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TREE–MULTIPARTITE GRAPH RAMSEY NUMBERS

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ABSTRACT The Ramsey number $r(T, K(n, n))$ is studied in the case where T is a fixed tree of order m and n is large. In particular, we find that $r(K(1, m-1), K(n, n))$ is bounded above and below by $cmn/\log(m)$ where in each bound c is an appropriate positive constant.

1. Introduction

Given graphs $G_1, \ldots, G_k$, the *Ramsey number* $r(G_1, \ldots, G_k)$ is the smallest integer r so that, if we color the edges of K_r by k colors, then for some i the ith color class contains a copy of G_i. The study of $r(G_1, \ldots, G_k)$ or *generalized Ramsey theory* was popularized by Harary, although there were earlier papers on this subject, in particular that of Gerencsér and Gyárfás [**4**].

In [**3**] we considered Ramsey numbers of the form $r(H, G)$ where H is a fixed multipartite graph and G is a large sparse graph. The present paper is a companion to [**3**]. In it we focus on Ramsey numbers of the form $r(T, G)$ where T is a fixed tree and G is a large multipartite graph.

Before presenting these rather special results, we first shall review some of the problems of generalized Ramsey theory which have been of great interest to us. It would be very desirable to have an asymptotic formula for $r(K_3, K_n)$. At present, we only know that

$$c_1\left(\frac{n^2}{(\log n)^2}\right) < r(K_3, K_n) < c_2\left(\frac{n^2}{\log n}\right) \tag{1}$$

for all sufficiently large n. One would expect that, for $m \geq 4$ fixed and n sufficiently large,

$$r(K_m, K_n) < n^{m-1-\varepsilon}, \tag{2}$$

but this is open even for $m = 4$. Perhaps

$$r(C_4, K_n) < n^{2-\varepsilon}. \tag{3}$$

Erdös strongly believes this but others disagree. All agree that the

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problem is likely to be difficult. No one doubts that

$$\lim_{n\to\infty} \frac{r(C_4, K_n)}{r(K_3, K_n)} = 0, \tag{4}$$

but even this is open at present. Szemerédi has observed that

$$r(C_4, K_n) < c\left(\frac{n^2}{(\log n)^2}\right), \tag{5}$$

which just fails to give (4). The argument is based on the following result, which is found in [**1**]. Let α, d and h denote the independence number, average degree and number of triangles respectively of a graph G of order N. Then

$$\alpha > c(N/d)\min\{\log(Nd^2/h), \log d\}. \tag{6}$$

(In (5) and (6) c stands for different absolute constants.) Now the desired result follows immediately by observing that in a graph G of order $N \geq c$ $(n/\log n)^2$ with no C_4 the average degree of G is $O(N^{1/2})$ and the number of triangles is at most as large as the number of edges, i.e. $Nd/2$.

Let G be a graph with q edges. Is it true that

$$r(K_3, G) \leq 2q + 1? \tag{7}$$

Equality holds in the case where G is a tree.

2. Results

Our first theorem gives a general upper bound for $r(T, K(n, n))$, where T is a tree of order m.

THEOREM 1 *Let T be a tree of order m. For all $n \geq 3m$,*

$$r(T, K(n, n)) \leq \lceil 4mn/\log(m) \rceil.$$

PROOF As the result is trivial in the case $m \leq 3$, we may assume that $m > 3$. Let (red, blue) be a two-coloring of K_N where $N = \lceil 4mn/\log(m) \rceil$. If there is no red copy of T, then the number of red edges is at most $N(m-2)$. (This is a well-known result which is easily proved by induction.) Thus, we may assume that there are at least $\binom{N}{2} - N(m-2)$ blue edges, so that the average degree of the blue graph is at least $N - 2m + 3$. Let $d_1, d_2, \ldots, d_N$ be the degree sequence of the blue graph and let d denote the average degree of this graph. By a well-known argument, the inequality

$$\sum_{k=1}^{N}\binom{d_k}{n}>(n-1)\binom{N}{n} \tag{8}$$

implies that there is a blue copy of $K(n, n)$. By convexity, (8) will be satisfied if

$$N\binom{d}{n}>(n-1)\binom{N}{n}, \tag{9}$$

and the latter certainly holds if

$$N\binom{N-2m}{n}>n\binom{N}{n}. \tag{10}$$

Note that (10) is equivalent to

$$N\binom{N-n}{2m}>n\binom{N}{2m} \tag{11}$$

and it certainly follows that there is a blue $K(n, n)$ if

$$\frac{N}{n}\left(1-\frac{(n+2m)}{N}\right)^{2m}>1. \tag{12}$$

With our choice of N and in view of the fact that $n\geq 3m$ we need only verify that

$$(4m/\log(m))\left(1-\frac{5\log(m)}{12m}\right)^{2m}>1 \tag{13}$$

for all $m>3$, and this is completely straightforward. □

Remarks Neither the constant 4 nor the inequality $n\geq 3m$ is a sharp condition. In fact, were we to set $N=\lceil cmn/\log(m)\rceil$ and assume n to be sufficiently large, then (11) would become

$$(cm/\log(m))\left(1-\frac{\log(m)}{cm}\right)^{2m}>1, \tag{14}$$

which is satisfied for all sufficiently large m by taking $c>2$. Further, the critical value c_0 so that $c>c_0$ will ensure that (14) holds for *all* m is approximately $2+1/e$.

The complete r-partite graph having n vertices in each part will be denoted by $K_r(n, \ldots, n)$. In the following theorem, $\log^{(r)}(n)$ denotes the r-times iterated logarithm, i.e. $\log^{(1)}(n)=\log(n)$ and $\log^{(r)}(n)=$

$\log(\log^{(r-1)}(n))$, $r=2,3,\ldots$. The theorem is proved by induction, with Theorem 1 constituting the first step.

THEOREM 2 *Let T be a tree of order m. For each $r\geq 2$ there exists a constant c_r such that*

$$r(T, K_r(n,\ldots,n)) \leq \lceil c_r mn/\log^{(r-1)}(m) \rceil$$

whenever m is sufficiently large and $n \geq 3m$.

The proof of this result is very similar to the proof of Theorem 1 and so it will be omitted. Suffice it to say that using the strategy of the proof of Theorem 1 one can verify that the blue graph contains a $K(n,p)$, where $p = \lceil c_{r-1} mn/\log^{(r-2)}(m) \rceil$. This fact, together with the induction hypothesis, completes the proof.

The next result shows that the result of Theorem 1 is, within a constant factor, the correct magnitude in the case where T is a star.

THEOREM 3 *Let m be fixed. There exists a positive constant c such that*

$$r(K(1,m-1), K(n,n)) \geq \lfloor cmn/\log(m) \rfloor$$

for all sufficiently large n. If m is sufficiently large, $c=\frac{1}{6}$ will suffice.

PROOF The proof uses the Lovász–Spencer method as developed in [**7**] and previously applied by the authors in [**2**]. We shall simply review the basic ideas of this method. Should additional details be needed, the reader is referred to the account given in [**7**]. Let $N=\lfloor cmn/\log(m) \rfloor$. We wish to show the existence of a two-colouring of the edges of K_N in which there is no red $K(1,m-1)$ and no blue $K(n,n)$. This will be accomplished by the probabilistic method, in particular by considering a random two-coloring in which each edge of the K_N is colored red with independent probability p. For each set S of m vertices of the K_N, let A_S denote the event that the red subgraph spanned by S contains $K(1,m-1)$. Similarly, for each set T of $2n$ vertices let B_T denote the event that the blue subgraph spanned by T contains $K(n,n)$. For a fixed A_S let N_{AA} denote the number of $S'\neq S$ such that A_S and $A_{S'}$ are dependent. Similarly, let N_{AB} denote the number of T such that A_S and B_T are dependent. In exactly the same way, define N_{BA} and N_{BB}. Letting A and B denote typical A_S and B_T respectively, the desired conclusion will follow from the fundamental lemma of Lovász if there exist constants a and b such that

$$aP(A)<1, \qquad bP(B)<1, \tag{15}$$

$$\log(a) > N_{AA}aP(A) + N_{AB}bP(B), \tag{16}$$

$$\log(b) > N_{BA}aP(A) + N_{BB}bP(B). \tag{17}$$

The following bounds are obvious:

$$N_{AA} \leqslant \binom{m}{2}\binom{N-2}{m-2}, \tag{18}$$

$$N_{AB}, N_{BB} \leqslant \binom{N}{2n}, \tag{19}$$

$$N_{BA} \leqslant \binom{2n}{2}\binom{N-2}{m-2}, \tag{20}$$

$$P(A) \leqslant mp^{m-1}, \tag{21}$$

$$P(B) \leqslant \binom{2n}{n}(1-p)^{n^2}. \tag{22}$$

With ε an appropriately small positive constant, set

$$p = (2+\varepsilon)\log(m)/n, \tag{23}$$

$$a = 1+\varepsilon, \tag{24}$$

$$b = m^{\varepsilon n}, \tag{25}$$

$$c = \tfrac{1}{6}. \tag{26}$$

Straightforward calculations verify that with these choices $N_{AA}aP(A)$ and $N_{AB}bP(B)$ tend to zero as $n \to \infty$ and that $\log(b)$ exceeds $N_{BA}aP(A)$, at least for all sufficiently large m. Thus with $n \to \infty$ and m taken to be sufficiently large, conditions (15)–(17) are satisfied and the proof is complete. □

Although the bound of Theorem 1 is, in a certain sense, sharp in the case where T is a star, this is certainly not the case in general. In particular, the behavior of $r(T, K(n, n))$ is quite different in the case where T is a path. Häggkvist reports that he has proved the following result [**5**]:

THEOREM (Häggkvist) $r(P_m, K(n, k)) < m+n+k-2$.

In any case, the crude upper bound $r(P_m, K(n, n)) \leqslant m+4n$ follows from a simple argument using a result of Pósa [**6**]. Let (red, blue) be a two-coloring of the edges of K_N, where $N = m+4n$. If there is no red P_m then Pósa's lemma yields a set of vertices X with its neighborhood in the red graph, $\Gamma(X)$, such that $|X| \leqslant m/3$ and $|\Gamma(X) \cup X| \leqslant 3|X|$. Repeated use of this result gives a set Y such that $n \leqslant |Y| \leqslant n+m/3$ and $|\Gamma(Y) \cup Y| \leqslant 3|Y| \leqslant 3n+m$. It follows that the blue graph contains a copy of $K(n, n)$.

3. Open questions and final remarks

What is the behavior of $r(T, K(n, n))$ when T has bounded degree? Perhaps the methods of Häggkvist will shed some light on this question.

We have seen that for a tree, T, the Ramsey number $r(T, K(n, n))$ is linear in n. However, if T is replaced by a graph containing a cycle this is no longer true. In [**7**] Spencer showed that $r(C_m, K_n) \geqslant c(n/\log(n))^{\alpha}$, where $\alpha = (m-1)/(m-2)$. By the same method, one obtains the same bound for $r(C_m, K(n, n))$, except for the value of the positive constant c.

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