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ENTIRE FUNCTIONS BOUNDED OUTSIDE A FINITE AREA

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Dedicated To G. Pólya and G. Szegő with respect and affection

0. Introduction

Let $f(z)$ be an entire function. Consider the (open) set of the z-plane defined by

$$\{z:\ |f(z)|>B\} \quad (B>0), \tag{1}$$

and let

$$\mu\big(|f(z)|>B\big) \tag{2}$$

denote its area (that is its 2-dimensional Lebesgue measure).

QUESTION. *When is it possible that*

$$\mu\big(|f(z)|>B\big)<+\infty, \tag{3}$$

for some suitable B $(0<B<+\infty)$?

Our answer is contained in

THEOREM 1. *Let* $f(z)$ *be entire, transcendental and such that*

$$\limsup_{r\to+\infty}\frac{\log\log\log M(r)}{\log r}<2 \quad \big(M(r)=\max_{|z|=r}|f(z)|\big). \tag{4}$$

Consider, in the z-plane, the set of points

$$E_R=\Big\{z:\ R<|z|<2R,\ \log|f(z)|>\tfrac{1}{2}T(R)\Big\} \quad (R>0), \tag{5}$$

where

$$T(R)=\frac{1}{2\pi}\int_0^{2\pi}\log|f(\mathrm{R}e^{i\theta})|\,d\theta \tag{6}$$

is the characteristic of Nevanlinna.

Then, the open set E_R *has a 2-dimensional Lebesgue measure* $\mu(E_R)$ *which satisfies the condition*

$$\mu(E_R)>R^{\delta} \quad \big(\delta>0,\ R>R_0(\delta)\big), \tag{7}$$

provided $\delta>0$ *has been chosen small enough.*

If (4) *is replaced by*

$$\liminf_{r\to+\infty}\frac{\log\log\log M(r)}{\log r}<2, \tag{8}$$

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we may only assert that (7) *holds if* R *is restricted to the values* $\{R_j\}_{j=1}^{\infty}$ *of some suitable, increasing, unbounded sequence.*

As an immediate consequence of Theorem 1, we find.

COROLLARY 1.1. *Any entire function* $f(z)$ *satisfying the condition* (8) *cannot satisfy* (3) *for any fixed positive* B.

To verify that Theorem 1 is sharp, we establish the

PROPERTIES OF A SPECIAL FUNCTION. *The entire function* $\Phi(z)$, *introduced below, is such that*

$$\lim_{r\to+\infty} \frac{\log\log\log M(r)}{\log r} = 2, \quad M(r) = \max_{|z|=r} |\Phi(z)|. \tag{9}$$

It satisfies the condition

$$\mu(|\Phi(z)| > B) < +\infty, \tag{10}$$

for some suitable finite B.

Our function $\Phi(z)$ shows that the assertions of Theorem 1 no longer hold if, in (4) and (8), the symbols <2 are replaced by $\leqq 2$.

The function $\Phi(z)$ is initially introduced as an integral:

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\exp(\exp((\zeta \log \zeta)^2))}{\zeta - z} d\zeta \quad (\operatorname{Re} z < e^2), \tag{11}$$

where the contour of integration Γ is the boundary of the open set

$$\Omega = \left\{ z = x+iy\colon x > e^2, \ -\frac{\pi}{2x(\log x)^2} < y < \frac{\pi}{2x(\log x)^2} \right\}. \tag{12}$$

The orientation on Γ is the one that always leaves Ω on the right-hand side.

By modifying Γ, in (11), we verify that $\Phi(z)$ may be continued throughout the complex plane and is therefore an entire function.

The properties of $\Phi(z)$, which may have some independent interest, are summarized in our

THEOREM 2. *The entire function* $\Phi(z)$ *is real for real values of* z *and has the following properties.*

I. *There exists some constant* B_1 *such that*

$$\left(\Phi(z) - \frac{B_1}{z}\right) z^2 \quad (z \neq 0) \tag{13}$$

remains bounded for

$$z \notin S = \{z = x+iy\colon x > 0, \ -1 < y < 1\}. \tag{14}$$

II. *The expression*

$$\Phi(z) \frac{z}{(\log |z|)^2} \tag{15}$$

remains bounded for

(16) $$|z| > e, \quad z \in S, \quad z \notin \Omega.$$

III. *The expression*

(17) $$\{\Phi(z) - \exp(\exp((z \log z)^2))\} \frac{z}{(\log |z|)^2}$$

remains bounded for $z \in \Omega$.

Our construction of $\Phi(z)$, and our proof of Theorem 2, are straightforward adaptations of a similar construction and a similar proof given by Pólya and Szegő [3; pp. 115—116, ex. 158, 159, 160].

It follows from Theorem 2 that

(18) $$\lim_{r \to \infty} \frac{\log \log M(r)}{(r \log r)^2} = 1,$$

which implies (9), and is clearly more precise. From assertions I and II of Theorem 2 we deduce the existence of a bound B $(0 < B < +\infty)$ such that $|\Phi(z)| \leqq B$ $(z \notin \Omega)$. As to the area of Ω, our definition (12) implies that it is equal to

(19) $$\pi \int_{e^2}^{+\infty} \frac{d\sigma}{\sigma (\log \sigma)^2} = \frac{\pi}{2}.$$

We have thus established the second property (stated above as (10)) of our special function $\Phi(z)$.

1. Proof of Theorem 1

We take for granted the following wellknown results of Nevanlinna's theory [2].

I. *The characteristic* $T(r)$, *introduced in* (6), *is a continuous, increasing function of* $r > 0$ *and*

(1.1) $$\frac{T(r)}{\log r} \to +\infty \quad (r \to +\infty),$$

provided $f(z)$ *does not reduce to a polynomial.*

II. *The functions* $T(r)$ *and* $\log M(r)$ *are connected by the double inequality* [2; p. 24]

(1.2) $$T(r) \leqq \log M(r) \leqq \frac{t+r}{t-r} T(t), \quad (0 < r < t).$$

In particular

(1.3) $$\frac{1}{3} \log M\left(\frac{R}{2}\right) \leqq T(R).$$

Let $U(r) > 1$ be a continuous, nondecreasing unbounded function of $r > 0$. A well-known fundamental result of E. Borel implies the following: given $\varepsilon > 0$, it is possible to find $R_0 = R_0(\varepsilon)$ such that if

(1.4) $$R_0 < R \leqq r \leqq 2R, \quad r \notin \mathscr{E}_1(R),$$

then

$$U\left(r+\frac{r}{\{\log U(r)\}^{1+\varepsilon}}\right) < eU(r). \tag{1.5}$$

The exceptional set $\mathscr{E}_1(R)$ is a measurable subset of the interval $[R, 2R]$ and its Lebesgue linear measure $\lambda(\mathscr{E}_1(R))$ is such that

$$\frac{\lambda(\mathscr{E}_1(R))}{R} \to 0 \quad (R \to +\infty). \tag{1.6}$$

The consequences of Borel's lemma stated in (1.4), (1.5) and (1.6) are found in a paper of Edrei and Fuchs [1; p. 341].

In the following proof we apply (1.5) with $U(r)$ replaced by $T(r)$ and always take R large enough to imply

$$\lambda(\mathscr{E}_1(R)) < \frac{R}{2}, \quad \log U(R) > 1. \tag{1.7}$$

Hence, taking

$$t = \frac{r}{\{\log T(r)\}^{1+\varepsilon}},$$

we deduce from (1.2), (1.5) and (1.7)

$$\log M(r) < 3e\, T(r)\{\log T(r)\}^{1+\varepsilon}, \tag{1.8}$$

provided

$$r \in D_R = \{r\colon\ R < r < 2R, \quad r \notin \mathscr{E}_1(R)\} \quad (R > R_0). \tag{1.9}$$

In view of (1.7), the one-dimensional set D_R has Lebesgue measure

$$\lambda(D_R) > \frac{R}{2}. \tag{1.10}$$

Introduce the set of values of θ defined by

$$\Lambda(r) = \left\{\theta\colon\ \log|f(re^{i\theta})| > \tfrac{1}{2}T(R),\ 0 < \theta < 2\pi\right\}; \tag{1.11}$$

for every $r>0$, $\Lambda(r)$ is an open subset of the interval $(0, 2\pi)$. Denote by $\lambda(\Lambda(r))$ the one-dimensional Lebesgue measure of $\Lambda(r)$. The definition of $\mu(E_R)$, as a two-dimensional Lebesgue measure, and Fubini's theorem yield

$$\mu(E_R) = \iint r\,dr\,d\theta = \int_R^{2R} r\,dr \int_{\Lambda(r)} d\theta = \int_R^{2R} r\lambda(\Lambda(r))\,dr, \tag{1.12}$$

where the double integral in (1.12) is extended to all points $z=re^{i\theta}\in E_R$.

By (1.9) and (1.12)

$$\mu(E_R) \geqq \int_{\Delta_R} r\lambda(\Lambda(r))\,dr. \tag{1.13}$$

To complete the proof we note that the definition of $T(r)$ (in (6)) and (1.11) imply

$$T(r) \leqq \frac{1}{2\pi} \int_{\Lambda(r)} \log M(r)\, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} T(R)\, d\theta.$$

Hence, in view of (1.8), (1.9) and the increasing character of $T(r)$, we find

$$\frac{1}{2} T(r) < \frac{3e}{2\pi} T(r) \left(\log T(r)\right)^{1+\varepsilon} \lambda\left(\Lambda(r)\right) \quad (r \in D_R,\ r > r_0),$$

$$\lambda\left(\Lambda(r)\right) > e^{-1} \left(\log T(r)\right)^{-1-\varepsilon} \quad (r \in D_R,\ r > r_0),$$

which used in (1.13) yields

$$\mu(E_R) \geqq e^{-1} \int_{\Delta_R} r \{\log T(2R)\}^{-1-\varepsilon}\, dr \geqq e^{-1} R \{\log T(2R)\}^{-1-\varepsilon} \lambda(D_R),$$

and finally by (1.10)

$$\mu(E_R) > \tfrac{1}{2} e^{-1} R^2 \{\log T(2R)\}^{-1-\varepsilon} \quad \left(R > R_0(\varepsilon)\right). \tag{1.14}$$

Up to this point we have not selected $\varepsilon > 0$, nor have we used (4) or the weaker assumption (8).

Assume for instance that (8) holds. Then, if $\eta > 0$ is small enough,

$$\log T(r) \leqq \log \log M(r) < r^{2(1-\eta)}, \tag{1.15}$$

as $r \to +\infty$ by values of a suitable increasing, unbounded sequence which we may write as $\{2R_j\}_{j=1}^{\infty}$. Take, in (1.14), $\eta = \varepsilon$, $R = R_j$ and note that since (1.15) now implies

$$\left(\log T(2R_j)\right)^{1+\varepsilon} < (2R_j)^{2(1-\eta^2)} \quad \left(j > j_0(\eta)\right),$$

we obtain

$$\mu(E_R) > (e^{-1}/8) R^{2\eta^2} \quad \left(R = R_j,\ j > j_0(\eta)\right). \tag{1.16}$$

This proves that, under the assumption (8), (7) holds with $R = R_j$, $j > j_0$.

The validity of (7) under the assumption (4) is obvious because then (1.16) holds for all sufficiently large values of R and not only for $R = R_j$. The proof of the Theorem is now complete.

2. Contours of integration

Let σ be a positive variable and γ a positive parameter which is restricted by the conditions

$$\frac{3}{4} \leqq \gamma \leqq \frac{5}{4}. \tag{2.1}$$

Assume that γ is fixed and consider, in the complex plane, the analytic arc described by

$$\zeta(\sigma;\, \gamma) = \sigma + i\tau(\sigma;\, \gamma), \quad \tau(\sigma;\, \gamma) = \frac{\pi\gamma}{2\sigma (\log \sigma)^2} \quad (e \leqq \sigma < +\infty). \tag{2.2}$$

We denote by $L_+(\alpha;\gamma)$ the arc described by $\zeta(\sigma;\gamma)$ as $\alpha\leqq\sigma<+\infty$, by $L_-(\alpha;\gamma)$ the symmetrical arc described by $\sigma-i\tau$ and by $V(\alpha;\gamma)$ the vertical segment

(2.3) $$V(\alpha;\,\gamma)=\{z=x+iy\colon\, x=\alpha,\ -\tau(\alpha;\,\gamma)\leqq y\leqq\tau(\alpha;\,\gamma)\}.$$

Denoting, as usual, opposite arcs by L and $-L$, we consider systematically contours of integration

(2.4) $$C(\alpha;\,\gamma)=-L_-(\alpha;\,\gamma)+V(\alpha;\,\gamma)+L_+(\alpha;\,\gamma)\quad\left(\alpha\geqq e,\ \frac{3}{4}\leqq\gamma\leqq\frac{5}{4}\right).$$

All the points $z\notin C(\alpha;\gamma)$ fall in two disjoint open regions. One of them:

(2.5) $$\Delta(\alpha;\gamma)=\{z=x+iy,\ x>\alpha,\ -\tau(x;\gamma)<y<\tau(x,\gamma)\}$$

has a finite area. (This fact is an obvious consequence of (19)).

The other one, which contains the whole negative axis, will be denoted by $\tilde{\Delta}(\alpha;\gamma)$.

3. The function $\Phi(z)$ is entire

Consider in the half-plane $\operatorname{Re} z\geqq 2$ the analytic function

(3.1) $$F(z)=\exp(e^{(z\log z)^2})\quad(\log e=1),$$

where the branch of $\log z$ is determined by its value at e.

We shall first verify that for any $\gamma\in[3/4,\,5/4]$

(3.2) $$\int\limits_{L_+(e^2,r)}|F(\zeta)|\,|d\zeta|=\int\limits_{e^2}^{+\infty}|F(\zeta)|\left|\frac{d\zeta}{d\sigma}\right|d\sigma<+\infty.$$

This follows at once from

(3.3) $$\frac{d\zeta}{d\sigma}\to 1\quad(\sigma\to+\infty,\ \gamma\text{ fixed})$$

and from the elementary estimates contained in

LEMMA 3.1. *If* $\zeta\in L_+(e^2;\gamma)3/4\leqq\gamma\leqq 5/4)$ *then*

(3.4) $$F(\zeta)=\exp\left(e^{(\sigma\log\sigma)^2}e^{i\pi\gamma}\left\{1+\frac{A\omega}{\log\sigma}\right\}\right)\quad(\operatorname{Re}\zeta=\sigma\geqq e^2,\quad\omega=\omega(\sigma,\gamma)),$$

where, in the error term,

$0<A=$*absolute const.*, $|\omega(\sigma,\gamma)|\leqq 1$.

Moreover, if $\alpha\geqq\alpha_0>e^2$ *and if* α_0 *is large enough, then*

(3.5) $$\left|F\left(\alpha+\frac{i\pi\gamma}{2\alpha(\log\alpha)^2}\right)\right|\leqq\exp\left(-\frac{1}{2}e^{(\alpha\log\alpha)^2}\right)\quad\left(\alpha\geqq\alpha_0,\ \frac{3}{4}\leqq\gamma\leqq\frac{5}{4}\right).$$

PROOF. An elementary evaluation shows that (2.1) and (2.2) imply

$$(3.6)\qquad (\zeta\log\zeta)^2=(\sigma\log\sigma)^2+i\pi\gamma+\frac{A\omega}{\log\sigma}\quad(\omega=\omega(\sigma,\gamma),\quad|\omega|\leqq 1).$$

In (3.6), and throughout the paper, we denote by ω a complex quantity, which may depend on all the parameters of the problem, but is always of modulus $\leqq 1$. The symbol ω, as well as A (our symbol for positive absolute constants), may assume different values at each occurrence.

We note that, with this convention,

$$(3.7)\qquad e^u=1+\omega u e^{|u|}.$$

It is obvious that (3.6) and (3.7) yield (3.4). Observing that

$$\operatorname{Re} e^{i\pi\gamma}\left\{1+\frac{A\omega}{\log\alpha}\right\}\leqq\cos(\pi\gamma)+\frac{A}{\log\alpha}<-\frac{1}{2}\quad\left(\frac{3}{4}\leqq\gamma\leqq\frac{5}{4},\ \alpha\geqq\alpha_0\right),$$

we deduce (3.5) from (3.4).

This completes the proof of Lemma 3.1.

Now the integrals in (3.2) are clearly convergent by (3.3) and (3.4). Noticing that the contour Γ, which appears in the definition (11) of $\Phi(z)$, coincides with $C(e^2;1)$ defined in (2.4), we may rewrite

$$(3.8)\qquad \Phi(z)=\frac{1}{2\pi i}\int_{C(e^2;1)}\frac{F(\zeta)}{\zeta-z}\,d\zeta\quad(\operatorname{Re} z<e^2).$$

This shows that $\Phi(z)$ is a function holomorphic in the half-plane

$$(3.9)\qquad \operatorname{Re} z<e^2.$$

The fact that $C(e^2;1)$ has the real axis for axis of symmetry, and that $F(z)$ is real for real z, shows that $\Phi(z)$ is real for real z.

By Cauchy's theorem, under the restriction (3.9), we may replace the representation (3.8) by

$$(3.10)\qquad \Phi(z)=\frac{1}{2\pi i}\int_{C(\alpha;1)}\frac{F(\zeta)}{\zeta-z}\,d\zeta\quad(\alpha>e^2)$$

and let $\alpha\to+\infty$. This step is certainly justified because $F(z)$ is holomorphic throughout $\operatorname{Re} z\geqq 2$. The form (3.10) shows that our original function, given by (3.8), may be continued throughout $\operatorname{Re} z<\alpha$. Hence $\Phi(z)$ is in fact an entire function.

4. Proof of assertions I and II of Theorem 2

If $z\in\tilde{\Delta}(e^2;1)$, Cauchy's theorem and (3.5) show that we may use the representation

$$(4.1)\qquad \Phi(z)=\frac{1}{2\pi i}\int_{C(e^2;3/4)}\frac{F(\zeta)}{\zeta-z}\,d\zeta,$$

instead of (3.8).

Using in (4.1) the identity

$$\frac{1}{\zeta-z} = -\frac{1}{z}-\frac{\zeta}{z^2}+\frac{\zeta^2}{z^2(\zeta-z)} \quad (z \neq 0), \tag{4.2}$$

and writing

$$B_1 = -\frac{1}{2\pi i}\int_{C_1} F(\zeta)\,d\zeta, \quad B_2 = -\frac{1}{2\pi i}\int_{C_1} \zeta F(\zeta)\,d\zeta, \quad C_1 = C\left(e^2, \frac{3}{4}\right), \tag{4.3}$$

we find

$$\Phi(z) = \frac{B_1}{z}+\frac{B_2}{z^2}+\frac{1}{2\pi i z^2}\int_{C_1} \frac{\zeta^2 F(\zeta)}{\zeta-z}\,d\zeta \quad \left(z\in\tilde{\Delta}(e^2\colon 1)\right). \tag{4.4}$$

To complete the proof of assertions I and II of Theorem 2, there only remains to estimate the integral in (4.4). It is clear that its modulus cannot exceed

$$\frac{1}{\delta_1(z)}\int_{C_1} |\zeta|^2 F(\zeta)\,|d\zeta|, \tag{4.5}$$

where $\delta_1(z)$ denotes the shortest distance between z and the contour C_1.

If $z \notin S$, an inspection of (12) and (14) shows that

$$\delta_1(z) > (9/10), \tag{4.6}$$

and hence (4.4) yields

$$\left|\Phi(z)-\frac{B_1}{z}-\frac{B_2}{z^2}\right| \leq \frac{(10/9)}{2\pi |z|^2}\int_{C_1} |\zeta|^2 F(\zeta)|\,d\zeta = \frac{B_3}{|z|^2}.$$

Assertion I of Theorem 2 is now obvious. To obtain assertion II of Theorem 2 it suffices to replace, in the previous proof, the inequality (4.6) by another one, valid under the restrictions (16).

If

$$\operatorname{Re} z = x > e^2+1, \quad y \geq \frac{\pi}{2x(\log x)^2},$$

we have

$$\delta_1(z) \geq \frac{\pi}{2x(\log x)^2} - \max_{x-1\leq\sigma\leq x+1} \frac{3\pi}{8\sigma \log \sigma)^2} = \tag{4.7}$$

$$= \frac{\pi}{2}\left(\frac{1}{x(\log x)^2}-\frac{3}{4(x-1)(\log(x-1))^2}\right),$$

$$\delta_1(z) \geq \frac{\pi}{10x(\log x)^2} \quad (x \geq x_0 > e^2+1) \tag{4.8}$$

provided x_0 is chosen large enough. Using (4.8) in (4.5) and returning to (4.4) we find, for some suitable constant $B_4 > 0$,

$$\Phi(z) = \frac{B_1}{z}+\frac{B_2}{z^2}+\frac{B_4\omega}{z^2}x(\log x)^2 \quad \left(z\in\tilde{\Delta}(x_0, 1)\right).$$

Hence the expression (15) remains bounded for

$$|z| \geqq x_0+1, \quad z\in S, \quad z\notin \Omega. \tag{4.9}$$

Since $\Phi(z)$ is entire it is also bounded in the disk $|z|\leqq x_0+1$. This enables us to replace the restrictions (4.9) by the less restrictive conditions (16). The proof of assertion II of Theorem 2 is now complete.

5. Proof of assertion III of Theorem 2

We first confine z to an open rectangle

$$\mathscr{R} = \left\{z = x+iy\colon\, e^2-1 < x < e^2, -\frac{\pi}{8e^2} < y < \frac{\pi}{8e^2}\right\}. \tag{5.1}$$

Let H be the contour of integration formed by the boundary of $\mathscr{R}$, taken in the positive sense. A first application of Cauchy's theorem yields

$$\frac{1}{2\pi i}\int_H \frac{F(\zeta)}{\zeta-z}\,d\zeta = \exp\left(\exp\left((z\log z)^2\right)\right), \tag{5.2}$$

and consequently

$$\Phi(z)-\exp\left(\exp\left((z\log z)^2\right)\right) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{F(\zeta)}{\zeta-z}\,d\zeta, \tag{5.3}$$

where Γ_1 is the contour formed by the juxtaposition of $-L_{-1}(e^2;1)$, three sides of $\mathscr{R}$, and $L_+(e^2;1)$.

It is obvious that the integral in (5.3) yields the analytic continuation of the left-hand side of (5.3) throughout the open region (of finite area) enclosed by Γ_1.

In particular (5.3) is valid for all points $z\in\Omega$. A new application of Cauchy's theorem and (3.5) enable us to replace (5.3) by

(5.4)

$$\Phi(z)-\exp\left(\exp\left((z\log z)^2\right) = \frac{1}{2\pi i}\int_{C_2}\frac{F(\zeta)}{\zeta-z}\,d\zeta \quad \left(C_2 = C\left(e^2;\frac{5}{4}\right),\ z\in\Omega\right).$$

We now repeat the argument in § 4: from (4.2) and (5.4) we see that, instead of (4.4), we obtain

$$\Phi(z)-\exp\left(\exp\left((z\log z)^2\right)\right) = \frac{B_1}{z}+\frac{B_2}{z^2}+\frac{1}{2\pi i z^2}\int_{C_2}\frac{\zeta^2F(\zeta)}{\zeta-z}\,d\zeta \quad (z\in\Omega). \tag{5.5}$$

The constants B_1 and B_2 are again given by (4.3) because, by Cauchy's theorem and (3.5), the values of the relevant integrals are not affected when the contour of integration C_1 is replaced by C_2.

To complete the proof of assertion III of Theorem 2 we need a lower bound for the distance $\delta_2(z)$ between z and C_2. As in (4.7), we find

$$\delta_2(z) \geqq \min_{x-1\leqq\sigma\leqq x+1}\left\{\frac{(5/4)\pi}{2\sigma(\log\sigma)^2}\right\}-\frac{\pi}{2x(\log x)^2}.$$

provided $x \geqq e^2+1$, $z \in \Omega$. Hence, if x_1 is choosen large enough

$$\delta_2(z) > \frac{\pi}{10x(\log x)^2} \quad (x \geqq x_1 \geqq e^2+1). \tag{5.6}$$

Using (5.6) in (5.5) we complete the proof of assertion III of Theorem 2 by the arguments which led to the proof of assertion II.

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