Annals of Discrete Mathematics 20 (1984) 129–136
North-Holland

SOME OLD AND NEW PROBLEMS IN COMBINATORIAL GEOMETRY

Paul ERDÖS

Hungarian Academy of Sciences, Budapest, Hungary

In this paper several unconnected old and new problems in combinatorial geometry are discussed. The reference list is as complete as possible, including two papers dealing with similar subjects [6, 7]. Many interesting problems and results are in Grünbaum [11], which also has an extensive and useful bibliography and interesting historical remarks.

1

The following problem is attributed to Heilbronn. Let $f(n)$ be the largest number for which there are n points $x_1, \ldots, x_n$ in the unit circle for which the areas of all the triangles $\{x_i, x_j, x_k\}$ are $\geq f(n)$. Determine or estimate $f(n)$ as accurately as possible. Trivially, $f(n) < c/n$ and Heilbronn conjectured

$$\frac{c_1}{n^2} < f(n) < \frac{c_2}{n^2}\,. \tag{1}$$

I observed that $f(n) > c_1/n^2$ holds. The first non-trivial upper bound,

$$f(n) < \frac{c}{n(\log\log n)^{1/2}}\,, \tag{2}$$

was proved by Roth [16], his result was improved by Schmidt [18] and later Roth proved $f(n) < n^{-11/10}$ [17]. Very recently Komlós, Pintz and Szemerédi disproved the conjecture of Heilbronn. In fact they proved

$$\frac{c_1 \log n}{n^2} < f(n) < \frac{c_2}{n^{8/7}}\,. \tag{3}$$

Their proof of the lower bound in (3) is extremely noteworthy and uses a very interesting new combinatorial idea. They seem to believe that the lower bound in (3) gives the correct order of magnitude of $f(n)$.

This problem led Szemeredi and Erdös to formulate the following conjecture. Let $z_1, \ldots, z_n$ be n points in the unit circle. Denote by $d(z_i, z_j)$ the distance between z_i and z_j. Put $D(z_i, \ldots, z_j) = \min_{1 \leq i < j \leq n} d(z_i, z_j)$. Denote by $\alpha(z_1, \ldots, z_n)$ the smallest angle determined by the n points. It is well known and easy to see that

$$\max \alpha(z_1, \ldots, z_n) = \frac{\pi}{n}$$

and for a certain c:

$$\max D(z_1, \ldots, z_n) = (c + o(1))n^{-1/2}.$$

Now we conjecture that

$$\alpha(z_1, \ldots, z_n)D(z_1, \ldots, z_n) = o\left(\frac{1}{n^{3/2}}\right), \tag{4}$$

and perhaps it is less than c/n^2. The regular polygon shows that if true, this is the best possible.

Corrádi, Hajnal and Erdös noticed more than 20 years ago that the following simple question seems to present some difficulties. Let $z_1, \ldots, z_n$ be n points not all on a line. Is it true that they determine a positive angle which is $\leqslant \pi/n$? We have not even been able to show that it is less than c/n for some absolute constant c.

Schmidt [18] proved that one can find n points in the unit circle so that the area of the least convex domain determined by any four of them is always greater than $c/n^{3/2}$. He also observed that it seems very difficult to show that among any of these n points there always are four of them so that the area of the convex domain determined by them is $o(1/n)$. Indeed, this beautiful problem is still open.

As far as I know the following question has not yet been investigated. Let $f_3(n)$ be the largest number for which there are n points in the unit sphere so that all the triangles determined by them have an area $\geqslant f_3(n)$. Determine or estimate $f_3(n)$ as accurately as possible. Trivially, $f_3(n) < (c/n)^{2/3}$ and I believe one can show that $f_3(n) > c/n$. The first problem would be to prove $f_3(n) = o(1/n^{2/3})$.

2

Straus, Purdy and Erdös [19] proved that if $x_1, \ldots, x_n$ are any n points in the plane not all on a line, then they always determine two triangles of non-zero area so that the ratio of their areas is not less than $[(n+1)/2]$, and it is easy to see that $[(n+1)/2]$ is the best possible. We further show that equality is only possible if the points are all situated on two parallel lines.

Several interesting generalizations are possible, e.g. for quadrilaterals instead of triangles and for the points being in higher dimensions. We also thought of the following possible sharpening of Roth's Theorem. Let $x_1, \ldots, x_n$ be n points in the unit circle, with at most $o(n^{1/2})$ of them on a line; then they always determine a triangle of non-zero area which is $o(1/n)$. The lattice points show immediately that the condition $o(n^{1/2})$ cannot be relaxed. Many further conjectures of this type can be stated but so far we have no results.

3

Let $x_1, \dots, x_n$ be n points in the plane. We join every pair of them to obtain the lines $L_1, \dots, L_m$. It is well known that if $m > 1$, then $m \geqslant n$. The possible values of $m \geqslant n$ are fairly well known. For $m < cn^{3/2}$ Kelly and Moser [13] have fairly complete results, and answering a question of Grünbaum Erdös [4] proved that every $cn^{3/2} < m < \binom{n}{2}$ can occur, except $\binom{n}{2} - 1$ and $\binom{n}{2} - 3$.

Denote by u_i the number of points on the line L_i, $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m = 2$. Very little is known about the possible choices of $\{u_1, \dots, u_n\}$. Trivially

$$\sum \binom{u_i}{2} = \binom{n}{2}$$

and by the well-known Gallai–Sylvester Theorem $u_m = 2$ (unless $m = 1$, $u_1 = n$). Erdös conjectured that the number of possible choices of $\{u_1, \dots, u_m\}$ is less than $\exp c \cdot n^{1/2}$. It is easy to see that this conjecture if true is best possible, apart from the value of c. Purdy and Erdös conjectured that

$$\sum_{u_i > cn^{1/2}} \binom{u_i}{3} < c' n^{3/2},$$

and perhaps the number of $u_i > n^{\alpha}$ is less than $C \cdot n^{1-\alpha}$, but so far no progress has been made with these conjectures.

Denote by $\alpha_n(k)$ the largest integer for which there are $\alpha_n(k)$ lines, with $u_i = k$, and by $\beta_n(k)$ the largest integer for which there are $\beta_n(k)$ lines, with $u_i = k$, under the assumption that $u_j = 0$ for all $u_j > k$ (i.e. there are no lines containing more than k points). Clearly $\alpha_n(k) \geqslant \beta_n(k)$.

As far as I know Sylvester was the first to investigate $\alpha_n(3)$. He proved

$$\alpha_n(3) > \frac{n^2}{6} - cn. \tag{5}$$

His results were improved in a recent paper by Burr, Grünbaum and Sloane but the exact value of $\alpha_n(3)$ is not yet known. It seems certain that $\alpha_n(3) = \beta_n(3)$.

Croft and Erdös observed that the example of the lattice points shows that for every k, $\alpha_n(k) > C_k n^2$. Trivially, $\alpha_n(k) < n(n-1)/k(k-1)$ for every $k > 2$ and we conjectured that

$$\lim_{n \to \infty} \frac{\alpha_n(k)}{\binom{n}{2}} < \frac{1}{\binom{k}{2}}. \tag{6}$$

Expression (6) should not be difficult to prove, but we have not yet done so. It would be of interest to determine the value of the limit in (6). Croft and Erdös further made the following much sharper conjecture. For every $\varepsilon > 0$ there is a $k_0(\varepsilon)$ and an $n = \eta(\varepsilon)$ so that

$$\sum_{k_0 < k < \eta n} \alpha_n(k) \binom{k}{2} < \varepsilon \binom{n}{2}. \tag{7}$$

In other words, the number of pairs of points (x_i, x_j) for which the line (x_i, x_j) has at least k_0 and fewer than ηn points is less than $\varepsilon \binom{n}{2}$.

I conjectured many years ago that for every $k > 3$:

$$\lim_{n\to\infty} \beta_n(k)/n = \infty, \qquad \lim_{n\to\infty} \beta_n(k)/n^2 = 0. \tag{8}$$

The first conjecture of (8) was proved by Karteszi who in fact proved that for every k:

$$\beta_n(k) > c_k n \log n,$$

and this was strengthened by Grünbaum to:

$$\beta_n(k) > c_k n^{1-1/k}, \tag{9}$$

which, as far as I know, is the strongest inequality known at present and is perhaps the best possible.

The second conjecture of (8) is still open and I offer 2000 shekels or 100 dollars to anyone who can prove or disprove it.

As stated previously, Gallai proved that unless all our points are on a line there is at least one ordinary line, i.e. a line with $|L_i| = 2$, and Dirac proved that there are at least three such lines. De Bruijn and Erdös conjectured that the number of ordinary lines tends to infinity. This was proved by Motzkin [15] who conjectured that the number of ordinary lines is, for $n > n_0$, at least $n/2$, and remarked that if true, this is the best possible. Kelly and Moser [13] proved that the number of ordinary lines is $\geqslant 3n/7$.

An old conjecture of Dirac states that if $x_1, \ldots, x_n$ are not all on a line and if one joins every two of them, then at least one of the points is incident to $n/2 - c$ lines, where c is an absolute constant. It is not even known that there is an $\varepsilon > 0$, so that there is a point incident to more than εn lines.

An old conjecture of mine states that if at most $n - k$ of the points $x_1, \ldots, x_n$ are on a line, then they determine at least ckn distinct lines. I offer (100 dollars or 2000 shekels) for a proof or disproof. If the conjecture holds, it would be of some interest to determine the largest value of c for which it holds. The results of Kelly and Moser [13] imply the conjecture for $k < c_1 n^{1/2}$.

To end this section I mention the following lovely conjecture of Kupitz, which is several years old but I only learned it at this conference. Let $z_1, \dots, z_n$ be any n points in the plane, then there always is a line l going through at least two of our points so that the difference between the number of points on the two sides of L is at most one. It is not even known that this difference can be made less than a bound independent of n.

4

An old problem of E. Klein (Mrs. Szekeres) states: Let $H(n)$ be the smallest integer so that every set of $H(n)$ points in the plane with no three on a line contains the vertices of a convex n-gon. Klein proved $H(4)=5$, Turán and Makai proved $H(5)=9$, and Erdös and Szekeres [5, 8, 9] proved

$$2^{n-2}+1 \leq H(n) \leq \binom{2n-4}{n-2}. \tag{10}$$

We conjectured $H(n)=2^{n-2}+1$, thus $H(6)=17$; but this is not yet known.

Recently, I found the following interesting modification of this problem. Let M_n be the smallest integer so that every set of M_n points in the plane no three on a line always contains the vertices of a convex n-gon which contains none of the other points in its interior. Trivially, $M_4=5$, Ehrenfeucht proved that M_5 is finite and Harborth [12] proved that $M_5=10$. It is not yet known if M_6 exists.

Denote by $f_k(n)$ the largest integer for which every set of n points $x_1, \dots, x_n$ no three on a line contains $f_k(n)$ convex k-gons. It is easy to see that for every $k \geq 4$

$$\lim_{n\to\infty} f_k(n) \Big/ \binom{n}{k} = c_k, \qquad 0 < c_k < 1.$$

It would be of some interest to determine the value of c_k. Guy conjectured more than 20 years ago that $c_4 = \frac{3}{8}$.

Denote by $F(n)$ the largest integer so that there are always at least $F(n)$ convex subsets of the x_i. It is not hard to deduce from (10) that:

$$n^{c_1 \log n} < F(n) < n^{c_2 \log n}. \tag{11}$$

It would be of interest to improve (11). No doubt $\lim_{n\to\infty} \log F(n)/(\log n)^2$ exists and I would like to know its value. To obtain an exact formula for $F(n)$ may be hopeless, but perhaps there is some chance for an asymptotic formula.

5. Miscellaneous problems

Denote by $\alpha_r(n)$ the largest angle not exceeding π so that every set of n

points in r-dimensional space determines an angle greater than $\alpha_r(n)$. Erdös and Szekeres proved that α_2:

$$\alpha_2(2^n) = \pi\left(1 - \frac{1}{n}\right). \tag{12}$$

The problem remains to determine the smallest $m_2(n)$ for which $\alpha_2(m_2(n)) = \pi(1-1/n)$. We could not disprove that for large n, $m_2(n) = 2^{n-1}+1$. Very likely $m_2(n) < 2^n$, but we only could prove that every set of $2^n - 1$ points in the plane determines an angle $\geqslant \pi(1-1/n)$, or for every $\varepsilon > 0$, $\alpha_2(2^n - 1) > \pi(1 - 1/n - \varepsilon)$.

Very much less is known about $\alpha_r(n)$ for $r > 2$. I conjectured and Danzer and Grünbaum [3] proved that every set of $2^n + 1$ points in n-dimensional space determines an angle $> \pi/2$, but the value of $\alpha_n(2^n + 1)$ is not yet known and may be difficult to determine. I asked: determine the smallest $h(n)$ for which $h(n)$ points in n-dimensional space always determine an angle $\geqslant \pi/2$. Trivially, $h(2) = 4$ and Croft proved $h(3) = 6$. His proof is not simple; as far as I know $h(4)$ is not yet known, and it is not at all impossible that $h(n)$ grows linearly, but I cannot exclude the possibility that $h(n)$ grows exponentially.

Denote by $f_r(n)$ the largest integer for which there are $f_r(n)$ points in r-dimensional space which determine n distinct distances. Trivially, $f_r(1) = r + 1$. As far as I know Coxeter first asked for the determination of $f_r(2)$. The sharpest current upper bound is due to Blokhuis [1]; he proved $f_2(r) \leqslant \frac{1}{2}(r+1)(r+2)$.

Is it true that there is an absolute constant c so that 2^n points in n-dimensional space determine more than cn distances? The n-dimensional cube determines $n+1$ distinct distances.

Denote by $f_r(n)$ the largest integer so that among any n points in r-dimensional space one can always select $f_r(n)$ of them so that all their distances should be different. I conjectured that for $r = 1$, $f_1(n) = (1 + o(1))n^{1/2}$, and that perhaps the minimum is assumed if the points are equidistant on the line. Komlós, Sulyok and Szemeredi [14] proved that $f_1(n) > cn^{1/2}$ for some absolute constant $c < 1$. In fact, they obtained a very much more general theorem which is very interesting in itself. $f_r(n) > n^{\varepsilon_r}$ is not hard to prove, but the value of

$$\lim_{n\to\infty} \log f_r(n)/\log n = c_r$$

is not known. Perhaps $c_r = 1/(r+1)$. All these problems become much easier for infinite sets. I proved more than thirty years ago (without using the continuum hypothesis) that if S is an infinite subset of r-dimensional space and the power of S is m, then S has a subset S' of power m so that all the distances of the points determined by the points of S' are different.

Let $x_1, \ldots, x_n$ be n (distinct) points in the plane. Denote by $d_1 > d_2 > \cdots > d_m$

the distances determined by these points and assume that the distance d_i occurs u_i times (i.e. there are u_i pairs (x_u, x_v) with $d(x_u, x_v) = u_i$). I have several problems and results on the largest possible value of $\max u_i$ and on the smallest possible value of m. I refer for a detailed account of these problems to my papers quoted earlier [6, 7]. Here I discuss related but slightly different problems. We evidently have

$$\sum_{i=1}^{m} u_i = \binom{n}{2}.$$

I conjectured that for $n > 4$ the set $u_1, \ldots, u_m$ cannot be a permutation of $1, 2, \ldots, n-1$ unless the x_i's are equidistant points on a line or a circle. For $n = 4$ this is clearly possible. Let x_1, x_2, x_3, be the vertices of an isosceles triangle and x_4 the centre of its circumscribed circle. Pomerance found an example which shows that my conjecture fails for $n = 5$. Let x_1, x_2, x_3 be the vertices of an equilateral triangle, x_4 the centre of the triangle, and x_5 one of the intersections of the circumbscribed circle of x_1, x_2, x_3 and the perpendicular bisector of the segment (x_3, x_4). Nevertheless I am fairly sure that my conjecture holds for sufficiently large n, perhaps for all $n > 5$. A recent communication from L. Berkes, a Hungarian high school student in Kecskemét, shows that my conjecture also fails for $n = 6$. More generally, one could ask: How many distinct values can the u_i's take? Clearly, at most $n - 1$, but perhaps for large n this is possible only if the x_i's are equidistant points on a line.

Not much is known about the possible values of $\{u_1, \ldots, u_m\}$. My previous conjecture can be restated as follows. Assume that the u_i's are all distinct. Then for $n \geq 7$, m cannot be $n - 1$ (not all the x_i's on a line or circle). If my conjecture is true, one could ask what is the largest possible value of m if we assume that the u_i's are all distinct. Observe that for the regular $(2k + 1)$-gon $m = k$ and all the u_i's are $2k + 1$. Clearly, many questions could be asked about the possible values of the u_i's. By an old result of Pannwitz the diameter of $x_1, \ldots, x_n$ can occur at most n times, thus $\min u_i \leq n$ equality holds when n is odd and the x_i's form a regular polygon. Many problems and conjectures can be formulated: e.g. let $x_1, \ldots, x_n$ be n points and assume all the u_i's are equal. For which values is this possible? Denote this value by t_n, then $t_n = 1$ is of course possible and $t_n = n$ is possible if and only if n is odd. Also by Pannwitz's result, $t_n \leq n$. What values are possible for t_n? Clearly,

$$t_n \,\bigg|\, \binom{n}{2}.$$

References

[1] A. Blokhuis, A new upper bound for the cardinality of 2-distance sets in Euclidean space, this volume.

[2] S.A. Burr, B. Grünbaum and N.J.A. Sloane, The orchard problem, Geometriae Dedicata 2 (1974) 397–424.

[3] L. Danzer and B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdös und V. Klee, Math. Z. 9 (1962) 90–99.

[4] P. Erdös, On a problem of Grünbaum, Can. Math. Bull. 15 (1972) 23–25.

[5] P. Erdös, The Art of Counting: Selected Writings (MIT Press, 1973).

[6] P. Erdös, On some problems of elementary and combinatorial geometry, Ann. Math. Ser. IV, V 103 (1975) 99–108.

[7] P. Erdös, Some combinatorial problems in geometry, in: Geometry and Differential Geometry, Lecture Notes in Mathematics 792 (1979) pp. 46–53.

[8] P. Erdös and G. Szekeres, A combinatorial problem in geometry, Compos. Math. 2 (1935) 463–470.

[9] P. Erdös and G. Szekeres, On some extremum problems in elementary geometry, Ann. Univ. Sci. Budapest Sect. Math. 3–4 (1961) 313–320.

[10] B. Grünbaum, New views on some old questions of combinatorial geometry, Coll. Intern. Sulla Teoria Comb., Rome, Vol. 1, Math. Rev. 57 10605 (1973).

[11] B. Grünbaum, Arrangements and Spreads, Conference Board on Mathematical Sciences, Amer. Math. Soc. No. 10.

[12] H. Harborth, Konvexe Funfecke in ebenen Punktmengen, Elem. Math. 33 (1978) 116–118.

[13] L.M. Kelly and W. Moser, On the number of ordinary lines determined by n points, Can. J. Math. 10 (1978) 210–219.

[14] J. Komlós, M. Sulyok and E. Szemerédi, Linear problems in combinatorial number theory, Acta Math. Acad. Sci. Hungary 26 (1975) 113–121.

[15] T.S. Motzkin, The lines and planes connecting the points of a finite set, Trans. Amer. Math. Soc. 70 (1951) 451–464.

[16] K.F. Roth, On a problem of Heilbronn, J. London Math. Soc. 26 (1951) 198–204.

[17] K.F. Roth, On a problem of Heilbronn, II and III, J. London Math. Soc. 25 (1972) 193–212.

[18] W. Schmidt, On a problem of Heilbronn, J. London Math. Soc. 4 (2) (1971/72) 545–550.

[19] E. Straus, G. Purdy and P. Erdös, Can. J. Math. (to appear).

Received March 1981