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Goodness of Trees for Generalized Books*

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Abstract. A connected graph G is said to be F-good if the Ramsey number $r(F, G)$ is equal to $(\chi(F) - 1)(p(G) - 1) + s(F)$, where $s(F)$ is the minimum number of vertices in some color class under all vertex colorings by $\chi(F)$ colors. It is of interest to know which graphs F have the property that all trees are F-good. It is shown that any large tree is $K(1, 1, m_1, m_2, \ldots, m_t)$-good.

1. Introduction

Let F be a graph with chromatic number $\chi(F)$. The *chromatic surplus* $s(F)$ is defined to be the smallest number of vertices in a color class under any $\chi(F)$-coloring of the vertices of F. For a pair of graphs (F, G) the *Ramsey number* $r(F, G)$ is the least number N such that in every two-coloring $(R, B) = (\text{red}, \text{blue})$ of the edges of K_N, there is either a red copy of F or a blue copy of G.

For a connected graph G, $r(F, G)$ satisfies

$$r(F,G) \geq (\chi(F) - 1)(p(G) - 1) + s(F), \qquad \text{if } p(G) \geq s(F). \tag{1}$$

This inequality follows by coloring red or blue the edges of a complete graph on $(\chi(F) - 1)(p(G) - 1) + s(F) - 1$ vertices such that the blue graph $\langle B \rangle$ is isomorphic to $(\chi(F) - 1)K_{p(G)-1} \cup K_{s(F)-1}$ and the red graph $\langle R \rangle$ is isomorphic to its complement. When equality occurs in (1) we say that G is *F-good*. The concept of F-goodness generalizes the classical simple result of Chvátal that $r(K_m, T_n) = (m-1)(n-1) + 1$ [5], where K_m denotes the complete graph on m vertices and T_n denotes a tree on n vertices.

Our purpose is to investigate those graphs F for which all large trees T_n are F-good. The importance of $\chi(F)$ in the value of $r(F, T_n)$ leads the investigation to a consideration of multipartite graphs F. It is known that T_n fails to be $K(m_1, m_2, \ldots, m_s)$-good when each $m_i \geq 2$ or when $m_1 = 1$ and $m_i \geq 2$ for $2 \leq i \leq s$

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[3,4]. For example in [3] it is shown that $r(K(2,2), K(1,n-1)) > n + n^{1/2} - 5n^{3/10}$ for n large. Thus, at this point the most general multipartite graph for which goodness of trees is unsettled is $K(1,1,m_1,m_2,\ldots,m_s)$. The principal result of the paper shows that each large order tree T_n is $K(1,1,m_1,m_2,\ldots,m_s)$-good, thus answering completely in one sense the F-goodness question for large trees.

Some known results, which we now list with appropriate references, will be needed in the proofs. However, we first introduce some special notation. For simplicity the multipartite graph $K(l_1,l_2,\ldots,l_k,m_1,m_2,\ldots,m_s)$ will be denoted by $K(l_1,l_2,\ldots,l_k,m;s)$ when $m = m_1 = m_2 = \cdots = m_s$. In particular we will frequently encounter this graph when $l_1 = l_2 = 1$ and $k = 2$, (i.e., the graph $K(1,1,m;s)$). If a graph G contains a vertex which is adjacent to k end vertices, we shall say that G has a *talon of degree k* or that G has a *talon* with k vertices. Additional notation will follow that used in standard texts, e.g. [1,8].

Theorem A. *Let m and s be fixed positive integers.*

(i) [4] *There is a nondecreasing function* $f_1(m,s)$ *of m and s such that* $r(K(1,m;s), K(1,n-1)) \le sn + f_1(m,s)$ *for all n.*

(ii) [4] *There is a function* $n_1(m,s)$ *of m and s such that* $r(K(1,1,m;s), K(1,n-1)) \le (s+1)(n-1)+1$ *for all* $n \ge n_1(m,s)$.

(iii) [6] *There are* $A = A(m,s)$, $\alpha = \alpha(m,s)$ $(0 < \alpha < 1)$ *and* $n_2(m,s)$ *such that* $r(K(m;s), T_n) \le (s-1)(n-1) + An^{\alpha}$ *for all* $n \ge n_2(m,s)$.

Theorem B. *Let* $X = \{x_1,x_2,\ldots,x_a\}$ *and* $Y = \{y_1,y_2,\ldots,y_b\}$ *be disjoint sets of vertices of a complete graph which is* (R,B) *2-colored.*

(i) [6] *If* $(x_1,x_2,\ldots,x_a)$ *is a maximal length path from* x_1 *to* x_a *in* $\langle B\rangle$ *and* $a \ge b(c-1)+d$, *then either* $\langle R\rangle \supseteq K_c$ *or Y is completely joined in* $\langle R\rangle$ *to a subset* $X' \subseteq X$ *with* $|X'| = d$.

(ii) [7] *There is either a blue matching of X into Y or for some c* $(0 \le c \le a-1)$ *all edges between a* $(c+1)$*-subset of X and a* $(b-c)$*-subset of Y are red (i.e.* $\langle R\rangle \supseteq K(c+1, b-c)$*).*

Theorem C. [2] *Let T be a tree with n vertices. If T has no suspended path with more than a vertices, then G has at least* $\lceil n/2a\rceil$ *vertices of degree one.*

2. Results

In order to prove our primary result we must first prove the following proposition.

Proposition 1. *For positive integers m and s, there is a function* $f(m,s)$ *such that for all n,*

$$r(K(1,m;s), T_n) \le sn + f(m,s).$$

Proof. Set $l = \max\{f_1(m,s), n_2(2m,s), (4Am^3s^2)^{1/(1-\alpha)}\}$, where f_1, n_2, m, s, $A = A(2m,s)$ and $\alpha = \alpha(2m,s)$ are as defined in Theorem A. The proof is by double induction on n and s.

The result is trivial when $s = 1$ since for any tree T_n, $r(K(1,m), T_n) \leq n + m - 1$. It is clear that for a proper choice of $f(m,s)$ the proposition is true for all $n \leq l$. Hence, we assume $s > 1$, $n > l$ and that the proposition is true for all smaller values of these parameters.

Let $N = sn + f(m,s)$, and suppose (R, B) is a 2-coloring of the edges of K_N in which $\langle R \rangle \not\supseteq K(1,m;s)$ and $\langle B \rangle \not\supseteq T_n$. We shall demonstrate that this assumption leads to a contradiction. The proof is divided into three cases.

(i) The tree T_n contains a suspended path with $ms(ms - m + 2)$ vertices

Let T' be the tree obtained from T_n by shortening the length of the suspended path by one. Since T' has $n - 1$ vertices, by induction there is a blue copy of T' and a disjoint red copy of $K(1,m;s-1)$. Now apply Theorem B(i) with X the vertex set of the suspended path and Y the vertex set of the red $K(1,m;s-1)$. Thus $a = ms(ms - m + 2) - 1$ and $b = m(s-1) + 1$, and let $c = ms + 1$ and $d = m$. The requisite inequality in Theorem B(i) is clearly satisfied, so since $\langle B \rangle \not\supseteq T_n$, either $\langle R \rangle \supseteq K(1,m;s)$ or $\langle R \rangle \supseteq K_{ms+1} \supseteq K(1,m;s)$, a contradiction.

(ii) The tree T_n contains $2m$ independent end-edges.

Since $n > l > \max\{n_2(2m,s), A^{1/(1-\alpha)}\}$, Theorem A(iii) implies $\langle R \rangle \supseteq K(2m;s)$. Let T' be the tree obtained by deleting the $2m$ independent end-edges from T_n. Thus T' has $n - 2m$ vertices, so by induction there is a blue copy of T' disjoint from the red copy of $K(2m;s)$. Apply Theorem B(ii) with X the vertices of T' incident to the independent end edges deleted from T_n and Y the vertices of the red $K(2m;s)$. Therefore either $\langle B \rangle \supseteq T_n$ or for some positive integer $c < 2m$ there is a $(c+1)$-element subset of T' which is adjacent in red to $2ms - c$ of the vertices of the red $K(2m;s)$. In the second case $\langle R \rangle \subseteq K(1,m;s)$, although there are two subcases to consider. The 1-element part of $K(1,m;s)$ will be in X if $c + 1 < m$ and will be in one of the parts of $K(m;s)$ if $c + 1 \geq m$.

(iii) Neither (i) nor (ii) occur.

By Theorem C the tree T_n contains at least $\lceil n/2m^2s^2 \rceil$ vertices of degree one. First suppose that T_n contains no talon with more than An^α vertices. Then, since T_n has fewer than $2m$ independent end-edges, T_n has at most $2Amn^\alpha$ vertices of degree one. Hence $2Amn^\alpha > n/2m^2s^2$. Therefore $n < (4Am^3s^2)^{1/(1-\alpha)} < l$, a contradiction.

The only possibility which remains is that T_n has a talon with more than An^α vertices. Let x be a center of such a talon. Since $n \geq f_1(m,s)$, Theorem A(i) implies that $\langle B \rangle \supseteq K(1, n-1)$. Let y denote the center of this blue $K(1, n-1)$.

Let T' denote the tree obtained from T_n by deleting the end vertices of the talon with center x. We will attempt to embed T' in $\langle B \rangle$ by mapping x onto y. Note that if T' could be so embedded, then T_n could also be embedded due to the large blue degree of y. Extend the embedding of T' one vertex at a time and always with a vertex of degree one as far as possible. Since $\langle B \rangle \not\supseteq T_n$, there exists a vertex z where the embedding stops. This means that z is adjacent in red to all vertices not in the embedded portion of T'. Thus z is adjacent in red to at least $(N - 1 - [(n-1) - An^\alpha]) = (s-1)n + An^\alpha + f(m,s)$ vertices. But $n > l \geq n_2(2m,s)$, so Theorem A(iii) implies that the graph induced by the red neighborhood of z contains a red copy of $K(m;s)$. This graph together with z is a red $K(1,m;s)$, again a contradiction.

Theorem 2. *For fixed positive integers m and s, there is a corresponding number $n_0(m, s)$ such that*

$$r(K(1,1,m;s), T_n) = (s+1)(n-1)+1 \quad \text{for all } n \geq n_0(m,s).$$

Proof. In light of inequality (1) we need only establish that $(s+1)(n-1)+1$ is an upper bound. The proof is very similar to that of Proposition 1 in that similar techniques will be applied to the same three cases. Therefore some details will be left to the reader.

We perform induction on s. The result is trivial when $s = 0$. Thus, we assume $s > 0$, set $N = (s+1)(n-1)+1$, and assume that (R, B) is a two-coloring of K_N in which $\langle R \rangle \not\supseteq K(1,1,m;s)$ and $\langle B \rangle \not\supseteq T_n$. In what follows $f(m, s)$ will denote the function whose existence was established in Proposition 1.

(i) The tree T_n contains a suspended path with $m^2s^2 + 2ms + 2 + f(m,s)$ vertices.

Let T' be the tree obtained from T_n by shortening this path by $f(m,s)$ vertices. The induction hypothesis together with Proposition 1 implies a blue copy of T' and, disjointly, a red copy of $K(1,1,m;s-1)$. By applying Theorem B(i) the argument for this case can be completed just as (i) in the proof of the Proposition.

(ii) The tree T_n contains $f(s,m) + s + m$ independent end-edges.

Let T' be the tree obtained from T_n by deleting $f(m,s) + s + m$ independent end edges. Since

$$(s+1)(n - [f(s,m) + s + m]) + f(s,m) \leq (s+1)(n-1)+1,$$

the Proposition implies that $\langle B \rangle \supseteq T'$. Apply Theorem B(ii) with X the vertices of T' which were incident to the edges deleted from T_n, and Y the vertices of K_N not in T'. Let X' be the vertices of X and Y' be the vertices of Y contained in the red $K(c+1, b-c)$ insured by Theorem B(ii).

There are two separate possibilities to consider, depending on the value of c. First consider the case $c + 1 \geq m$. Since $c \leq f(m,s) + s + m - 1$,

$$b - c \geq N - (n - f(m,s) - s - m) - c \geq s(n-1) + 1.$$

The induction assumption implies the graph $\langle Y' \rangle$ contains a red copy of $K(1,1,m;s-1)$. This red $K(1,1,m;s-1)$ along with m vertices of X' gives a red $K(1,1,m;s)$. Next consider the case $c < m - 1$. Thus

$$b - c \geq N - (n - f(m,s) - s - m) - c > sn + f(m,s) + 1.$$

The Proposition implies that $\langle Y' \rangle$ contains a red copy of $K(1,m;s)$, which along with a vertex in X' gives a red $K(1,1,m;s)$. Thus for each of the two possibilities we reach a contradiction.

(iii) Neither (i) nor (ii) occur.

We assume without loss of generality that $n_0(m,s) \geq n_1(m,s)$, where n_1 is as given in Theorem A(ii). Hence by Theorem A(ii), $\langle B \rangle \supseteq K(1, n-1)$. Also for $l = m^2s^2 + 2ms + 2 + f(m,s)$ Theorem C implies that the tree T_n contains at least $\lceil n/2l \rceil$ vertices of degree one.

Again two possibilities must be considered. The first is that T_n contains no talon with more than $f(m,s) + s$ vertices. Since T_n has fewer than $f(s,m) + s + m$ independent end-edges, this tree has at most $[f(m,s)+s][f(s,m)+s+m]$ vertices of degree one. But then

$$[f(m,s)+s][f(s,m)+s+m] \geq n/2l,$$

which clearly fails for all $n \geq n^*$ for an appropriate n^* depending only on s and m.

The second possibility is that T_n contains a talon with more than $f(m,s)+s$ vertices. Let x be a vertex of T_n incident with more than $f(m,s)+s$ end vertices. Since $\langle B\rangle \supseteq K(1,n-1)$, let y denote the center of this blue star in K_N. A contradiction to complete the proof of this case can be reached just as in case (iii) of the Proposition, except that the Proposition is applied in the last step instead of Theorem A(iii). This completes the proof of the Theorem 2.

There are statements that are equivalent to Theorem 2 which appear to be more general. We state two of them.

Theorem 3. *Let $m_1, m_2, \ldots, m_s$ be fixed positive integers. There exists a integer n_0 such that*

$$r(K(1,1,m_1,m_2,\ldots,m_s),T_n)=(s+1)(n-1)+1$$

for all $n \geq n_0$.

Theorem 4. *Let G be a graph with $\chi(G)\geq 2$ which has a vertex $\chi(G)$-coloring with at least two color classes consisting of a single vertex. Then for sufficiently large n,*

$$r(G,T_n)=(\chi(G)-1)(n-1)+1.$$

3. Comments

There are many questions left unanswered concerning connected graphs G which are F-good. In light of the results of this paper it would be interesting to be able to determine the value of $r(K(m_1,m_2,\ldots,m_s),T_n)$ when n is large and $m_1 \leq m_2 \leq \cdots \leq m_s$ are arbitrary, but fixed. However, this appears to be a very difficult question.

In particular it is true that for n large,

$$r(K(1,m_2,\ldots,m_s),T_n) \leq (s-1)(M-1)+1$$

where $M=r(K(1,m_2),T_n)$. There is equality when T_n is a star [3], and a forthcoming paper will show that there is in fact equality for almost all large trees. It is quite possible that $r(K(m_1,m_2),K(1,\Delta(T_n))$ is involved in the value of $r(K(m_1,m_2,\ldots,m_s),T_n)$ for large n. Also, a reasonable conjecture would be

$$r(K(m_1,m_2,\ldots,m_s),T_n) \leq (s-1)(r(K(m_1,m_2),T_n)-1)+m_1.$$

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