MINIMAL DECOMPOSITION OF ALL GRAPHS WITH EQUINUMEROUS VERTICES AND EDGES INTO MUTUALLY ISOMORPHIC SUBGRAPHS

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I. INTRODUCTION

Suppose $\mathbf{G} = \{G_1, G_2, \ldots, G_k\}$ is a collection of graphs*, all having the same number of edges. By a U-decomposition of $\mathbf{G}$ we mean a set of partitions of the edge set $E(G_i)$ of the G_i, say $E(G_i) = \sum_{j=1}^{r} E_{ij}$, such that for each j, all the $E_{i,j}$ $(1 \leqslant i \leqslant k)$, are isomorphic as graphs. Define the function $U(\mathbf{G})$ to be the least possible value of r any U-decomposition of $\mathbf{G}$ can have. Finally, let $U_k(n)$ denote the largest possible value $U(\mathbf{G})$ can assume where $\mathbf{G}$ ranges over all sets of k graphs each having n vertices and the same number of edges.

In previous work [3], [4], it was shown that

$$U_2(n) = \frac{2}{3} n + o(n) \quad \text{and} \quad U_k(n) = \frac{3}{4} n + o(n)$$

for any fixed $k \geqslant 3$.

*In general, we follow the terminology of [1].

In this paper we consider the family, denoted by $\mathbf{G}(n, e)$, of all graphs on n vertices and e edges. Let $U(n, e)$ denote $U(\mathbf{G}(n, e))$, and let $U(n)$ denote the maximum value of $U(n, e)$ over all values of e. It is easily seen that $U_k(n) \leqslant U(n)$. We will prove that

$$U(n) = \frac{3}{4} n + O(1).$$

In particular,

$$U(n, e) = o(n) \quad \text{if} \quad e >> n \qquad (\text{i.e., } \frac{n}{e} = o(1)).$$

II. PRELIMINARIES

Before we study U-decompositions of $\mathbf{G}(n, e)$, we will state some auxiliary facts on unavoidable graphs, which were first investigated by two of the authors in [2]. A graph contained in every graph on n vertices and e edges is called an (n, e)-unavoidable graph. Let $f(n, e)$ denote the largest integer m with the property that there exists an (n, e)-unavoidable graph on m edges. It was proved in [2] that

(i) $f(n, e) = 1$ if $e \leqslant \lfloor \frac{n}{2} \rfloor$;

(ii) $f(n, e) = 2$ if $\lfloor \frac{n}{2} \rfloor < e \leqslant n$;

(iii) $f(n, e) = \left(\frac{e}{n}\right)^2 + O\left(\frac{e}{n}\right)$ if $n \leqslant e \leqslant n^{\frac{4}{3}}$;

(iv) $c_1 \dfrac{\sqrt{e} \log n}{\log \binom{n}{2} - \log e} < f(n, e) < c_2 \dfrac{\sqrt{e} \log n}{\log \binom{n}{2} - \log e}$

for $d_1 n^2 < e < \binom{n}{2} - n^{1 + d_2}$

where c_1 and c_2 are appropriate constants where d_1 and d_2 are any constants satisfying $0 < d_1 < \frac{1}{2}$, $0 < d_2 < 1$. In particular,

(v) $f(n, e) \geqslant (1 + o(1)) \sqrt{2e}$ for $\frac{n}{e} = o(1)$.

The unavoidable graphs in (i), (ii) and (iii) are disjoint unions of stars.

In (iv) and (v) the unvoidable graphs involved are disjoint unions of complete bipartite graphs.

Let S_i denote a star with i edges and let jS_i denote the vertex disjoint union of j copies of S_i. We need the following useful facts.

Lemma 1. *Suppose G has n vertices and e edges, and has maximum degree d. For any two integers t and r, if we have*

$$e \geqslant \frac{r-1}{2} n + (t-1)d + t^2 r^2$$

then G contains tS_r.

Proof. Suppose k is the largest integer such that kS_r is embedded in G and suppose $k < t$. Let X denote the image of k centers of S_i's. Let Y denote the image of kr leaves. Because of the maximality of k, the induced subgraph of S on $Z = V(G) - X - Y$ does not contain any vertex with degree r or more. At most k vertices in $X \cup Y$ are adjacent to at least kr vertices in Z. The total number of edges in G is then bounded above by

$$\binom{(k+1)r}{2} + \frac{(n-(k+1)r)(r-1)}{2} + kd + k^2 r^2 <$$

$$< \frac{r-1}{2} n + (t-1)d + t^2 r^2.$$

This is a contradiction and Lemma 1 is proved. ∎

Lemma 2. *Suppose G has n vertices and e edges with*

$$o(n^{\frac{4}{3}}) = e = mn + s \quad (n > s \geqslant 0).$$

Then G has the following properties:

(i) *If $s > \frac{n}{2}$, G contains $\lfloor \frac{n-s-m^2}{2} \rfloor$ (edge-disjoint) copies of mS_2. After removing $\lfloor \frac{n-s-m^2}{2} \rfloor$ copies of mS_2, the remaining graph G' has maximum degree $s+m^2$. G' contains $\lfloor \frac{s}{2} - \frac{n}{4} - m^2 \rfloor$ copies of*

$(m+1)S_2$. After removing $\lfloor \frac{s}{2} - \frac{n}{4} - m^2 \rfloor$ copies of $(m+1)S_2$ from G' the remaining graph G'' has maximum degree at most $\frac{n}{2} + 2m^2$. G'' contains $\lfloor \frac{n}{2} - m^2 \rfloor$ copies of $(m+1)S_1$. After removing these $(m+1)S_1$ from G'' the remaining graph has maximum degree $4(m+1)^2$ and has at most $20(m+1)^3$ edges.

(ii) *If $s \leqslant \frac{n}{2}$, G contains $\lfloor \frac{n}{4} - m^2 \rfloor$ copies of mS_2. After removing $\lfloor \frac{n}{4} - m^2 \rfloor$ copies of mS_2, the remaining graph $\bar{G}'$ contains $\lfloor \frac{n}{2} - s - m^2 \rfloor$ copies of mS_1. After removing $\lfloor \frac{n}{2} - s - m^2 \rfloor$ copies of mS_1, the remaining graph $\bar{G}''$ contains $s - m^2$ copies of $(m+1)S_1$. After removing $s - m^2$ copies of $(m+1)S_1$, the remaining graph has maximum degree $4(m+1)$ and $20(m+1)^2$ edges.*

Proof. The proof proceeds by using Lemma 1 iteratively. We first prove (i) by proving the following stronger statement.

By removing i copies of mS_2 from G, $i < \lceil \frac{n - s - m^2}{2} \rceil$, the remaining graph G_i contains mS_2 and G_i has maximum degree $\leqslant n - 2i + 2$.

It is clearly true for $i = 1$ by Lemma 1 (we may assume $m \geqslant 1$ in (i)). Suppose it is true for $j < i$. We note that

$$|E(G_i)| \geqslant e - 2im \geqslant \frac{n}{2} + (m-i)(n-2i+4) + 4m^2.$$

Thus by Lemma 1, G_i contains mS_2. We now embed mS_2 into G_i such that centers are mapped into vertices with highest degrees if possible. If there are more than m vertices with degree $n - 2i + 3$ or more, the total number of edges in G_{i-1} is then at least $(n-2i+3)(m+1) - \binom{m+1}{2}$. Since G_{i-1} has $e - 2(i-1)m$ edges, we then have

$$e - 2(i-1)m \geqslant (n-2i+3)(m+1) - \binom{m+1}{2},$$

i.e. $s \geq n - 2i + 3 - \binom{m+1}{2}$.

This yields a contradiction. The rest of (1) can be proved by using Lemma 1 repeatedly. (ii) can be proved in a similar fashion. ∎

Lemma 3. *Suppose G has n vertices and e edges with $e = mn + s = o(n^{\frac{4}{3}})$ and $m > c$ for some constant c. G contains $\frac{4n}{c} - cm$ copies of $\lfloor \frac{m}{2} \rfloor S_{\lceil \frac{c}{2} \rceil}$. After removing $\frac{4n}{c} - cm$ copies of $\lfloor \frac{m}{2} \rfloor S_{\lceil \frac{c}{2} \rceil}$, the remaining graph has at most cm^3 edges.*

Proof. It can again be proved by induction that after removing $2i$ copies of $\lfloor \frac{m}{2} \rfloor S_{\lceil \frac{c}{2} \rceil}$ the remaining graph has degree at most $n - \frac{ic}{2}$. ∎

III. ESTIMATING $U(n)$

We are now ready to tackle the problem of determining $U(n)$. In [4] it is proved that $U_3(n) \geq \frac{3}{4} n - \sqrt{n} - 1$. Thus, $U(n) \geq U_3(n) \geq \frac{3}{4} n - \sqrt{n} - 1$. We will first prove the following:

Theorem 1. $U(n, e) < \alpha n$ *if* $e > \frac{10n}{\alpha}$.

Proof. We consider all graphs on n vertices and e_0 edges. We will remove an (n, e)-unavoidable graph from each graph of edges currently remaining in each of the graphs. We consider the following cases.

Case 1. $n^{2-\epsilon} < e \leq \binom{n}{2}$, where $\epsilon = \frac{\alpha}{10}$.

In this case, we remove a common subgraph having at least $\frac{1}{\epsilon} \sqrt{e}$ edges. Thus, if e_i denotes the number of edges remaining in each graph after i repetitions have been performed then

$$e_{i+1} \leq e_i - \frac{1}{\epsilon} \sqrt{e_i}.$$

It can then be proved by induction that $e_i \leqslant \left(\sqrt{e_0} - \frac{i}{2\epsilon}\right)^2$ since

$$e_{i+1} \leqslant e_i - \frac{1}{\epsilon}\sqrt{e_i} \leqslant \left(\sqrt{e_0} - \frac{i}{2\epsilon}\right)^2 - \frac{1}{\epsilon}\left(\sqrt{e_0} - \frac{i}{2\epsilon}\right) \leqslant$$
$$\leqslant \left(\sqrt{e_0} - \frac{i+1}{2\epsilon}\right)^2.$$

We apply this process as long as $e_i > n^{2-\epsilon}$ so that at most $2\epsilon n$ subgraphs are removed from each graph.

Case 2. $n^{\frac{4}{3}} < e < n^{2-\epsilon}$.

In this range, the unavoidable graph has at least $c_1 \sqrt{e}$ edges (see [2]). Let e_i denote the number of edges remaining in each graph after i subgraphs are removed. We have

$$e_{i+1} \leqslant e_i - c_1 \sqrt{e_i}.$$

It can be proved by induction that

$$e_i \leqslant \left(n^{1-\frac{\epsilon}{2}} - \frac{2i}{c_1}\right)^2.$$

We apply this process as long as $e_i > n^{\frac{4}{3}}$ so that at most $c_1 n^{1-\frac{\epsilon}{2}}$ subgraphs are removed.

Case 3. $\frac{n}{\epsilon} < e \leqslant n^{\frac{4}{3}}$.

In this step, we repeatedly remove unavoidable graphs with $(1-\epsilon)\left(\frac{e}{n}\right)^2$ edges. Then

$$e_{i+1} \leqslant e_i - \left(\frac{e_i}{n}\right)^2.$$

It can be proved by induction that

$$\frac{e_i}{n^2} \leqslant \frac{1}{i}.$$

Hence, to reach $e \leqslant \frac{n}{\epsilon}$ requires the removal of at most ϵn subgraphs.

Case 4. $\frac{n}{2\epsilon} < e < \frac{n}{\epsilon}$.

We now use Lemma 3 by choosing $c = \lceil \frac{1}{2\epsilon} \rceil$. After removing at most $3\epsilon n$ graphs, at most c^2 edges are left. We then remove one edge at a time.

Since $e_0 >> n$, then $e > \frac{n}{\epsilon}$ and $c^2 < \epsilon n$. Therefore we require at most $\alpha n = \frac{10n}{\epsilon}$ steps in the U-decomposition of $\mathbf{G}(n, e_0)$. Theorem 1 is proved. ∎

Theorem 2. $U(n, cn^2) \leqslant n \log n$ *for some constant* c.

Proof. The proof is similar to that in Theorem 1 except for taking ϵ to be $\frac{1}{100 \log n}$ in the proof of Theorem 1. ∎

Theorem 3. $U(n) < \frac{3}{4} n + O(1)$.

Proof. We consider graphs on n vertices and e edges. From Theorem 1 we only have to consider the case that $e < 15n$. We now use Lemma 2. Let c be equal to 225 and $e = mn + r$. We consider the following cases.

Case 1. $s > \frac{n}{2}$.

Each G in $\mathbf{G}(n, e)$ can be decomposed into $\lfloor \frac{n-s-c}{2} \rfloor$ copies of mS_2, $\lfloor \frac{s}{2} - \frac{n}{4} - c \rfloor$ copies of $(m+1)S_2$ and $\lfloor \frac{n}{2} - c \rfloor$ copies of $(m+1)S_1$. After removing these star-forests, only $4c^2$ edges are left. Thus we have

$$U(n, e) \leqslant \lfloor \frac{n-s-c}{2} \rfloor + \lfloor \frac{s}{2} - \frac{n}{4} - c \rfloor + \lfloor \frac{n}{2} - c \rfloor + 4c^2 \leqslant$$

$$\leqslant \frac{3n}{4} + 4c^2.$$

Case 2. $s \leq \frac{n}{2}$.

Each G in $\mathbf{G}(n, e)$ can be decomposed into $\lfloor \frac{n}{4} - c \rfloor$ copies of mS_2, $\lfloor \frac{n}{2} - s - c \rfloor$ copies of mS_1 and $s - c$ copies of $(m+1)S_1$. After removing these star-forests, only $4c^2$ edges are left. Thus we have

$$U(n, e) \leq \lfloor \frac{n}{4} - c \rfloor + \lfloor \frac{n}{2} - s - c \rfloor + s - c' + 4c^2 \leq \frac{3n}{4} + 4c^2.$$

Therefore $U(n) \leq \frac{3n}{4} + 4c^2$ and the proof of Theorem 3 is completed. ∎

IV. CONCLUDING REMARKS

Let c_i denote some appropriate constants. From Theorem 2 we know that $U(n, c_1 n^2) \leq c_1 n \log n$. If we insist that only unavoidable graphs can be used in the U-decomposition, then $\frac{c_3}{\log n}$ subgraphs are required since an $(n, c_1 n^2)$-unavoidable graph can have at most $c_4 n \log n$ edges. Is it true that $U(n, c_1 n^2) = c_5 n \log n$? Can we do better by using graphs other than unavoidable graphs in finding minimal U-decompositions of $\mathbf{G}(n, e)$?

In this paper we actually prove that

$$\frac{3}{4} n - \sqrt{n-1} < U(n) < \frac{3}{4} n + c_6.$$

There is still room for improvement.

For $U_2(n)$, it can be shown in a similar manner that

$$\frac{2}{3} n - \frac{1}{3} < U_2(n) < \frac{2}{3} n + c_7.$$

It would be of interest to get the exact value for $U_2(n)$ (and $U(n)$, for that matter).

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