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COLORING GRAPHS WITH LOCALLY FEW COLORS

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Let G be a graph, $m > r \geq 1$ integers. Suppose that it has a good-coloring with m colors which uses at most r colors in the neighborhood of every vertex. We investigate these so-called local r-colorings. One of our results (Theorem 2.4) states: The chromatic number of G, $\mathrm{Chr}(G) \leq r2^r \log_2 \log_2 m$ (and this value is the best possible in a certain sense). We consider infinite graphs as well.

Introduction

Assume that a graph G has a good-coloring which uses at most r colors in the neighborhood of every vertex. We call this kind of coloring a local r-coloring. Is it true that the chromatic number of G is bounded? For $r = 1$ the answer is easy, G is bipartite, as it cannot have an odd circuit. For $r = 2$, however, the situation is completely different. A graph can be given with arbitrarily large (infinite) chromatic number: The vertex set is the set of all triples $\{x_0, x_1, x_2\}$ with $x_0, x_1, x_2 \in X$, here X is an arbitrary ordered set. If $x_0 < x_1 < x_2$ and $y_0 < y_1 < y_2$, $x_1 = y_0$, $x_2 = y_1$, then $\{x_0, x_1, x_2\}$ and $\{y_0, y_1, y_2\}$ are joined. If the cardinality of X is large enough then this graph has large chromatic number (by Ramsey's or the Erdös–Rado Theorem, in the finite or in the infinite case, respectively). But $f(\{x_0, x_1, x_2\}) = x_1$ (where $x_0 < x_1 < x_2$) is a good coloring, and the neighbors of $\{x_0, x_1, x_2\}$ are colored with x_0, x_2.

In this paper we investigate the most general problems of this kind:

$(*)$ Assume that G is a graph which has a good coloring with m colors which uses at most r colors for the neighborhood of every point (for a technical reason we count the point itself as an element of its neighborhood); is it true that the chromatic number of G is at most n?

In the discussion we get sharp or almost sharp answers in both the finite and infinite cases. If n, r are finite, the smallest m with a negative answer is something about $2 \uparrow (2 \uparrow (n/2^r))$. We have exact result for $r \geq \sqrt{n}$, the weakest estimates are in the interval $\log n < r < \sqrt{n}$. If n is infinite, the threshold m is 2^{2^n}. Under the generalized continuum hypothesis we have a full answer to the main problem.

We also investigate the problem whether a (finite) graph with large girth and large local chromatic number can be found (this generalizes an old result of

Erdös) and the problem that in infinite graphs establishing a negative answer to (∗) which finite subgraphs must occur. We also find analogous results for k-neighborhoods in place of neighborhoods.

The organization of the paper is as follows. In Section 1 the basic definitions, a universal graph and a very useful matrix-equivalent form of the problem are given. The basic results for the finite and infinite cases are given in Sections 2 and 3 respectively. Section 4 gives the results for k-neighborhoods.

In this paper we adapt the usual set theory notation, i.e., a cardinal is the set of smaller ordinals, ${}^{\kappa}\lambda$ denotes the functions from κ to λ, $\kappa^{\underline{\lambda}}$ is the cardinal $\sum_{\alpha<\lambda}\kappa^{\alpha}$ $f''A$ is $\{f(x): x\in A\}$. If A is a set, $[A]^r$ is the system of r-element subsets, $P(A)$ is the system of *all* subsets of A. A *graph* G is a pair (V, E) with $E\subseteq[V]^2$. A *good coloring* for G is a function f from V into a cardinal with $f(x)\neq f(y)$ if x, y are joined. The *chromatic number* of G, in short, $\mathrm{Chr}(G)$ is the smallest cardinal κ such that a good coloring into κ exists.

1. Definition and preliminary results

In this Section m, n, r all can be both finite and infinite cardinals. If $G=(V, E)$ is a graph, put $d_G(x, y)$ for the distance of x, $y\in V$. Let us define $\Gamma(x)=\{y\in V: d_G(x, y)\leq 1\}$ for $x\in V$. As we have already mentioned in the introduction, a cardinal is the smallest ordinal of this cardinality, thus every finite n equals to $\{0, 1, \ldots, n-1\}$.

Definition 1.1. A function $f: V\to m$ is a local (m, r)-coloring (a local $(m, <r)$-coloring) of the graph $G=(V, E)$ if it is a good coloring (i.e., $f(x)\neq f(y)$ whenever x and y are joined) and $|\{f(y): y\in\Gamma(x)\}|\leq r$ ($|\{f(y): y\in\Gamma(x)\}|<r$) holds for every $x\in V$.

Notice that the concept of $(m, <r)$-coloring is slightly more general as gives some new cases if r is a limit cardinal. We shall, however, mostly deal with local (m, r)-colorings and leave the generalizations for $(m, <r)$ to the reader.

Definition 1.2. $P(m, n, r)$ abbreviates the following statement: there exists a graph $G=(V, E)$ with $f: V\to m$, a local (m, r)-coloring, and $\mathrm{Chr}(G)>n$.

Some easy remarks are in order. $P(m, n, r)$ always holds if $n<r$. If $P(m, n, r)$ holds, then $P(m', n', r')$ also holds if $m\leq m'$, $n'\leq n$ and $r\leq r'$.

As one can observe there exists a universal graph among those with local (m, r)-coloring.

Definition 1.3. $U(m, r)$ is the following graph (V, E):

$$V = \{\langle \alpha, A\rangle : \alpha < m,\ A \subseteq m,\ \alpha \notin A,\ |\{\alpha\} \cup A| \leq r\},\text{ and}$$
$$E = \{\{\langle \alpha, A\rangle, \langle \beta, B\rangle\} : \alpha \in B \text{ and } \beta \in A\}.$$

Lemma 1.1. *$P(m, n, r)$ holds if and only if* $\mathrm{Chr}(U(m, r)) > n$.

Proof. Clearly the function $f: V \to m$, $f(\alpha, A) = \alpha$ is a local (m, r)-coloring, so one direction is clear. Suppose, on the other hand, $\mathrm{Chr}(U(m, r)) \leq n$ and let $G = (V_G, E_G)$ be an arbitrary graph with $f: V_G \to m$, a local (m, r)-coloring. We need to show that $\mathrm{Chr}(G) \leq n$. For $x \in V_G$ put $g(x) = \langle f(x), (f''\Gamma(x)) - \{f(x)\}\rangle \in V_{U(m,r)}$. Obviously, $\{x, y\} \in E_G$ implies $\{g(x), g(y)\} \in E_{U(m,r)}$, so g is a graph homomorphism. Now, the composition of g with a good coloring of $U(m, r)$ with n colors also colors G. □

Definition 1.4. The system $\{A_{\alpha,\beta} : \alpha < \beta < m\} \subseteq P(n)$ is *(m, n, r)-independent* if and only if the following holds:
for every $B \in [m]^r$ and every $\alpha \in B$ the set

$$[\bigcap \{A_{\beta,\alpha} : \beta < \alpha,\ \beta \in B\}] - [\bigcup \{A_{\alpha,\gamma} : \alpha < \gamma,\ \gamma \in B\}]$$

is non-empty.

Lemma 1.2. *$P(m, n, r)$ holds if and only if (m, n, r)-independent systems do not exist.*

Proof. Assume that $\{A_{\alpha,\beta} : \alpha < \beta < m\} \subseteq P(n)$ is an independent system. We are going to show that $\mathrm{Chr}(U(m, r)) \leq n$. For $\langle \alpha, A\rangle \in V_{U(m,r)}$ put

$$g(\alpha, A) = \min\{\bigcap\{A_{\beta,\alpha} : \beta < \alpha,\ \beta \in A\} - \bigcup\{A_{\alpha,\gamma} : \gamma \in A,\ \alpha < \gamma\}\}.$$

This function $g: V_{U(m,r)} \to n$ is a good coloring of $U(m, r)$ since $\{\langle \alpha, A\rangle, \langle \beta, B\rangle\} \in E_{U(m,r)}$, $\alpha < \beta$ imply $g(\beta, B) \in A_{\alpha,\beta}$, $g(\alpha, A) \notin A_{\alpha,\beta}$.

For the reverse implication assume that $g: V_{U(m,r)} \to n$ witnesses $\mathrm{Chr}(U(m, r)) \leq n$. Put $A_{\alpha,\beta} = \{g(\beta, B) : \alpha \in B\}$ for $\alpha < \beta < m$, we show that this system is (m, n, r)-independent. If not, there is a set $A \in [m]^r$ and an $\alpha \in A$ with $[\bigcap\{A_{\beta,\alpha} : \beta < \alpha,\ \beta \in A\}] - [\bigcup\{A_{\alpha,\gamma} : \alpha < \gamma,\ \gamma \in A\}] = \emptyset$. Put $\xi = g(\alpha, A \setminus \{a\})$, then $\xi \in \bigcap\{A_{\beta,\alpha} : \beta < \alpha,\ \beta \in A\}$ by the choice of the system. Hence there exists a $\gamma \in A$ with $\alpha < \gamma$ satisfying $\xi \in A_{\alpha,\gamma}$, i.e., $\xi = g(\gamma, C)$ for some $\langle \gamma, C\rangle \in V_{U(m,r)}$ with $\alpha \in C$. But then g assigns ξ to $\langle \alpha, A - \{\alpha\}\rangle$ and $\langle \gamma, C\rangle$ and they are joined, a contradiction. □

2. Finite graphs

In this section m, n, r are finite cardinals, i.e., natrual numbers. As we already mentioned non-$P(m, 2, 2)$ holds for every m, hence the first problem is finding the smallest m with $P(m, n, 3)$.

Definition 2.1. $S \subseteq P(n)$ is an *intersecting Sperner family* if $A, B \in S$, $A \neq B$ implies $A \nsubseteq B$, $A \cap B \neq \emptyset$. $S(n)$ denotes the number of intersecting Sperner families on n points.

Theorem 2.1. *$P(S(n)+1, n, 3)$ holds.*

Proof. By Lemma 1.2 it is enough to show that no $(S(n)+1, n, 3)$-independent systems exist. Assume, on the contrary, that $\mathcal{I} = \{A_{ij}: 0 \leq i < j \leq S(n)\}$ is such a system. Let $\mathcal{S}_j$ be the system of those sets in $\{A_{i,j}: i<j\}$ which are minimal under inclusion, i.e., for which $A_{i',j} \nsubseteq A_{i,j}$ does not hold if $i' < j$. Clearly, $\mathcal{S}_j$ is a Sperner family. It is also intersecting, for $\mathcal{I}$ is $(S(n)+1, n, 3)$-independent. To reach a contradiction we only need to show $\mathcal{S}_i \neq \mathcal{S}_j$ for $i \neq j$. Assume, therefore, $\mathcal{S}_i = \mathcal{S}_j$ and $i<j$. By the definition of $\mathcal{S}_j$, there exists a $B \in \mathcal{S}_j$ with $B \subset A_{i,j}$. As $\mathcal{S}_i = \mathcal{S}_j$, there is a $k<i$ satisfying $B = A_{k,i}$. Now, $A_{k,i} - A_{i,j} = \emptyset$ contradicting the $(S(n)+1, n, 3)$-independence of $\mathcal{I}$. □

By a recent result of Erdös and Hindman ([5]) $S(n) = 2 \uparrow (\binom{n}{[n/2]})(\frac{1}{2} + o(1)))$. On the other hand, we prove

Theorem 2.2. *Non-$P(2 \uparrow \binom{n-2}{[(n-2)/2]}, n, 3)$ holds for all n.*

Proof. First notice that $k = \binom{n-2}{[(n-2)/2]} = \frac{1}{4}\binom{n}{[n/2]}(1+o(1))$. We are going to construct a $(2^k, n, 3)$-independent system. Enumerate the subsets of $[n-2]^{[(n-2)/n]}$ as $\{X_i: 0 \leq i < 2^k\}$ and put $Y_i = \{A \cup \{n-1\}: A \in X_i\}$. We can assume $|Y_i| \leq |Y_j|$ when $i<j$. By this, we can also choose $A_{i,j} \in Y_j - Y_i$. We claim that the system $\mathcal{I} = \{A_{i,j}: 0 \leq i < j < 2^k\}$ is $(2^k, n, 3)$-independent. To this end, let $\{i, j, l\} \in [2^k]^3$. Then $n-1 \in A_{i,l} \cap A_{j,l}$, $n-2 \in n - (A_{i,j} \cup A_{i,l})$, and also $A_{i,j} - A_{j,l} \neq \emptyset$ as $A_{i,j} - \{n-1\}$ and $A_{j,l} - \{n-1\}$ are different $[(n-2)/2]$-element sets. □

Although the next theorem is true for all values of n and r, it gives useful estimates only in case $r = O(\log n)$.

Theorem 2.3. *$P(2 \uparrow (2n + 2 \uparrow (n/2^{r-3})), n, r)$ holds.*

Proof. By induction on n. The case $n=3$ is trivial if $r>3$ and $P(2^{14}, 3, 3)$ holds by Theorem 2.1. Assume our theorem is true for every $n' < n$ and an (m, n, r)-independent system $\mathcal{I} = \{A_{i,j}: 0 \leq i < j < m\}$ is given. As Theorem 2.1 treats the case $r=3$, we can assume $r>3$. We call $j<n$ of type A, where

$A \in [n]^{<[n/2]}$, if either there is an $i<j$ with $A_{i,j}=A$ or there exists $l>j$ with $A_{j,l}=n-A$. If $i<j<m$, then either i is of type $n \setminus A_{i,j}$ or j is of type $A_{i,j}$ depending whether $|A_{i,j}| \geq [n/2]$ holds or not. This argument shows that all but possibly one $j<m$ is of type A for some $A \in [n]^{<[n/2]}$. There exist on $M \subseteq m$ with $m' = |M| > (m-1)/2^n$ and a fixed A such that every $i \in M$ is of type A. We claim that $\{A_{i,j} \cap A: 0 \leq i<j<m,\ i \in M,\ j \in M\}$ is an $(m', |A|, r-1)$-independent system. If not, assume that $X \in [M]^{r-1}$, $j \in X$, and

$$[\bigcap \{A_{i,j} \cap A: i \in X, i<j\}] - [\bigcup \{A_{j,l} \cap A: l \in X, j<l\}] = \emptyset.$$

As j is of type A, either there is a $k<j$ with $A_{k,j}=A$ or else there is a $k>j$ with $A_{j,k}=n-A$, hence choosing $X' = X \cup \{k\}$ and j the (m, n, r)-independence of $\mathcal{I}$ is refuted. By the induction hypothesis $m' < 2\uparrow(2|A| + 2\uparrow(|A|/2^{r-4})) \leq 2\uparrow(n + 2\uparrow(n/2^{r-3}))$, $m' \cdot 2^n < 2\uparrow(2n + 2\uparrow(n/2^{r-3}))$. On the other hand, $m-1 < m'2^n$, so $m \leq m'2^n$, and we are done. □

Theorem 2.4. *Non-*$P(2\uparrow(2\uparrow(n/((r-1)2^{r-1}))), n, r)$.

Proof. Let $k = 2\uparrow(n/((r-1)2^{r-1}))$ and $|B| = k$, $B \subseteq P(n)$ be an $(r-1)$-independent system, i.e.,

$$B_1 \cap B_2 \cap \cdots \cap B_s \cap (n - B_{s+1}) \cap \cdots \cap (n - B_{r-1}) \neq \emptyset,$$

whenever $B_1, B_2, \ldots, B_{r-1}$ are different members of $\mathcal{B}$ and $1 \leq s \leq r-1$. The existence of such a family was proved by Kleitman and Spencer [9]. Let $\{Y_i: 0 \leq i < 2^k\}$ be an enumeration of $P(\mathcal{B})$ with $|Y_i| \leq |Y_j|$ for $i<j$. Put $\mathcal{I} = \{A_{i,j}: i<j<2^k\}$, where $A_{i,j} \in Y_j - Y_i$. $\mathcal{I}$ is $(2^k, n, r)$-independent, as, if $A \in [2^k]^r$ and $j \in A$, for $i<j<l$, $A_{i,j} \neq A_{j,l}$ holds by the construction of $\mathcal{I}$, and so $[\bigcap \{A_{i,j}: i<j, i \in A\}] - [\bigcup \{A_{j,l}: j<l, l \in A\}]$ is non-empty by the $r-1$-independence of $\mathcal{B}$. □

The next Theorem gives lower estimates in case $\log n < r < \sqrt{n}$. We don't have useful upper estimates in this interval.

Theorem 2.5. (a) $non-P((1+1/4r^2)^{n-1}, n, r)$;
(b) $non-P((\sqrt[4]{n}-1)^{[\sqrt{n}-1/r]}, n, r)$.

Proof. Let $f(n, r)$ be the maximum size of a system $\mathcal{S} \subseteq P(n)$ such that no member is covered by $r-1$ other members. If $\{S_i: 0 \leq i < f(n, r)\}$ enumerates $\mathcal{S}$, put $\mathcal{I} = \{A_{i,j}: 0 \leq i<j<f(n, r)\}$, $A_{i,j} = S_j$. Obviously, $\mathcal{I}$ is $(f(n, r), n+1, r+1)$-independent. The estimates $f(n-1, r-1) > (1+1/4r^2)^{n-1}$ and $f(n-1, r-1) > (\sqrt[4]{n}-1)^{[\sqrt{n}-1/r]}$ by Erdös, Frankl and Füredi [2], finish the proof. □

Theorem 2.6. *For every* n, k, $P(n+k+1, n, [n/(k+1)]+k+1)$ *holds.*

Proof. Suppose, on the contrary, that $\{A_{i,j}: 0 \leq i<j<n+k+1\}$ is an

$(n+k+1, n, [n/(k+1)]+k+1)$-independent system, and put $\Pi_j = [\bigcap_{i<j} A_{i,j}] - [\bigcup_{j<l} A_{j,l}]$. As for $i<j$, $\Pi_i \cap A_{i,j} = \emptyset$ and $\Pi_j \subseteq A_{i,j}$, these Π_i's are pairwise disjoint. Hence, there exists an $X \in [n+k+1]^{k+1}$ with $\Pi_j = \emptyset$ for $j \in X$. Put $\Pi'_j = [\bigcap \{A_{i,j}: i<j,\ i \in X\}] - [\bigcup \{A_{j,l}: j<l,\ l \in X\}]$ for $j \in X$. Again, $\Pi'_i \cap \Pi'_j = \emptyset$, whenever $i \neq j$. Therefore, there exists an $l \in X$ with $|\Pi'_l| \leq [n/(k+1)]$. As $\Pi_l = \emptyset$, for every $j \in \Pi'_l$, there is a $g_j < n+k+1$ with either $g_j < l$ and $j \in A_{g_j,l}$ or $g_j > l$ and $j \in A_{l,g_j}$. Then $Y = X \cup \{g_j: j \in \Pi'_l\}$ and $l \in Y$ witnesses that our system is not $(n+k+1, n, [n/(k+1)]+k+1)$-independent, a contradiction. □

This theorem is surprisingly sharp, when k is small, i.e., r is relatively large compared to n.

Theorem 2.7. *Non* $- P(n+k+1, n, [n/(k+1)]+k)$ *holds for* $n \geq k^2+k$.

Proof. We are going to construct an $(n+k+1, n, [n/(k+1)]+k)$-independent system. Put $A_{i,j} = \{j\}$ for $0 \leq i<j<n$ and $A_{j,h} = n - \{j\}$ for $j<n \leq h<n+k+1$. We have to define $A_{n+p,n+q}$, with $0 \leq p<q<k+1$. Put $X_h = \{i: h[n/(k+1)] \leq i < (h+1)[n/(k+1)]\}$ for $0 \leq h < k+1$ and pick k different elements, $\{X_{h,l}: l < k+1, l \neq h\}$ from X_h (possible, as $n \geq k^2+k$). Put $A_{n+p,n+q} = X_q \cup \{X_{h,q}: h \neq p\}$. We claim that our system is $(n+k+1, n, [n/(n+k)]+k)$-independent. Assume that $A \in [n+k+1]^{[n/(k+1)]+k}$, $j \in A$. We have to show $Y = [\bigcap \{A_{i,j}: i<j, i \in A\} - [\bigcup \{A_{j,l}: j<l \in A\}] \neq \emptyset$. If $j<n$, then $j \in Y$. If $j = n+p$ with $0 \leq p$, define $X = X_p \cup \{X_{h,p}: 0 \leq h < k+1\}$. Clearly, $|x| = [n/(k+1)]+k$, $|A_{i,j} \cap X| \geq |X|-1$ for all $i<j$ and $|X - A_{j,l}| = |X|-1$ if $l>j$, hence $X \cap Y \neq \emptyset$. □

Our next topic is how large the girth of a graph with local coloring can be. Let us notice, that by a well-known result of Erdös ([1]), for given g and $\delta < 1/g$ and n large enough there exists a graph G on n point, with girth at least g, and $\mathrm{Chr}(G) \geq n^{\delta}$. By Theorem 2.4 this graph has no local (n, r)-coloring if n is large enough, depending on r. On the contrary, we show

Theorem 2.8. *Given* n, g *there exists a graph* G *with a local* $(m, 3)$*-coloring for a certain* m, $\mathrm{Chr}(G) \geq n$, *the girth of* G *is at least* g.

Our graph will be a random subgraph of the shift graph on $[m]^3$ with m large enough. It has a local $(m, 3)$-coloring, anyway. First we need a lemma.

Lemma 2.9. *For every* n *there exists a* $c(n) > 0$, *such that for every* m, *if* $f: [m]^3 \to n$ *is a coloring, there exist* $c(n)m^2$ *pairs* $\{a, b\}$ *such that there are* X, Y *with* $|X|, |Y| > c(n)m$, $X < a < b < Y$ *and a color* $\chi < n$, *such that* $f(x, a, b) = f(a, b, y) = \chi$ *if* $x \in X$, $y \in Y$.

Proof. For every pair $i<j<m$ define A_{ij} as the set of those $\chi<n$ for which

$$|\{k<i: f(k,i,j)=\chi\}|>\varepsilon m,$$

where $\varepsilon>0$ will be chosen later. The number of triples with a color not counted is at most $\binom{m}{2}\varepsilon mn<\binom{m}{3}4\varepsilon n$.

As $A_{ij}\subseteq n$, by the Erdös–Szekeres theorem ([6]) on every $2^{2^n}+1$ points there is a triple $k<i<j$, with $A_{ki}=A_{ij}$. By a result of Katona–Nemetz–Simonovits ([8]) the number of these triples is at least $\binom{m}{3}/\binom{2\uparrow(2\uparrow n)+1}{3}$. Summing up, there are at least $1/(\binom{2\uparrow(2\uparrow n)+1}{3}-4\varepsilon n)\binom{m}{3}$ triples $k<i<j$ with $f(k,i,j)\in A_{ki}=A_{ij}$. If ε is small enough this is at least $c\binom{m}{3}$ with $c>0$. Counting again, there are at least cm^2 pairs $\{a,b\}$ such that for each pair $a<b$ there are at least $cm\cdot y>b$ with $\{a,b,y\}$ as described above, with a certain $c>0$. For every such $\{a,b\}$ there is a set Y with $A_{ab}=A_{by}$ $f(a,b,y)\in A_{ab}$. Thinning again, there is an $Y'\subseteq Y$, such that $|Y'|>c'm$ and $f(a,b,y)=\chi$ with a certain $\chi\in A_{ab}$ for $y\in Y$, $Y\geqslant cm$. By the definition of A_{ab} we can also choose $X\subseteq a$ $|X|\geqslant cm$ with $f(x,a,b)=\chi$ for $x\in X$. □

Proof of Theorem 2.8. Fix n, g. Let G be the random graph on $[m]^3$, choosing the edge $\{\{a,b,c\},\{b,c,d\}\}$ into G with probability p, independently of each other. m will grow to infinity with $pm<m^{\delta}$ where $\delta>0$ is small enough. If $\{X_1,X_2,\ldots,X_l\}$ is a circuit of length l in the shift-graph on $[m]^3$, then $|\bigcup\{X_i: 1\leqslant i\leqslant l\}|\leqslant l+2$ (by an easy induction). The number of circuits with length l is therefore $O(m^{l+2})$. The average number of circuits of length l in our random graph is $O(m^{l+2}p^l)=O(m^2(mp)^l)$, the average number of circuits of length at most g is $O(m^2(mp)^g)$. Remove the edges of these circuits. The remaining graphs has girth at least $g+1$. Assume that almost all of these graphs have chromatic number at most n. By Lemma 2.9 in each of these graphs we can exhibit cm^2 pairwise edge-disjoint bipartite graphs (X^*,Y^*), where $X^*=X\times\{a,b\}$, $Y^*=\{a,b\}\times Y$, where $X<a<b<Y$. For every graph of the above kind there are $X<a<b<Y$ with the property that only $O((mp)^g)$ of the edges from (X^*,Y^*) were omitted. As f is supposed to be a good coloring, no edge can go between X^* and Y^* in the graph. This means that almost every graph has $X<a<b<Y$, $|X|$, $|Y|>cm$ such that the number of edges between X^* and Y^* is $O((mp)^g)$. But the probability of this event is $o(e^{-c^2pm^2+Am})=o(1)$ if $\delta<1/g$. □

3. Infinite graphs

In this Section κ, λ, ρ, τ denote infinite cardinals. First we restate a result mentioned in the Introduction.

Theorem 3.1. *For $\kappa\geqslant\omega$, $P((2^{2^\kappa})^+,\kappa,3)$ holds.*

Proof. This is given by the shift graph on $[(2^{2^\kappa})^+]^3$. □

Theorem 3.2. *For λ, $\rho \geq \omega$ non-$P(2\uparrow(2\uparrow\lambda^{\rho}), \lambda^{\rho}, <\rho)$ holds.*

Proof. By a theorem of Hausdorff ([7]) there exists a $<\rho$-independent system $\mathscr{S} \subseteq P(\lambda^{\rho})$ with $|\mathscr{S}| = 2^{\lambda^{\rho}} = \tau$. There exists a system of 2^{τ} sets $Y_i \subseteq \mathscr{S}$ with $Y_i \nsubseteq Y_j$ $(i \neq j)$. Now choose $A_{i,j} \in Y_j - Y_i$, the system $\{A_{i,j}: i<j<2^{\tau}\}$ is $(2^{\tau}, \lambda^{\rho}, <\rho)$-independent, similarly to the proof in Theorem 2.4. □

Theorem 3.3. *Assume $\lambda > cf(\lambda)$ and that for $\tau < \lambda$, $2^{2^{\tau}} < \kappa = cf(\kappa)$ holds, then $P(\kappa, \lambda, cf(\lambda))$ is true.*

Proof. Put $\tau = cf(\lambda)$ and choose a sequence $\langle \lambda_{\xi}: \xi < \tau\rangle$ converging to λ. Assume that $\{A_{\alpha,\beta}: \alpha < \beta < \kappa\}$ is a (κ, λ, τ)-independent family. For $\xi < \tau$ put

$$S_{\xi} = \{\alpha < \kappa: \text{there are no } \gamma < \alpha < \delta \text{ with } A_{\gamma,\alpha} \cap \lambda_{\xi} = A_{\alpha,\delta} \cap \lambda_{\xi}\}.$$

Now for $\alpha \in S_{\xi}$, $f(\alpha) = \{A_{\gamma,\alpha} \cap \lambda_{\xi}: \gamma < \alpha\}$ is a function from S_{ξ} into $P(P(\lambda_{\xi}))$. If $|S_{\xi}| > 2\uparrow(2\uparrow\lambda_{\xi})$, there are $\alpha < \delta$ in S_{ξ} with $f(\alpha) = f(\delta)$, so, by the definition of f there is a $\gamma < \alpha$ with $A_{\gamma,\alpha} \cap \lambda_{\xi} = A_{\alpha,\delta} \cap \lambda_{\xi}$, a contradiction. As $|S_{\xi}| \leq 2\uparrow(2\uparrow\lambda_{\xi})$ for $\xi < \tau$, there is an $\alpha < \kappa$ such that $\alpha \notin \bigcup\{S_{\xi}: \xi < \tau\}$, so $A_{\gamma_{\xi},\alpha} \cap \lambda_{\xi} = A_{\alpha,\delta_{\xi}} \cap \lambda_{\xi}$ with $\gamma_{\xi} < \alpha < \delta_{\xi}$ $(\xi < \tau)$. But then $[\bigcap\{A_{\gamma_{\xi},\alpha}: \xi < \tau\}] - [\bigcup\{A_{\alpha,\delta_{\xi}}: \xi < \tau\}] = \emptyset$. □

Theorem 3.4. *For $\kappa \geq \omega$, $P(\kappa^{+}, \kappa, \kappa)$ holds.*

Proof. We invole a construction of Erdős–Hajnal ([3]). Let $G = (V, E)$ be the following graph: $V = \{\langle\alpha, \beta\rangle: \alpha < \beta < \kappa^{+}\}$, $\langle\alpha, \beta\rangle$ and $\langle\alpha', \beta'\rangle$ are joined for $\alpha < \alpha' < \beta < \beta'$. It is shown in [3] that $\mathrm{Chr}(G) = \kappa^{+}$, and the function $f(\alpha, \beta) = \alpha$ is obviously a local (κ^{+}, κ) – coloring: $f''\Gamma(\langle\alpha, \beta\rangle) \subseteq \beta$.

Assuming GCH these last three results give that $P(\kappa, \lambda, \rho)$ holds if and only if $\rho \geq 3$, $\kappa \geq \lambda^{+++}$ or $\rho \geq cf(\lambda)$, $\kappa \geq \lambda^{+}$. □

4. k-neighborhoods

In this section we generalize our original problem: How is the chromatic number of G affected by the existence of a good coloring which uses few colors for the k-neighborhood of every vertex?

Let us start with some notation. In the following discussion k is always a natural number, m, n, r denote cardinals (both finite and infinite), κ, λ, ρ are infinite cardinals. If $G = (V, E)$ is a graph, $x \in V$, then $\Gamma^{k}(xc) = \{y \in V: d_G(x, y) \leq k\}$. $\exp_k(m)$ is defined by induction: $\exp_0(m) = m$, $\exp_{k+1}(m) = 2^{\exp_k(m)}$.

Definition 4.1. A function $f: V \to m$ is a *local $(m, r)^{k}$-coloring* of $G = (V, E)$ if it is a good coloring and $|f''\Gamma^{k}(x)| \leq r$ for every $x \in V$.

Definition 4.2. $P^k(m, n, r)$ stands for the following statement: There exists a graph $G=(V, E)$ with a local $(m, r)^k$-coloring, but $\mathrm{Chr}(G)>n$.

The immediate generalizations of the facts mentioned in the introduction are true.

Theorem 4.1. $P^k((\exp_{2k}(\kappa))^+, \kappa, 2k+1)$ *holds for every* $\kappa \geq \omega$.

Theorem 4.2. *non-*$P^k(\kappa, 2^{(2k)^k}, 2k)$ *for every* κ.

Definition 4.3. The *k-shift graphs on X* has the vertex-set $\{\langle x_0, x_1, \ldots, x_{k-1}\rangle : x_i \neq x_{i+1} \ (0 \leq i < k-1)\}$. $\langle x_0, x_1, \ldots, x_{k-1}\rangle$ and $\langle y_0, \ldots, y_{k-1}\rangle$ are joined if and only if $x_i = y_{i+1}$ $(0 \leq i < k-1)$ or vice versa.

Proof of Theorem 4.1. On the $(2k+1)$-shift graph G on $(\exp_{2k}(\kappa))^+$, $f(x_0, x_1, \ldots, x_{2k}) = x_k$ is a local $((\exp_{2k}(\kappa))^+, 2k+1)^k$-coloring and $\mathrm{Chr}(G) > \kappa$ by a result of Erdös and Hajnal ([4]). □

Proof of Theorem 4.2. Assume that $f: V \to \kappa$ is a local $(\kappa, 2k)^k$-coloring of $G=(V, E)$. Consider all walks (paths with not necessarily distinct vertices) of length k starting in a fixed vertex $x \in V$. As f is sufficiently local, it colors all points in these walks by at most $2k$ colors. As these colors are ordinals, they are ordered by the usual ordering between ordinals, so, we can re-number them increasingly by $0, 1, \ldots, l$ $(<2k)$. By this, each walk mentioned above gives a mapping from k to $2k$. Summing up, we can define $g(x) \subseteq {}^k(2k)$ as the set of these maps. For $\mathrm{Chr}(G) \leq 2^{(2k)^k}$ it suffices to show that f is a good coloring of G. Suppose, in order to reach a contradiction that $g(x) = g(y)$ and $(x, y) \in E$. Put $f''\Gamma^k(x) = \{\alpha_0, \ldots, \alpha_l\}$, $\alpha_0 < \alpha_1 < \cdots < \alpha_l$, $f''\Gamma^k(y) = \{\beta_0, \ldots, \beta_l\}$, $\beta_0 < \cdots < \beta_l$, $f(x) = \alpha_{i_0}$, $f(y) = \beta_{i_0}$. As f is a good coloring, $\alpha_{i_0} \neq \beta_{i_0}$, assume $\alpha_{i_0} < \beta_{i_0}$. There are $i_{-1} < i_0 < i_1$ such that $\beta_{i_0} = \alpha_{i_1}$, $\alpha_{i_0} = \beta_{i_{-1}}$. There is a walk starting from x with the first two vertices colored $\alpha_{i_0}, \alpha_{i_1}$, so, as $g(x) = g(y)$, there is a corresponding walk from y with β_{i_0}, β_{i_1} as the first two colors. As, by assumption, $(x, y) \in E$ there is a walk from x with the respective colors $\alpha_{i_0}, \alpha_{i_1} = \beta_{i_0}, \beta_{i_1}$ so there is an $i_2 > i_1$ with $\beta_{i_1} = \alpha_{i_2}$. Similarly, $\alpha_{i_{-1}} = \beta_{i_{-2}}$ for some $i_{-2} < i_{-1}$. Continuing this process we obtain $2k+1$ different indices $i_{-k} < i_{-k+1} < \cdots < i_0 < \cdots < i_k$ so that $\alpha_{i_j} = \beta_{i_{j-1}}$ for $-(k-1) \leq j \leq k$, $\{\alpha_{i_j} : -k \leq j \leq k\} \subseteq f''\Gamma^k(x)$, a contradiction. □

A universal graph like the one in Section 1 can also be defined.

Definition 4.4. $U^k(m, r)$ is the following graph: The vertex-set is the set of all

$(k+1)$-sequences $\langle A_0, A_1, \ldots, A_k\rangle$ satisfying

(i) $A_i \subset m$;

(ii) $|A_0| = 1$;

(iii) $A_0 \subset A_2 \subset A_4 \subset \cdots$;

(iv) $A_1 \subset A_3 \subset A_5 \subset \cdots$;

(v) $A_0 \not\subseteq A_1$;

(vi) $|A_0 \cup A_1 \cup \cdots \cup A_k| \leq r$.

$\langle A_0, \ldots, A_k\rangle$ and $\langle B_0, \ldots, B_k\rangle$ are joined iff $A_i \subset B_{i+1}$ and $B_i \subset A_{i+1}$ for all $i < k$.

Lemma 4.3. *$P^k(m, n, r)$ holds if and only if* $\mathrm{Chr}(U^k(m, r)) > n$.

Proof. If $\mathrm{Chr}(U^k(m, r)) > n$, then the graph $G = U^k(m, r)$ witnesses $P^k(m, n, r)$: put $f(\langle A_0, \ldots, A_k\rangle) = \bigcup A_0$, i.e., α where $A_0 = \{\alpha\}$. If $\langle B_0, \ldots, B_k\rangle \Gamma^k(\langle A_0, \ldots, A_k\rangle)$, then $B_0 \subseteq A_0 \cup \cdots \cup A_k$, so $f(\langle B_0, \ldots, B_k\rangle)$ has r possible values. For the other direction, assume that $\mathrm{Chr}(U^k(m, r)) \leq n$ and $G = (V, E)$ is a graph with $f: V \to m$, a local $(m, r)^K$-coloring. Put $A_i^x = f''\{y \in V$: there is an (x, y)-walk of length i in $G\}$ for $x \in V$, $i \leq k$. The mapping $g(x) = \langle A_0^x, \ldots, A_k^x\rangle$ is a graph homomorphism from G to $U^k(m, r)$. Composing g with the n-coloring of $U^k(m, r)$ we get a good coloring of G with n colors. □

Definition 4.5. The system $\{A_X: X \in E(U^{k-1}(m, r))\} \subseteq P(n)$ is $(m, n, r)^k$-independent if and only if the following holds: For every $\{A_0, \ldots, A_{k-1}\} \in V(U^{k-1}(m, r))$ and $A_{k-2} \subset X \subset m$ if $|X \cup A_{k-1}| \leq r$, then

$$[\bigcap\{A_{\langle B_0, \ldots, B_{k-1}\rangle, \langle A_0, \ldots, A_{k-1}\rangle}: \{\langle B_0, \ldots, B_{k-1}\rangle, \langle A_0, \ldots, A_{k-1}\rangle\} \in E(U^{k-1}(m, r)), \bigcup B_0 < \bigcup A_0, B_{k-1} \subset X\}]$$
$$-[\bigcup\{A_{\langle A_0, \ldots, A_{k-1}\rangle, \langle C_0, \ldots, C_{k-1}\rangle}: \{\langle A_0, \ldots, A_{k-1}\rangle, \langle C_0, \ldots, C_{k-1}\rangle\} \in E(U^{k-1}(m, r)), \bigcup A_0 < \bigcup C_0, C_{k-1} \subset X\}]$$

is non-empty.

Lemma 4.4. *$P^k(m, n, r)$ holds if and only if no $(m, n, r)^k$-independent set exists.*

Proof. Assume that $\{A_X: X \in E(U^{k-1}(m, r))\} \subseteq P(n)$ is $(m, n, r)^k$-independent. We have to show that $\mathrm{Chr}(U^k(m, r)) \leq n$. Whenever $\langle A_0, \ldots, A_k\rangle \in V(U^k(m, r))$, choose $g(\langle A_0, \ldots, A_k\rangle)$ as the minimal element in

$$[\bigcap\{A_{\langle B_0, \ldots, B_{k-1}\rangle, \langle A_0, \ldots, A_{k-1}\rangle}: \{\langle B_0, \ldots, B_{k-1}\rangle, \langle A_0, \ldots, A_{k-1}\rangle\} \in E(U^{k-1}(m, r)), \bigcup B_0 < \bigcup A_0, B_{k-1} \subset A_k\}]$$
$$-[\bigcup\{A_{\langle A_0, \ldots, A_{k-1}\rangle, \langle C_0, \ldots, C_{k-1}\rangle}: \{\langle A_0, \ldots, A_{k-1}\rangle, \langle C_0, \ldots, C_{k-1}\rangle\} \in E(U^{k-1}(m, r)), \cup A_0 < \bigcup C_0, C_{k-1} \subset A_k\}],$$

which is non-empty by Definition 4.5 with A_k in place of X. We have to show that $g: V(U^k(m, r)) \to n$ is a good coloring. If $\{\langle A_0, \ldots, A_k\rangle, \langle B_0, \ldots, B_k\rangle\} \in E(U^k(m, r))$, $A_0 < \bigcup B_0$ then $g(\langle B_0, \ldots, B_k\rangle) \in A_{\langle A_0, \ldots, A_{k-1}\rangle, \langle B_0, \ldots, B_{k-1}\rangle}$ and $g(\langle A_0, \ldots, A_k\rangle) \notin A_{\langle A_0, \ldots, A_{k-1}\rangle, \langle B_0, \ldots, B_{k-1}\rangle}$ so they are different.

For the other implication assume that $g: V(U^k(m, r)) \to n$ is a good coloring. Put $A_{\langle A_0, \ldots, A_{k-1}\rangle, \langle B_0, \ldots, B_{k-1}\rangle} = \{g(\langle B_0, \ldots, B_k\rangle): A_{k-1} \subset B_k\}$ for $\{\langle A_0, \ldots, A_{k-1}\rangle, \langle B_0, \ldots, B_{k-1}\rangle\} \in E(U^{k-1}(m, r))$, $\bigcup A_0 < \bigcup B_0$. We only need to show that the system just defined is $(m, n, r)^k$-independent. If not, there are an $\langle A_0, \ldots, A_{k-1}\rangle \in V(U^{k-1}(m, r))$ and an $X \supset A_{k-2}$ with $|X \cup A_{k-1}| \leq 1$ and the difference in Definition 4.5 empty. Put $\xi = g(\langle A_0, \ldots, A_{k-1}, X\rangle)$. Clearly,

$$\xi \in \bigcap \{A_{\langle B_0, \ldots, B_{k-1}\rangle, \langle A_0, \ldots, A_{k-1}\rangle}: B_{k-1} \subset X, \bigcup B_0 < \bigcup A_0\}$$

by the above definition. By the indirect assumption, there is a $\langle C_0, \ldots, C_{k-1}\rangle$ with $\xi \in A_{\langle A_0, \ldots, A_{k-1}\rangle, \langle C_0, \ldots, C_{k-1}\rangle}$, $\{\langle A_0, \ldots, A_{k-1}\rangle, \langle C_0, \ldots, C_{k-1}\rangle\} \in E(U^{k-1}(m, r))$, $C_{k-1} \subset X$, $\bigcup A_0 < \bigcup C_0$. By the choice of the system, there is a C_k with $A_{k-1} \subset C_k$, $\xi = g(\langle C_0, \ldots, C_{k-1}\rangle)$, so the color ξ is assigned to $\langle A_0, \ldots, A_{k-1}, X\rangle$ and $\langle C_0, C_1, \ldots, C_{k-1}, C_k\rangle$ and they are joined, a contradiction. □

Theorem 4.5. *If $\kappa > \lambda > cf(\lambda)$, λ is a strong limit cardinal, then $P^k(\kappa, \lambda, cf(\lambda))$ holds.*

Proof. By induction on k. Put $\tau = cf(\lambda)$ and choose a sequence $\langle \lambda_\xi: \xi < \tau\rangle$ converging to λ. The case $k = 1$ is Theorem 3.3. Assume that $P^{k-1}(\kappa, \lambda, cf(\lambda))$ holds, i.e., $\mathrm{Chr}(U^{k-1}(\kappa, \tau)) > \lambda$ and let $\{A_X: X \in E(U^{k-1}(\kappa, \tau))\} \subset P(\lambda)$ be a $(\kappa, \lambda, \tau)^k$-independent system. For $\xi < \tau$ put

$$\begin{aligned} S_\xi = \{\langle A_0, \ldots, A_{k-1}\rangle \in V(U^{k-1}(\kappa, \tau))&: \text{there are no} \\ \langle B_0, \ldots, B_{k-1}\rangle, \langle C_0, \ldots, C_{k-1}\rangle &\in V(U^{k-1}(\kappa, \tau)) \text{ with} \\ \bigcup B_0 < \bigcup A_0 < \bigcup C_0 \text{ and } & A_{\langle B_0, \ldots, B_{k-1}\rangle \langle A_0, \ldots, A_{k-1}\rangle} \cap \lambda_\xi = \\ & A_{\langle A_0, \ldots, A_{k-1}\rangle, \langle C_0, \ldots, C_{k-1}\rangle} \cap \lambda_\xi\}. \end{aligned}$$

If there is an $\langle A_0, \ldots, A_{k-1}\rangle \notin \bigcup \{S_\xi: \xi < \tau\}$ then for $\xi < \tau$ there are $\langle B_0^\xi, \ldots, B_{k-1}^\xi\rangle, \langle C_0^\xi, \ldots, C_{k-1}^\xi\rangle \in V(U^{k-1}(\kappa, \tau))$ such that

$$A_{\langle B_0^\xi, \ldots, B_{k-1}^\xi\rangle, \langle A_0, \ldots, A_{k-1}\rangle} \cap \lambda_\xi = A_{\langle A_0, \ldots, A_{k-1}\rangle, \langle C_0^\xi, \ldots, C_{k-1}^\xi\rangle} \cap \lambda_\xi,$$
$$\bigcup B_0^\xi < \bigcup A_0 < \bigcup C_0^\xi.$$

Now the choice of this $\langle A_0, \ldots, A_{k-1}\rangle$ and $X = \bigcup \{B_{k-1}^\xi \cup C_{k-1}^\xi: \xi < \tau\}$ disproves $(\kappa, \lambda, \tau)^k$-independence of our system. Hence we assume that $f(\langle A_0, \ldots, A_{k-1}\rangle) = \min\{\xi < \tau: \langle A_0, \ldots, A_{k-1}\rangle \in S_\xi\}$ is well defined on

$V(U^{k-1}(\kappa, \tau))$. Put

$$g(\langle A_0, \ldots, A_{k-1}\rangle) = \langle f(\langle A_0, \ldots, A_{k-1}\rangle), \{A_{\langle B_0,\ldots,B_{k-1}\rangle,\langle A_0,\ldots,A_{k-1}\rangle} \cap \lambda_{f(\langle A_0,\ldots,A_{k-1}\rangle)}: \bigcup B_0 < \bigcup A_0\}\rangle.$$

g constitutes a coloring of $V(U^{k-1}(\kappa, \tau))$ with $\sum\{2\uparrow(2\uparrow\lambda_\xi): \xi < \tau\} = \lambda$ colors, so by our inductive assumption there exists an

$$\{\langle A_0, \ldots, A_{k-1}\rangle, \langle C_0, \ldots, C_{k-1}\rangle\} \in E(U^{k-1}(\kappa, \tau)),$$

with $g(\langle A_0, \ldots, A_{k-1}\rangle) = g(\langle C_0, \ldots, C_{k-1}\rangle)$ and $\bigcup A_0 < \bigcup C_0$. Put $\xi = f(\langle A_0, \ldots, A_{k-1}\rangle) = f(\langle C_0, \ldots, C_{k-1}\rangle)$ and we know that

$$\{A_{\langle B_0,\ldots,B_{k-1}\rangle,\langle A_0,\ldots,A_{k-1}\rangle} \cap \lambda_\xi: \bigcup B_0 < \bigcup A_0\} = \{A_{\langle B_0,\ldots,B_{k-1}\rangle,\langle C_0,\ldots,C_{k-1}\rangle} \cap \lambda_\xi: \bigcup B_0 < \bigcup C_0\}$$

so there exists a $\langle B_0, \ldots, B_{k-1}\rangle \in V(U^{k-1}(\kappa, \tau))$ with

$$A_{\langle B_0,\ldots,B_{k-1}\rangle,\langle A_0,\ldots,A_{k-1}\rangle} \cap \lambda_\xi = A_{\langle A_0,\ldots,A_{k-1}\rangle,\langle C_0,\ldots,C_{k-1}\rangle} \cap \lambda_\xi$$

which contradicts $\langle A_0, \ldots, A_{k-1}\rangle \in S_\xi$. □

Theorem 4.6. *$P^k(\kappa^+, \kappa, \kappa)$ holds for $\kappa \geq \omega$.*

Proof. Our graph is the direct generalization of the one described in Theorem 3.4. Put $V = \{\langle \alpha_0, \ldots, \alpha_k\rangle: \alpha_0 < \alpha_1 < \cdots < \alpha_k < \kappa^+\}$, $\{\alpha_0, \ldots, \alpha_k\}$, $\{\beta_0, \ldots, \beta_k\}$ are joined if $\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots < \alpha_k < \beta_k$. $\mathrm{Chr}(G) = \kappa^+$ (see [3]) and G has a local $(\kappa^+, \kappa)^k$-coloring since $f''\Gamma^k(\langle \alpha_0, \ldots, \alpha_n\rangle)$ if f is chosen as $f(\langle \alpha_0, \ldots, \alpha_n\rangle) = \alpha_0$. □

In the next part we investigate the finite subgraphs of large local chormatic graphs.

Lemma 4.7. *A graph on $|V(G)| = \kappa^+$ has a local $(\kappa^+, \kappa)^k$-coloring if and only if $\mathrm{Chr}(\Gamma^k(x)) \leq \kappa$ for every $x \in V(G)$.*

Proof. One direction is trivial. For the other assume that $V(G) = \kappa^+$, $f_\alpha: \Gamma^k(\alpha) \to \kappa$ witnesses $\mathrm{Chr}(\Gamma^k(\alpha)) \leq \kappa$, for $\alpha < \kappa^+$. Whenever $\xi < \kappa^+$, put $\gamma(\xi) = \min\{\alpha < \kappa^+: \alpha \neq \xi$ and $\xi \in \Gamma^k(\alpha)\}$ and take $g(\xi) = \langle \gamma(\xi), f_{\gamma(\xi)}(\xi)\rangle$. We show that g is a local $(\kappa^+, \kappa)^k$-coloring. If α is fixed, $g''\Gamma^k(\alpha) \subseteq \{g(\alpha)\} \cup \{\langle \beta, \tau\rangle: \beta \leq \alpha, \tau < \kappa\}$ which is of size $\leq \kappa$. Assume that $\xi \neq \eta$ and $g(\xi) = g(\eta)$. Then $\gamma(\xi) = \gamma(\eta) = \gamma$, so $\xi, \eta \in \Gamma^k(\gamma)$, $f_\gamma(\xi) = f_\gamma(\eta)$, ξ, η are not joined. □

Corollary 4.8. (a) *Let H be a finite graph with a vertex x such that* $\mathrm{Chr}(H -$

$\{x\}) \leq 2$ (e.g. *any circuit*). *If G is a graph, $|V(G)| = \kappa^+$, G has no local (κ^+, κ)-coloring, then G contains a copy of H.*

(b) *If H is a finite graph such that for every $x \in V(H)$ $\mathrm{Chr}(H - \{x\}) \geq 3$, then there is a graph G on κ^+ with no local (κ^+, κ)-coloring and with no H as subgraph.*

(c) *If G is a graph on κ^+ and G does not contain odd circuits of length $\leq 2k+1$, then G has a local $(\kappa^+, \kappa)^k$-coloring.*

Proof. (a) By Lemma 4.7 there is an $\alpha < \kappa^+$ with $\mathrm{Chr}(\Gamma(\alpha)) = \kappa^+$ and an old theorem of Erdös–Hajnal ([3]) states that $\Gamma(\alpha)$ must contain every finite bipartite graph.

(b) Let s be so large that for every $x \in V(H)$, $H - \{x\}$ contains odd circuits of length $\leq 2s+1$. By another theorem of Erdös–Hajnal ([3]), there is a graph K with $\mathrm{Chr}(K) = |V(K)| = \kappa^+$ and without odd circuits of length $\leq 2s+1$. Join a point $y \notin V(K)$ to every point of $V(K)$. The resulting graph on $\{y\} \cup V(K)$ has no local (κ^+, κ)-coloring and does not contain H, either.

(c) In this case $\mathrm{Chr}(\Gamma^k(\alpha)) \leq 2$ for $\alpha < \kappa^+$, so we are done by Lemma 4.7. □

For larger cardinals the situation is different.

Theorem 4.9. *For $j < \omega \leq \kappa$ there is a graph on κ^{++} with no local (κ^{++}, κ)-coloring and without odd circuits, of length $\leq 2j+1$.*

Proof. Our graph will be the Specker graph: $V(G) = [\kappa^{++}]^{2j^2+1}$ and $x_0 < \cdots < x_{2j^2}$ is joined to $y_0 < \cdots < y_{2j^2}$ if $x_{j+i} < y_i < x_{j+i+1}$ for every $0 \leq i \leq 2j^2 - j$. This graph has no odd circuits of length $\leq 2j+1$ (see [3]), we show that it has no local (κ^{++}, κ)-coloring, either. Assume that $f\colon [\kappa^{++}]^{2j^2+1} \to \kappa^{++}$ is one. Let $r \leq 2j^2 - j$ and fix a sequence $\alpha_0 < \alpha_1 < \cdots < \alpha_r < \kappa^{++}$. Put

$$A = \{\{\beta_0, \ldots, \beta_{2j^2}\} \colon \alpha_{t+j} < \beta_t < \alpha_{t+j+1} \text{ for } t \leq r\}.$$

We show that $|f''A| \leq \kappa$. Once this is proved for $r = 0$, we get that the graph on $[\kappa^{++} - \alpha_j]^{2j^2+1}$ is κ-chromatic, a contradiction to [3]. Also, the claim is true for $r = 2j^2 + j$, by the properties of local coloring. For general r we prove the assertion by reverse induction, assume it is true for $r+1$. Put

$$A = \{\{\beta_0, \ldots, \beta_{2j^2}\} \colon \alpha_{t+j} < \beta_t < \alpha_{t+j+1} \text{ for } t < r \text{ and } \alpha_{j+r} < \beta_r < \alpha\}.$$

$|f''A_\alpha| \leq \kappa$ by hypothesis, and A is the increasing union of $\{A_\alpha \colon \alpha < \kappa^{++}\}$. If $|f''A| \geq \kappa^+$, there is a $\beta < \kappa^{++}$ with $|f''A_\beta| \geq \kappa^+$, a contradiction. □

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