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RESEARCH PROBLEMS

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In this column Periodica Mathematica Hungarica publishes current research problems whose proposers believe them to be within reach of existing methods. Manuscripts should preferably contain the background of the problem and all references known to the author. The length of the manuscripts should not exceed two type-written pages.

36. Let $X_n = \{x_1, \ldots, x_n\}$ be n points in the plane. I will say that the set X_n has property P_k if no line contains more than k of the points. Thus property P_{n-1} means that not all the points are on a line. I stated many conjectures on the number of lines determined by the set X_n. Many of my conjectures have recently been proved by Beck, F. Chung, Spencer, Szemerédi, Trotter and others [1], [2], [11]. But one of my old conjectures remained open. Let X_n have property P_k, $k > 3$. Denote by $f_k(n)$ the maximal number of lines which contain k points of X_n. I conjectured that for fixed k if $n \to \infty$

$$\frac{f_k(n)}{n} \to \infty, \quad \frac{f_k(n)}{n^2} \to 0. \tag{1}$$

The first conjecture in (1) was proved by F. Kárteszi who showed that $f_k(n) > > c_k\, n \log n$ is possible. Later Grünbaum [6] improved this to

$$f_k(n) > c_n' \, n^{1+\frac{1}{k-2}}. \tag{2}$$

Perhaps (2) is best possible, but even the second conjecture of (1) remains open and I offer 100 dollars for a proof or disproof of it.

The case $k = 3$ has already been considered by Sylvester. He proved in sharp contrast to $k \geq 4$ that [3]

$$\frac{n^2}{6} - c_1 n < f_3(n) < \frac{n^2}{6} - e_2 n. \tag{3}$$

Sylvester conjectured that if X_n has property P_{n-1} (i.e. if not all the x_i are on a line) then there is at least one line which contains exactly two of our points. These lines will be called ordinary lines. I rediscovered this conjecture in 1933 and told it to Gallai who soon proved it. Denote by $g_2(n)$ the largest integer for which there are at least $g_2(n)$ ordinary lines. Hansen in his Copenhagen dissertation recently proved that for even $n > n_1$, $g_2(n) = n/2$. This was conjectured by Motzkin [10], weaker results were proved by Kelly and Moser and others.

I thought for a moment that if X_n has property P_3 then there is an ordinary triangly, i.e., there are three of our points x_i, x_j, x_l so that all the lines determined by them are ordinary ones. Füredi and Palásti [5] pointed it out by a simple and elegant construction that this certainly is not so. On the other hand, perhaps the following problem is of some interest: Let $k(n; r, k)$ be the smallest integer so that if X_n has property P_k and if there are at least $k(n; r, k)$ ordinary lines then there always is an ordinary r-tuple, i.e., r of our points so that all the $\binom{r}{2}$ lines determined by them are ordinary lines. Perhaps it is best to assume that r and k are fixed and n tends to infinity.

It follows trivially from Turán's theorem that

$$k(n; r, k) \leq \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right) + 1,$$

but I hope that

$$k(n; r, k) = o(n^2)$$

and perhaps

$$k(n; r, k) < c_{r,k}\ n. \tag{4}$$

Let $\alpha_1 < \alpha_2 < \dots$ be the set of integers for which there is a set X_n which determines exactly α_i distinct lines. Several results are known about the possible values of the α_i [4], e.g., $\alpha_1 = 1$, $\alpha_2 = n$. As far as I know the number of possible values of the α_i has not yet been determined. Also very much less seems to be known about the possible values of the ordinary lines determined by our set X_n, or about the possible values of the number of lines which contain exactly r of our points x_i.

Two final problems: Let X_n have property P_k. Denote by $g(n;, k, l)$ $(l < k)$ the size of the largest subset of X_n which has property P_l. The most interesting case is $k = 3$, $l = 2$. The greedy algorithm trivially gives

$$g(n; 3, 2) \geq (2n)^{1/2}. \tag{5}$$

I could not improve (5) and I could not disprove $h(n; 3, 2) > vn$. I am sure that for every X_n $3 \leq l < k$ and $n \to \infty$ $g(n; k, l) > cn$.

I conjectured and Beck proved[1] [1] that there is an absolute constant c so that if X_n has property P_{n-k} then X_n determines at least ckn distinct

[1] The conjecture is a consequence of the results of Szemerédi and Trotter [11], too.

lines (here we only assume $2 \leq k \leq n$). If $3k^2 < n$, Kelly and Moser [8] obtained the exact value for the number of these lines. The value given for c by Beck seems too small. It would be tempting to conjecture that $c = 1/6$. That $c \leq 1/6$ follows from Sylvester's result [3], perhaps the conjecture $c = 1/6$ is too optimistic and one should first look for a counter-example.

The papers [3], [7], [10] contain many interesting historical facts and have extensive references.

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