A PROPERTY OF RANDOM GRAPHS

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1. INTRODUCTION AND SUMMARY

A graph G is said to have property $P(k)$ if for any two sets A and B of vertices of G with $A \cap B = \phi$ and $|A \cup B| = k$, there is a vertex $u \notin A \cup B$ which is joined to every vertex of A and not joined to any vertex of B. Note that if a graph G has property $P(k)$, then the complement G^c of G also has property $P(k)$. If we specify the sizes of A and B as x and y respectively $(x + y = k)$ then we denote the above property by $P(k:x,y)$. Let $f(n)$ be the largest integer for which there exists a graph on n vertices having property $P(f(n))$.

In this paper we prove, using probabilistic methods (see Erdös [3], Erdös and Moser [4], Erdös and Spencer [5] and [6]) that

$$\frac{\log n - (2 + o(1)) \log\log n}{\log 2} < f(n) < \frac{\log n}{\log 2} .$$

We do not construct our graphs explicitly.

In a previous paper, Caccetta, Vijayan and Wallis [2] studied graphs having property $P(4:2,2)$. That is, they considered graphs with the property that for any four vertices of the graph there is always another vertex joined to the first two and not joined to the last two. In particular, they showed that the order of such a graph must be at least 34 and verified that all Paley graphs of prime order between 61 and 173 have this property. A consequence of our result on $f(n)$ is that graphs with property $P(4:2,2)$ exist for large n $(n \geq 345)$.

We conclude this paper by giving, for every $n \geq 9$, a class of graphs on n vertices having property $P(2)$. The graph constructed has a certain monotonic property namely an m-vertex graph with property $P(2)$ is obtained from an $(m - 1)$-vertex graph with the same property by adding a new vertex and some edges incident to it.

Graphs with property $P(k:x,y)$ have been studied by other authors; see Exoo [7] for details.

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2. MAIN RESULTS

THEOREM 1: For n sufficiently large

$$\frac{\log n - (2 + o(1)) \log\log n}{\log 2} \le f(n) < \frac{\log n}{\log 2} .$$

PROOF. We first establish the upper bound. Denote by $G(n)$, a graph on n vertices with property $P(k)$. Let $v_1, v_2, \ldots, v_k$ be any set of k vertices from $G(n)$. We can divide this set in 2^k ways into two disjoint subsets A and B. To every such division there corresponds a vertex u_{AB} which is joined to every vertex of A and not to any vertex of B. Thus there are more than $2^k + k$ vertices in our graph. Hence the upper bound.

The proof of the lower bound is slightly more complicated. We would use a probabilistic argument. We choose a random graph G of n vertices in which each of the $\frac{n(n-1)}{2}$ possible edges is chosen with probability $\frac{1}{2}$. We show that if

$$k \le \frac{\log n - (2 + o(1)) \log\log n}{\log 2}$$

then G has the property $P(k)$ with probability greater than 0.

First we observe that, the number of ways of choosing disjoint sets A and B with $|A| + |B| = k$ is less than n^k.

For a fixed A and B with $|A| + |B| = k$, the probability that a given vertex u is joined to every vertex of A and to no vertex of B is clearly

$$\frac{1}{2^{|A|+|B|}} = \frac{1}{2^k} .$$

Therefore the probability that none of the $n - k$ vertices of G not in $A \cup B$ has the required property is

$$\left(1 - \frac{1}{2^k}\right)^{n-k} .$$

G does not satisfy $P(k)$ only if there is at least one choice of A and B for which the required property is not satisfied and this probability is at the most equal to

$$n^k \left(1 - \frac{1}{2^k}\right)^{n-k} .$$

Now,

$$n^k\left(1 - \frac{1}{2^k}\right)^{n-k} < 2n^k\left(1 - \frac{1}{2^k}\right)^n$$
$$< 2n^k e^{-n/2^k} .$$

And if

$$k \leq \frac{\log n - (2 + o(1)) \log\log n}{\log 2}$$

then $2n^k e^{-n/2^k} < 1$.

Hence for all k less than or equal to the lower bound, there exists a graph satisfying $P(k)$ and hence $f(n)$ should at least equal to the lower bound. □

It would be of interest to decide whether the bounds in Theorem 1 can be improved or not. In particular whether the lower bound could be improved to $\frac{\log n}{\log 2} - c$ for some absolute constant c . This problem is similar to an old and slightly related problem of Erdös and Spencer [6]:

> Let S be a set of n elements for which there corresponds to every subset $S_1 \subset S$ an element $h(S_1)$ of S . Denote by $g(n)$ the smallest integer for which there is such a function $h(S_1)$ so that for every subset $S' \subset S$, $|S'| \geq g(n)$ and
>
> $$\bigcup_{S'' \subset S'} h(S'') = S .$$
>
> They proved, by a similar method, that
>
> $$\frac{\log n}{\log 2} - c \log\log n < g(n) < \frac{\log n}{\log 2}$$
>
> and they could never improve this.

Let $G(n,2)$ denote the class of graphs on n vertices having property $P(2)$. Also the class of graphs on n vertices having property $P(4 : 2,2)$ be denoted by $G(n,2,2)$. It was the class $G(n,2,2)$ that was studied in [2]. Using an argument similar to the one employed in the proof of Theorem 1 we can establish the following result concerning the classes $G(n,2)$ and $G(n,2,2)$.

THEOREM 2:

(a) $G(n,2) \neq \phi$ for all $n \geq 28$

(b) $G(n,2,2) \neq \phi$ for all $n \geq 345$ □

It may be of interest to reformulate our general problem in terms of the inverse function $h(n)$ of $f(n)$. That is, $h(n)$ is the smallest integer for which $f(h(n)) \geqq n$, or, in other words, $h(n)$ is the smallest integer for which there exists a graph on $h(n)$ vertices with $f(h(n)) \geqq n$. Our Theorem 1 can then be restated as follows:

THEOREM 3:

$$2^n + n < h(n) < 2^{n+(2+o(1))\log n}$$ □

Remark. Let $a(m,m)$ denote the minimum order among all graphs satisfying property $P(2m : m,m)$. Exoo[7] obtained the following bounds on $a(m,m)$:

$$\frac{m+1}{2} 2^{2m} \leq a(m,m) \leq cm^2 2^{2m}$$

for some positive constant c. Since a graph with property $P(2m)$ or $P(2m+1)$ also has property $P(2m : m,m)$ it follows immediately that

$$h(n) \geq \frac{n}{4} 2^n \quad ,$$

thus improving our lower bound in Theorem 3.

We conclude this section with the following problems.

Problem 1. Is it true that $h(n)/n2^n \to \infty$ or is it the case that $h(n) < cn2^n$ for some constant c?

We feel that

$$\lim_{n \to \infty} \frac{h(n)}{n2^n}$$

exists.

Problem 2. Is it true that if there is a graph $G(n,k)$ on n vertices with property $P(k)$, then there is a graph $G(n+1, k)$ with the same property? If so, can we get a family of graphs $G(n,k)$ such that a $G(m+1, k)$ is obtained by adding a vertex from $G(m,k)$ and some edges incident to it.

In the next section we give a class of graphs for which $G(m+1, k)$ is obtained from $G(m,k)$ for $k = 2$.

3. THE CASE $f(n) = 2$

In the previous section we noted that $G(n,2) \neq \phi$ for all $n \geqq 28$. In this section we establish that $G(n,2) \neq \phi$ for all $n \geqq 9$ through construction and $G(n,2) = \phi$ for all $n < 9$.

The graph H displayed in Figure 1, is a basic building block in our construction.

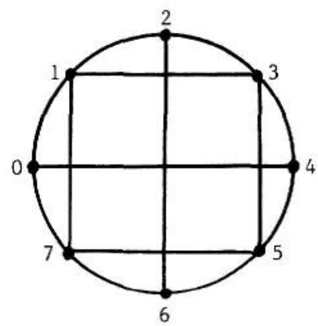


Figure 1. The Graph H.

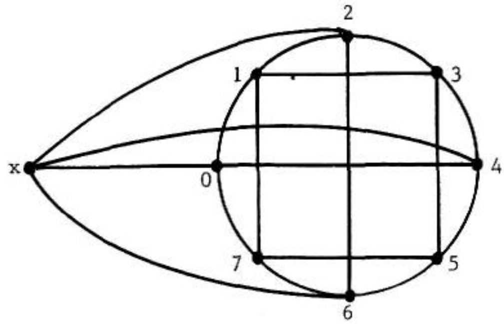
We identify a vertex as odd or even according to its label. The graph H has the following properties:

1. every vertex of H has two odd neighbours;
2. every vertex of H has two even non-neighbours;
3. every odd vertex of H has two even neighbours.

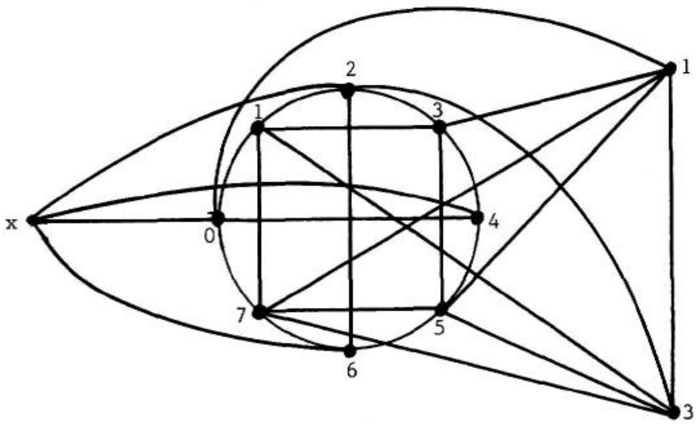
For $n \geqq 9$, let $n = 8m + 1 + \theta$, where $0 \leqq \theta \leqq 7$. We define $\delta(t)$ as 0 or 1 according to whether t is odd or even. The graph $L(8m + 9)$ on $8m + 9$ vertices is formed as follows. Take $m + 1$ copies $H_1, H_2, \ldots, H_{m+1}$ of H. Join vertex i of H_ℓ, $\ell > 1$ to vertices

$$i - 1 + \delta(i)\ ,\quad i + 2 - \delta(i)\ ,\quad i + 4 - \delta(i)\ ,\quad i + 6 - \delta(i) \pmod{8}$$

of each H_k, $1 \leqq k < \ell$. To the resulting graph we add a new vertex x and join x to all even labelled vertices. The graph $G(n)$ is obtained from $L(8m + 9)$ by deleting $8 - \theta$ vertices of H_{m+1} with even vertices being deleted (in the order 6, 4, 2, 0) before any odd vertex is deleted. Some members of $G(n)$ are displayed in Figure 2 below.



G(9)



G(11)

Figure 2.

Each subgraph H_i , $1 \leq i \leq m$, of $G(n)$ has the properties (1) to (3) mentioned above. In addition, we make the following observations concerning our graph $G(n)$:

4. every vertex of H_ℓ , $\ell > 1$, is joined to three odd vertices and one even vertex of each H_k , $1 \leq k < \ell$;
5. two vertices of H_ℓ , $\ell > 1$, have at most three common neighbours in any H_k , $1 \leq k < \ell$

We now establish the following Theorem.

THEOREM 4: $G(n,2) \neq \phi$ if and only if $n \geq 9$.

PROOF. We first establish that $G(n,2) \neq \phi$ only if $n \geq 9$. Let u be a vertex of $G \in \mathcal{G}(n,2)$ and N_u the set of vertices joined to u. Any vertex of N_u should have both a neighbour and a non-neighbour (i.e. a vertex not adjacent) amongst vertices of N_u if $P(2)$ is to be satisfied. This implies that $|N_u| \geq 4$. Since $G^c \in \mathcal{G}(n,2)$, it follows that $n \geq 9$.

We now establish that $G(n) \in \mathcal{G}(n,2)$ for every $n \geq 9$. Observations (1) - (3) together with the fact that x is joined to all even labelled vertices imply that the subgraph $H_k \cup \{x\} \in \mathcal{G}(9,2)$ for each $1 \leq k \leq m$. We note that if a pair of vertices i and j has the desired property in a subgraph of $G(n)$, then the pair has the desired property in $G(n)$. Thus it suffices to consider only pairs of vertices left in H_{m+1}, and pairs in different H_k's.

Suppose $\theta \geq 2$ and let i and j be a pair of vertices in H_{m+1}. Observation (4) ensures that i and j have a common neighbour, and a common non-neighbour in $G(n)$. Observation (5) ensures that i has a neighbour which is not joined to j and that j has a neighbour which is not joined to i. Thus the pair i and j have the desired property in $G(n)$.

Finally, consider the pair i,j with $i \in H_\ell$ and $j \in H_k$, $\ell < k$. Observations (1), (2) and (4) ensure that i and j share a common neighbour and a common non-neighbour in $G(n)$. So we now need only establish that there is always a vertex joined to i but not to j, and a vertex joined to j but not to i. Observations (3) and (4) together with the fact that x is joined to all even vertices ensures that this is the case providing i and j are not both even. Suppose then that i and j are both even. Certainly j is joined to an odd vertex of H_ℓ which is not joined to i. So we need to show that there is a vertex, t, say, joined to i but not to j. By definition i is joined to $i + 4$ (mod.8) in H_ℓ. If j is joined to i then we can take t as $i + 4$ (mod.8) since j can only be joined to one even vertex of H_ℓ. So suppose j is not joined to i. The only case that needs attention is that when j is joined to vertices $i + 4$, $i + 1$ and $i + 7$ (mod.8) of H_k. But then vertex $j + 5$ (mod.8) of H_k is joined to vertex i of H_ℓ and not joined to vertex j of H_k, and so we can take $t = j + 5$ (mod.8). Note that this argument holds when $k = m + 1$, since all odd vertices are present when H_{m+1} has even vertices. This completes our proof. □

Remark. The class of graphs G(n) constructed above possess the monotonicity property mentioned at the end of the previous section (Problem 2). That is, G(n + 1) can always be obtained from G(n) by adding a vertex.

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