 3].

It has been shown [cf. 2] that the vertex degree distribution in $R_{(f)}$ has probability generating function (pgf):

$$F_0(\theta) = (1 - a + a\theta)^f = \sum_{j=0}^{f} \binom{f}{j} (1 - a)^{f-j} a^j \theta^j, \tag{1}$$

that is, the probability that a random point in $R_{(f)}$ has degree j is the coefficient of θ^j in $F_0(\theta)$, where in the asymptotic limit $n \to \infty$:

$$a \sim \frac{2N}{nf} = \text{probability of an edge in } R_{(f)}. \tag{2}$$

Let d be the mean vertex degree in $R_{(f)}$, then:

$$d = \frac{2N}{n} \sim af. \tag{3}$$

In the double limit that $f \to \infty$ and $a \to 0$, but such that the mean vertex degree d is preserved, degree restrictions are removed and $R_{(f),n,N} \to R_{n,N}$. From eqn (1) we obtain

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[cf. 2] the vertex degree pgf for R as:

$$F_0(\theta) = e^{d(\theta-1)} = e^{-d} \sum_{j\geq 0} \frac{d^j \theta^j}{j!}. \tag{4}$$

Quite generally, the degree restriction of random f-graphs can be relaxed by applying the double limit $f \to \infty$ and $a \to 0$ and therebye any result obtained for $R_{(f)}$ furnishes an analogous result for R.

3. 1-Connectivity in Random Graphs and Random f-Graphs

Consider a point p picked at random in $R_{(f)}$ and let it be the root, on generation g_0, of a rooted component of $R_{(f)}$ whose points fall on generation g_s if these points are distance s from p ($s = 1, 2, 3, \ldots$). The point p has degree j with probability given by eqn (1), thus with this probability it has j successors on generation g_1. Obviously, each of these successors has degree at least 1 and at most f so that the degree distribution for points on g_1 has pgf $\theta F_1(\theta)$ where:

$$F_1(\theta) = (1 - a + a\theta)^{f-1} \tag{5}$$

is the pgf for the number of successors (on g_2) of a point on g_1. Similarly, a point on g_s has a pgf for its number of successors (on g_{s+1})

$$F_s(\theta) = F_1(\theta), \qquad s > 0 \tag{6}$$

As $R_{(f),n,N}$ evolves (that is as N or as a increases) almost all components are initially trees with the order of the largest component growing smoothly until for some value $N = N_1$ ($a = a_1$) the structure of $R_{(f)}$ changes abruptly and the order of the largest component exhibits a double jump, or discontinuity, in the limit $n \to \infty$. The unique largest component in a random graph following this *abrupt change* was termed the *giant component* by Erdös and Rényi [7] who also discuss its properties in some detail. The phenomenon has also been noted in the chemical and physical literature [1, 2, 3, 4] where the *abrupt change* has been likened to such processes as phase transitions and polymer gelation [9]. It was shown that [2]:

$$N_1 \sim \frac{f}{2(f-1)} n \qquad \text{or} \qquad a_1 \sim \frac{1}{(f-1)}. \tag{7}$$

To prepare for what follows we sketch the cascade theory proof of this result (for details see [2] and references therein).

Since, prior to the transition, almost all components are *trees* we obtain (by cascade substitution) the pgf for the order of components in $R_{(f)}$:

$$W(\theta) = \theta F_0(\theta F_1(U)) \equiv \sum_{j\geq 0} w_j \theta^j, \tag{8}$$

where $U(\theta) = \theta F_1(U)$. Since:

$$\frac{dW(\theta)}{d\theta} = F_0(U) + \theta \frac{dF_0(U)}{dU} \left(\frac{F_1(U)}{1 - \theta(dF_1/dU)} \right), \tag{9}$$

the expected order $\langle w \rangle$ of the component (tree) of which the random point p in $R_{(f)}$ is root, is:

$$\langle w \rangle \equiv \Sigma_j j w_j = \left.\frac{dW}{d\theta}\right)_{\theta=1} = \frac{1 - F_1'(1) + F_0'(1)}{1 - F_1'(1)}, \tag{10}$$

where $F'(1) \equiv dF(\theta)/d\theta)_{\theta=1}$.

The expected order $\langle w \rangle$ diverges when:

$$1 - F_1'(1) = 0, \tag{11}$$

and it is at this stage in the evolution of a random f-graph that the giant component suddenly appears. Since, furthermore, a maximal 1-connected subgraph of $R_{(f)}$ is just a component of the random graph, eqn (11) gives the critical value a_1 at which there is an *abrupt* increase in the order of the largest *1-connected subgraph* of $R_{(f)}$. From eqn (5):

$$F'(1) = (f - 1)a, \tag{12}$$

so that the critical (or threshold) value for 1-connected subgraphs of $R_{(f)}$ is

$$a_1 = 1/(f - 1). \tag{13}$$

If $R_{(f),n,N}$ has n points and N edges then:

$$N \sim f a n/2. \tag{14}$$

Thus, for the evolving random f-graph, the critical number of edges N_1 for a giant 1-connected subgraph is [cf. 2]:

$$N_1 \sim \frac{f}{2(f - 1)} n. \tag{15}$$

Obviously for $f \to \infty$, $N_1 \sim n/2$ as obtained by Erdös and Rényi [7], and as can be obtained directly from eqn (11) using:

$$F_1(\theta) = e^{d(\theta - 1)}, \tag{16}$$

to which eqn (5) leads in the double limit $f \to \infty$ and $a \to 0$ ($fa = d$ is fixed).

4. k-Connectivity in Random Graphs and Random f-Graphs ($k = 2, 3, \ldots$)

We now follow a similar construction but *discount* all points of degree $< k$. That is, choose a random point p_k from among the points in $R_{(f)}$ known (with probability given by eqn (1)) to have degree $\geqslant k$. Next examine the degrees of the successors of p_k through generations $g_1, g_2, \ldots$ discarding any successors on g_s whose degree is less than k. By similar arguments to those used in Section 3 for 1-connected subgraphs of $R_{(f)}$, the order of the maximal *k-connected subgraph* of which a random point (of degree $\geqslant k$) in $R_{(f)}$ is root, has pgf:

$$\begin{aligned} W_k(\theta) &= \theta H_k(U) = \Sigma_j w_{k,j}\theta^j, \\ U_k(\theta) &= G_{k,1} + \theta G_{k,2}(U), \end{aligned} \tag{17}$$

where $H_k(\theta)$ is the renormalised pgf for degrees of points in $R_{(f)}$ known to have degree $\geqslant k$ [cf. eqn (1)]. Thus:

$$H_k(\theta) = \sum_{j=k}^{f} \binom{f}{j} (1 - a)^{f-j} a^j \theta^j \Big/ \sum_{j=k}^{f} \binom{f}{j} (1 - a)^{f-j} a^j. \tag{18}$$

Also

$$\begin{aligned} G_{k,1} &= \sum_{j=0}^{k-2} \binom{f-1}{j} (1 - a)^{f-j-1} a^j, \\ G_{k,2} &= \sum_{j=k-1}^{f-1} \binom{f-1}{j} (1 - a)^{f-j-1} a^j \theta^j \\ &= F_1(\theta) - \sum_{j=0}^{k-2} \binom{f-1}{j} (1 - a)^{f-j-1} a^j \theta^j. \end{aligned} \tag{19}$$

The expected order of the maximal k-connected subgraph of which a random point (of degree $\geq k$) in $R_{(f)}$ is root is:

$$\langle w_k \rangle = \left.\frac{\mathrm{d}W_k(\theta)}{\mathrm{d}\theta}\right)_{\theta=1} = \frac{1 - G'_{k,2}(1) + H'_k(1)G_{k,2}(1)}{1 - G'_{k,2}(1)}, \tag{20}$$

which diverges when:

$$1 - G'_{k,2}(1) = 0. \tag{21}$$

On substituting for $G_{k,2}(\theta)$ from eqn (19) we have proved:

THEOREM 1. *In the evolution of a random f-graph $R_{(f),n,N}$ the order of the largest k-connected ($k = 1, 2, \ldots$) subgraph increases abruptly at a critical edge probability a_k given by the root (between zero and unity) to*

$$\sum_{j=k-1}^{f-1} j\binom{f-1}{j}(1-a_k)^{f-1-j}a_k^j = 1$$

COROLLARY 1. *As is easily seen by rewriting Theorem 1, a_k is also the solution to:*

$$(f-1)\,a_k - \sum_{j=0}^{k-2} j\binom{f-1}{j}(1-a_k)^{f-1-j}a_k^j = 1$$

or to

$$\sum_{j=k-1}^{f-1}(-1)^{j-k+1} j\binom{j-2}{j-k+1}\binom{f-1}{j}a_k^j = 1$$

THEOREM 2. *The asymptotic critical size N_k of a random f-graph $R_{(f),n,N}$ for the appearance of a giant k-connected subgraph is*

$$N_k \sim f a_k n/2$$

PROOF. Obvious from eqn (2) and Theorem 1.

In the double limit $f \to \infty$ and $a \to 0$, but with fixed mean vertex degree $d = af$, eqn (19) [cf. eqn (16)] becomes:

$$G_k(\theta) = \mathrm{e}^{-d}\sum_{j=0}^{k-2}\frac{d^j}{j!} + \sum_{j=k-1}^{\infty}\frac{d^j}{j!}\theta^j. \tag{22}$$

Thus,

THEOREM 3. *The asymptotic critical size of a random graph $R_{n,N}$ for the appearance of a giant k-connected subgraph is*

$$N_k \sim d_k n/2$$

where the critical mean vertex degree d_k is the solution to

$$\sum_{j=k-1}^{\infty} j\frac{d_k^j}{j!} = \mathrm{e}^{d_k} \quad \text{or to} \quad d_k - \mathrm{e}^{-d_k}\sum_{j=0}^{k-2} j\frac{d_k^j}{j!} = 1.$$

PROOF. Substitute eqn (22) into eqn (21). Alternatively, replace a_k in Theorem 1 by d_k/f and then pass to the limit $f \to \infty$ with d_k fixed.

5. SPECIAL CASES

The critical parameters (a_k and N_k) for k-connectivity in random graphs can be obtained explicitly from the foregoing for a few special cases. Thus:

$$k = 1 \text{ (all } f\text{)}: \qquad a_1 = (f-1)^{-1} \qquad N_1 \sim \frac{f}{2(f-1)}\, n, \tag{23}$$

$$k = 2 \text{ (all } f\text{)}: \qquad a_2 = (f-1)^{-1} \qquad N_2 \sim \frac{f}{2(f-1)}\, n, \tag{24}$$

$$k = f \text{ (all } f\text{)}: \qquad a_f = (f-1)^{-1/(f-1)}, \quad N_f \sim \tfrac{1}{2} f (f-1)^{-1/(f-1)} n \tag{25}$$

For $f \to \infty$:

$$N_1 = N_2 \sim n/2, \qquad N_f \sim \binom{n}{2}$$

The equivalence between 1-connected and 2-connected subgraphs is easily explained. As soon as the giant component (1-connected subgraph) appears in the evolution of random graphs, closure of many infinite cycles is possible. An infinite cycle is a giant 2-connected subgraph.

For all f it is reasonably obvious that:

$$(f-1)^{-1} = a_1 = a_2 < a_3 < \cdots < a_{f-1} < a_f = (f-1)^{-1/(f-1)}, \tag{26}$$

$$\frac{fn}{2(f-1)} = N_1 = N_2 < N_3 < \cdots < N_{f-1} < N_f = \tfrac{1}{2} f (f-1)^{-1/(f-1)} n. \tag{27}$$

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