AN EXTREMUM PROBLEM CONCERNING ALGEBRAIC POLYNOMIALS

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Let S_n be the set of all polynomials whose degree does not exceed n and whose all zeros are real but lie outside $(-1, 1)$. Similarly, we say $p_n \in Q_n$ if $p_n(x)$ is a real polynomial whose all zeros lie outside the open disk with center at the origin and radius 1. Further we will denote by H_n the set of all polynomials of degree $\leqq n$ and of the form

$$p_n(x) = \sum_{k=0}^{n} a_k q_{nk}(x), \quad \text{with} \quad a_k \geqq 0, \quad k = 0, 1, 2, \ldots, n, \tag{1.1}$$

where $q_{nk}(x) = (1+x)^k(1-x)^{n-k}$. Elements of H_n are called polynomials with positive coefficients (in $1-x$ and $1+x$) by G. G. Lorentz.

The following inequalities for derivatives of polynomials of special type are known:

THEOREM A (P. Erdős). *Let $p_n \in S_n$ then*

$$\max_{-1 \leqq x \leqq 1} |p_n'(x)| \leqq \frac{1}{2} en \max_{-1 \leqq x \leqq 1} |p_n(x)|.$$

Further, the constant $\frac{1}{2}e$ can not be replaced by a smaller one.

THEOREM B (G. G. Lorentz). *Let $p_n \in H_n$ then for each $r = 1, 2, \ldots$ there exists a constant C_r for which*

$$\max_{-1 \leqq x \leqq 1} |p_n^{(r)}(x)| \leqq C_r n^r \max_{-1 \leqq x \leqq 1} |p_n(x)|. \tag{1.2}$$

THEOREM C (J. T. Scheick). *If $p_n \in H_n$ and $n \geqq 1$ then*

$$\max_{-1 \leqq x \leqq 1} |p_n'(x)| \leqq \frac{1}{2} en \max_{-1 \leqq x \leqq 1} |p_n(x)|, \tag{1.3}$$

$$\max_{-1 \leqq x \leqq 1} |p_n''(x)| \leqq en(n-1) \max_{-1 \leqq x \leqq 1} |p_n(x)|. \tag{1.4}$$

THEOREM D (A. K. Varma). *Let $p_n \in S_n$, then we have*

$$\int_{-1}^{1} (1-x^2)(p_n'(x))^2 \, dx \leqq \frac{n(n+1)(2n+3)}{4(2n+1)} \int_{-1}^{1} (1-x^2) p_n^2(x) \, dx \tag{1.5}$$

with equality for $p_n(x)=(1+x)^n$ *or* $p_n(x)=(1-x)^n$. *Moreover if* $p_n(1)=p_n(-1)=0$ *then for* $n\geqq 2$

$$\int_{-1}^{1}(p_n'(x))^2\,dx \leqq \frac{n(2n+1)(n-1)}{4(2n-3)}\int_{-1}^{1}(p_n(x))^2\,dx, \tag{1.6}$$

equality holds for only $p_n(x)=c(1+x)(1-x)^{n-1}$ *or* $p_n(x)=c(1-x)(1+x)^{n-1}$.

It is known [2] that if $p_n\in S_n$ (or $p_n\in Q_n$) then $p_n\in H_n$ or $-p_n\in H_n$. Thus Theorem B as well Theorem C can be looked as a generalization of Theorem A. Similarly Theorem D is an extension of Theorem A in L_2 norm for $p_n\in S_n$. The object of this paper is to extend Theorem B as well as Theorem D in L_2 norm for $p_n\in H_n$.

THEOREM 1. *Let* $p_n\in H_n$ *then for* $n\geqq 2$

$$\int_{-1}^{1}(p_n'(x))^2\,dx \leqq \frac{n(n-1)(2n+1)}{4(2n-3)}\int_{-1}^{1}(p_n(x))^2\,dx, \tag{1.7}$$

equality holds iff $p_n(x)=c(1+x)^{n-1}(1-x)$ *or* $p_n(x)=c(1+x)(1-x)^{n-1}$.

THEOREM 2. *Let* $p_n\in H_n$ *then*

$$\int_{-1}^{1}(1-x^2)(p_n'(x))^2\,dx \leqq \frac{n(n+1)(2n+3)}{4(2n+1)}\int_{-1}^{1}(1-x^2)\,p_n^2(x)\,dx \tag{1.8}$$

with equality for $p_n(x)=(1+x)^n$ *or* $p_n(x)=(1-x)^n$.

COROLLARY. *If* $p_n\in Q_n$ *then* (1.7) *and* (1.8) *are valid.*

2. Some lemmas. For the proof of Theorem 1 and Theorem 2 we need the following lemmas.

LEMMA 2.1. *Let* $p_n\in H_n$. *Then we have*

$$\int_{-1}^{1}(1-x^2)\,p_n^2(x)\,dx \geqq \frac{2(2n+1)}{(n+1)(2n+3)}\int_{-1}^{1}p_n^2(x)\,dx. \tag{2.1}$$

PROOF. From (1.1) we have

$$p_n^2(x) = \sum_{p+q=2n} a_{pq}(1+x)^p(1-x)^q, \quad a_{pq}\geqq 0. \tag{2.2}$$

Hence we may write

$$\int_{-1}^{1}(1-x^2)\,p_n^2(x)\,dx = \sum_{p+q=2n} a_{pq}\int_{-1}^{1}(1+x)^{p+1}(1-x)^{q+1}\,dx.$$

But on using

$$\frac{\int_{-1}^{1}(1+x)^{p+1}(1-x)^{q+1}\,dx}{\int_{-1}^{1}(1+x)^{p}(1-x)^{q}\,dx} = \frac{4(p+1)(q+1)}{(p+q+3)(p+q+2)} \tag{2.3}$$

and simple computation the lemma follows. Note that equality in (2.1) holds for $p_n^2(x)=(1+x)^{2n}$ or $p_n^2(x)=(1-x)^{2n}$.

LEMMA 2.2. *Let* $p_n \in H_n$ *and suppose that*

$$p_n(1) = p_n(-1) = 0. \tag{2.4}$$

Then for $n \geqq 2$ *we have*

$$\frac{\int_{-1}^{1} (p_n'(x))^2\,dx}{\int_{-1}^{1} (p_n(x))^2\,dx} \leqq \frac{n(n-1)(2n+1)}{4(2n-3)}, \tag{2.5}$$

equality iff $p_n(x)=(1+x)(1-x)^{n-1}$ *or* $p_n(x)=(1-x)(1+x)^{n-1}$.

PROOF. From (1.1) and (2.4) we may write

$$p_n(x) = \sum_{k=1}^{n-1} a_{kn}(1-x)^k(1+x)^{n-k}, \quad a_{kn} \geqq 0, \quad 1 \leqq k \leqq n-1. \tag{2.6}$$

Therefore

$$\int_{-1}^{1} p_n^2(x)\,dx = \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} a_{kn}a_{jn} \int_{-1}^{1} (1+x)^{2n-k-j}(1-x)^{k+j}\,dx.$$

On using the known formula

$$\int_{-1}^{1} (1-x)^p(1+x)^q\,dx = \frac{2^{p+q+1}\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}, \tag{2.7}$$

we have

$$\int_{-1}^{1} p_n^2(x)\,dx = \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{a_{kn}a_{jn}2^{2n+1}\Gamma(k+j+1)\Gamma(2n-k-j+1)}{\Gamma(2n+2)}. \tag{2.8}$$

Next, we turn to prove that

$$\int_{-1}^{1} (p_n'(x))^2\,dx \leqq \frac{2^{2n-2}(n-1)}{(2n-3)} \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{a_{kn}a_{jn}\Gamma(k+j+1)\Gamma(2n-k-j)}{\Gamma(2n)}. \tag{2.9}$$

To prove (2.9) we first note that if

$$q_{kn}(x) = (1-x)^k(1+x)^{n-k}, \tag{2.10}$$

then

$$q_{kn}'(x) = -k(1-x)^{k-1}(1+x)^{n-k} + (n-k)(1-x)^k(1+x)^{n-k-1}, \tag{2.11}$$

and on using (2.7) we have

$$\begin{aligned} I_{k,j} = \int_{-1}^{1} q_{kn}'(x)q_{jn}'(x)\,dx = \frac{2^{2n-1}}{\Gamma(2n)}\big[&kj\Gamma(k+j-1)\Gamma(2n-j-k+1) + \\ &+(n-k)(n-j)\Gamma(k+j+1)\Gamma(2n-k-j-1) - \\ &-((j(n-k)+k(n-j))\Gamma(k+j)\Gamma(2n-j-k)\big]. \end{aligned} \tag{2.12}$$

After a simple computation it can be shown that

$$I_{k,j} = \frac{2^{2n-1}(n-1)\Gamma(k+j+1)\Gamma(2n-j-k+1)}{\Gamma(2n)}\mu_{k,j} \tag{2.13}$$

where

$$\mu_{k,j} = \frac{n(k+j)-2kj-n(k-j)^2}{(k+j)(k+j-1)(2n-j-k)(2n-j-k-1)}.$$

Next, we will show that for $k, j = 1, 2, \ldots, n-1$

$$\mu_{k,j} \leqq \frac{1}{2(2n-3)}, \tag{2.14}$$

equality holds only for $k=1$, $j=1$, or $k=n-1$, $j=n-1$. In (2.13) let $k+j=l$ then (2.14) is equivalent to

$$l(l-1)(2n-l)(2n-l-1) \geqq (2n-3)[2nl-2n(k-j)^2-4kj]$$

or

$$l(l-1)(2n-l)(2n-l-1) \geqq (2n-3)\{2nl-l^2-(2n-1)(k-j)^2\}$$

or

$$l(2n-l)(l-2)(2n-l-2)+(2n-1)(2n-3)(k-j)^2 \geqq 0.$$

This proves (2.14). Now, one using (2.13) and (2.14) we have

$$\int_{-1}^{1} q'_{kn}(x)q'_{jn}(x)\,dx \leqq \frac{2^{2n-2}(n-1)\Gamma(k+j+1)\Gamma(2n-j-k+1)}{(2n-3)\Gamma(2n)}. \tag{2.15}$$

Now, on using (2.15), (2.10), (2.11), we obtain (2.9). Further from (2.9) and (2.8) we have (2.5). This proves Lemma 2.2.

3. Proof of Theorem 1. Let $p_n \in H_n$. Then from (1.1) we have

$$p_n(x) = a_0(1+x)^n + a_n(1-x)^n + q_n(x) \tag{3.1}$$

where

$$q_n(x) = \sum_{k=1}^{n-1} a_k(1+x)^{n-k}(1-x)^k, \quad a_k \geqq 0. \tag{3.2}$$

We note that $q_n(1) = q_n(-1) = 0$, therefore on using Lemma 2.2 we have

$$\frac{\int_{-1}^{1} q'_n(x)^2\,dx}{\int_{-1}^{1} q_n(x)^2\,dx} \leqq \frac{n}{4}\frac{(2n+1)(n-1)}{2n-3}, \quad n \geqq 2. \tag{3.3}$$

Next, from (3.1) and (3.2) we have

$$\int_{-1}^{1} p_n'(x)^2\,dx = \frac{n^2 2^{2n-1}}{2n-1}(a_0^2+a_n^2) + \int_{-1}^{1} q_n'(x)^2\,dx + \tag{3.4}$$

$$+2n\int_{-1}^{1}\left(a_0(1+x)^{n-1}-a_n(1-x)^{n-1}\right)q_n'(x)\,dx - 2a_0a_nn^2\int_{-1}^{1}(1-x^2)^{n-1}\,dx.$$

By integrating by parts we obtain

$$\int_{-1}^{1} q_n'(x)\{a_0(1+x)^{n-1}-a_n(1-x)^{n-1}\}\,dx = \tag{3.5}$$

$$= -(n-1)\int_{-1}^{1} q_n(x)\{a_0(1+x)^{n-2}+a_n(1-x)^{n-2}\}\,dx \leqq 0.$$

From (3.4) and (3.5) we obtain

$$\int_{-1}^{1} p_n'(x)^2\,dx \leqq \int_{-1}^{1} q_n'(x)^2\,dx + \frac{2^{2n-1}n^2(a_0^2+a_n^2)}{2n-1}. \tag{3.6}$$

Also from (3.1) it follows that

$$\int_{-1}^{1} p_n^2(x)\,dx \geqq \frac{(a_0^2+a_n^2)2^{2n+1}}{2n+1} + \int_{-1}^{1} q_n^2(x)\,dx. \tag{3.7}$$

Therefore by (3.6) and (3.7) we have

$$\frac{\int_{-1}^{1} p_n'(x)^2\,dx}{\int_{-1}^{1} p_n^2(x)\,dx} \leqq \frac{\int_{-1}^{1} q_n'(x)^2\,dx + \frac{2^{2n-1}n^2(a_0^2+a_n^2)}{2n-1}}{\int_{-1}^{1} q_n^2(x)\,dx + \frac{(a_0^2+a_n^2)2^{2n+1}}{2n+1}}. \tag{3.8}$$

It is easy to verify that

$$\frac{2^{2n-1}n^2(a_0^2+a_n^2)}{2n-1} < \frac{2^{2n+1}(a_0^2+a_n^2)}{(2n+1)}\,\frac{n}{4}\,\frac{(2n+1)(n-1)}{(2n-3)}. \tag{3.9}$$

Using (3.9) and (3.3) we obtain

$$\frac{\int_{-1}^{1} p_n'(x)^2\,dx}{\int_{-1}^{1} p_n^2(x)\,dx} \leqq \frac{n}{4}\,\frac{(2n+1)(n-1)}{(2n-3)}.$$

This proves Theorem 1.

4. Proof of Theorem 2. Let $p_n \in H_n$. First we write

(4.1)
$$\frac{\int_{-1}^{1}(1-x^2)p_n'(x)^2\,dx}{\int_{-1}^{1}(1-x^2)p_n^2(x)\,dx} = \frac{\int_{-1}^{1}(1-x^2)p_n'(x)^2\,dx}{\int_{-1}^{1}p_n^2(x)\,dx}\,\frac{\int_{-1}^{1}p_n^2(x)\,dx}{\int_{-1}^{1}(1-x^2)p_n^2(x)\,dx}.$$

On using Lemma (2.1) we obtain

$$\text{(4.2)}\qquad \frac{\int_{-1}^{1}p_n^2(x)\,dx}{\int_{-1}^{1}(1-x^2)p_n^2(x)\,dx} \le \frac{(n+1)(2n+3)}{2(2n+1)},$$

equality holds for $p_n(x)=(1+x)^n$ or $p_n(x)=(1-x)^n$. Next, we will prove that for $p_n \in H_n$

$$\text{(4.3)}\qquad \frac{\int_{-1}^{1}(1-x^2)p_n'(x)^2\,dx}{\int_{-1}^{1}p_n^2(x)\,dx} \le \frac{n}{2},$$

equality holds for $p_n(x)=(1+x)^k(1-x)^{n-k}$ $k=0, 1, \ldots, n$. Let $p_n \in H_n$. Then we may write

$$\text{(4.4)}\qquad p_n(x) = \sum_{k=0}^{n} a_{kn}(1-x)^k(1+x)^{n-k} \equiv \sum_{k=0}^{n} a_{kn}\,q_{kn}(x).$$

Following the proof of Lemma 2.2 we first note that

$$\text{(4.5)}\qquad \int_{-1}^{1} q_{kn}'(x)\,q_{jn}'(x)(1-x^2)\,dx = \frac{2^{2n+1}\Gamma(k+j+1)\Gamma(2n-k-j+1)}{\Gamma(2n+2)}\,\mu_{kj}$$

where by $k+j=l$,

$$\text{(4.6)}\quad \mu_{kj} = \frac{(2n-l)(2n-l+1)kj+(n^2-nl+kj)\,l(l+1)-l(2n-l)(nl-2kj)}{l(2n-l)} =$$

$$= \frac{\frac{n}{2}\,l(2n-l)-\frac{n}{2}(2n+1)(k-j)^2}{l(2n-l)} \le \frac{n}{2}$$

equality holds iff $k=j$, $k=0, 1, \ldots, n$. Therefore

$$\text{(4.7)}\qquad \int_{-1}^{1} q_{kn}'(x)q_{jn}'(x)(1-x^2)\,dx \le 2^{2n+1}\frac{n}{2}\,\frac{\Gamma(k+j+1)\Gamma(2n-k-j+1)}{\Gamma(2n+2)}.$$

By using (4.4), (4.7) we have

$$\int_{-1}^{1} (1-x^2)p_n'(x)^2\,dx \leqq \frac{2^{2n+1}}{\Gamma(2n+2)}\frac{n}{2}\sum_{k=0}^{n}\sum_{j=0}^{n} a_{kn}a_{jn}\Gamma(k+j+1)\Gamma(2n-k-j+1) =$$

$$= \frac{n}{2}\int_{-1}^{1} p_n^2(x)\,dx.$$

This proves (4.3). Now, using (4.1)—(4.3) we have

$$\frac{\int_{-1}^{1} p_n'(x)^2(1-x^2)\,dx}{\int_{-1}^{1} p_n(x)^2(1-x^2)\,dx} \leqq \frac{n}{2}\frac{(n+1)(2n+3)}{2(2n+1)}.$$

This proves Theorem 2 as well.

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