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Cross-Cuts in the Power Set of an Infinite Set*

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Abstract. In the power set $P(E)$ of a set E, the sets of a fixed finite cardinality k form a *cross-cut*, that is, a maximal unordered set C such that if $X, Y \subseteq E$ satisfy $X \subseteq Y$, $X \subseteq$ some X' in C, and $Y \supseteq$ some Y' in C, then $X \subseteq Z \subseteq Y$ for some Z in C. For $E = \omega$, ω_1, and ω_2, it is shown with the aid of the continuum hypothesis that $P(E)$ has cross-cuts consisting of infinite sets with infinite complements, and somewhat stronger results are proved for ω and ω_1.

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A *cross-cut* of a partially ordered set P is a maximal unordered subset C of P satisfying the following interpolation condition: if x and y are elements of P such that $x \leqslant y$, $x \leqslant$ some x' in C, and $y \geqslant$ some y' in C then $x \leqslant z \leqslant y$ for some z in C. For example, if the power set $P(E)$ of a set E is ordered by inclusion, then the set of all k-element subsets of E is a cross-cut of $P(E)$ for any natural number $k \leqslant$ the cardinality $|E|$ of E. For E finite such cross-cuts are the only ones, whereas if E is infinite this is no longer the case since the complements of the k-element subsets of E also form a cross-cut. Let us say that a cross-cut of $P(E)$ is *trivial* if it consists either of all k-element subsets of E or of their complements. Problem 7 in [1] asks whether $P(E)$ has any nontrivial cross-cuts when E is infinite. Assuming the continuum hypothesis (CH), we are able to give a positive answer to this question in the cases $E = \omega, \omega_1, \omega_2$ and prove somewhat stronger results for ω and ω_1. We note that instead of CH, Martin's Axiom (MA) could be used here (inductions up to ω_1 are then replaced by inductions up to 2^{ω}). Incidentally, it is not difficult to show using the generalized continuum hypothesis that the sets in a cross-cut of $P(E)$ all have the same cardinality (see [2]).

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It is convenient (though not essential) to define the notions of cross-cut etc. not just for partially ordered sets but for quasi-ordered sets, in which $\leqslant$ is reflexive and transitive but not necessarily antisymmetric.

Let P be a quasi-ordered set. We write $x \equiv y$ if $x \leqslant y$ and $y \leqslant x$, $x < y$ if $x \leqslant y$ but $y \nleqslant x$, $x \geqslant y$ if $y \leqslant x$, and $x > y$ if $y < x$. A subset C of P is *unordered* if $\forall x, y \in C$ $(x \leqslant y \rightarrow x = y)$, and a *cross-cut* of P is a maximal unordered subset C of P such that $\forall x, y \in P, x', y' \in C\ (x \leqslant y, x \leqslant x', y' \leqslant y \rightarrow \exists z \in C\ (x \leqslant z \leqslant y))$. It is easily verified that a subset C of P is a cross-cut of P iff it is unordered and meets every subset of P of the form $S(a, b) = \{x \in P : x \leqslant a, \text{ or } a \leqslant x \leqslant b, \text{ or } b \leqslant x\}$, where $a \leqslant b$ in P. A set $\mathscr{G}$ of subsets of P is *acyclic* if there do not exist distinct $C_0, \ldots, C_n$ in $\mathscr{G}$, $n \geqslant 1$, and elements x_i, y_i of C_i, $i = 0, \ldots, n$, such that $y_i \leqslant x_{i+1}$ for $i = 0, \ldots, n-1$, and $y_n \leqslant x_0$. If $\mathscr{G}$ is acyclic then the sets in $\mathscr{G}$ are necessarily pairwise disjoint and $\mathscr{G}$ is partially ordered by the relation $\leqslant$ defined as follows: $C \leqslant C'$ iff there exist $C_0, \ldots, C_n$ in $\mathscr{G}$, x_i in C_i, $i = 1, \ldots, n$, and y_i in C_i, $i = 0, \ldots, n-1$, such that $C_0 = C, C_n = C'$, and $y_i \leqslant x_{i+1}$ for $i = 0, \ldots, n-1$. A *grading* of P is an acyclic set $\mathscr{G}$ consisting entirely of cross-cuts of P such that every element x of P is $\equiv$ some element y of $\cup \mathscr{G}$. Then y and the cross-cut C in $\mathscr{G}$ to which y belongs are uniquely determined by x and we denote C by $C(x)$; also $\mathscr{G}$ is totally ordered under the ordering for acyclic sets just defined.

Let E be an infinite set and κ an infinite cardinal. Then $P_\kappa(E)$ denotes the set of all subsets of E of cardinality less than κ and for X, Y in $P(E)$, $X \leqslant Y$ *mod* κ means that $|X \backslash Y| < \kappa$; $\leqslant$ mod κ is a quasi-order on $P(E)$ and $\equiv$ mod κ, $<$ mod κ, etc., are defined as above. A cross-cut with respect to the mod κ ordering will be called a *mod* κ *cross-cut*, and similarly for the other notions described in the previous paragraph (an $S(A, B)$ in the mod κ sense will be written as $S_\kappa(A, B)$). A mod κ cross-cut of $P(E)$ is *trivial* if it either consists of a single set in $P_\kappa(E)$ or of the complement of such a set; cross-cuts etc. without qualification are understood to be with respect to inclusion. A set X will be said to κ-*split* the sets in a family of sets $\mathscr{W}$ if $|W \cap X| \geqslant \kappa$ and $|W \backslash X| \geqslant \kappa$ for all W in $\mathscr{W}$ with $|W| \geqslant \kappa$. The following weakened form of a result of Sierpiński ([3], p. 113, Théorème 1) is the essential tool used in constructing cross-cuts and gradings.

LEMMA 1 (Sierpiński). *If* $|\mathscr{W}| \leqslant \kappa$ *then there exists a set* X *which* κ-*splits the sets in* $\mathscr{W}$.

LEMMA 2. *Assume* $2^\kappa = \kappa^+$. *Then there exists a* mod κ *grading of* $P(\kappa)$.

Proof. Arrange the elements of $P(\kappa)$ in a list of type κ^+ and do the same for the subsets $S_\kappa(A, B)$ of $P(\kappa)$ and for the ordinals $\alpha < \kappa^+$, where in the last list each α is required to occur κ^+ times. We define subsets $C_\alpha(\beta)$ of $P(\kappa)$ for $\alpha, \beta < \kappa^+$ by induction on β such that for each β the following condition holds:

(*) The $C_\alpha(\beta)$'s are mod κ unordered subsets of $P(\kappa)$ of cardinality $\leqslant \kappa$, at most κ of them are nonempty, and they form a mod κ acyclic set.

Note that then $\{C_\alpha(\beta) : \alpha < \kappa^+\}$ will be partially ordered by the $\leqslant$ relation defined earlier on.

First we put $C_\alpha(0) = \phi$ for all α.

Next let β (which remains fixed in what follows) be such that $C_\alpha(\beta)$ has been defined for all α and let the βth ordinal in our list of ordinals $<\kappa^+$ be α_0. Then we put $C_\alpha(\beta+1) = C_\alpha(\beta)$ for all $\alpha \neq \alpha_0$ and only have to define $C_{\alpha_0}(\beta+1)$. Write C for $C_{\alpha_0}(\beta)$. If $C = \phi$ and X is the first element of $P(\kappa)$ not $\equiv$ mod κ to any member of any $C_\alpha(\beta)$ we put $C_{\alpha_0}(\beta+1) = \{X\}$, and if C meets every $S_\kappa(X, Y)$ we put $C_{\alpha_0}(\beta+1) = C$. So suppose that C is non-empty but does not meet every $S_\kappa(X, Y)$, and let $S_\kappa(A, B)$ be the first such. We wish to find X in $S_\kappa(A, B)$ so that (*) will continue to hold when we put $C_{\alpha_0}(\beta+1) = C \cup \{X\}$.

Let U be the union of all $C_\alpha(\beta)$'s $< C$ and let V be the union of all $C_\alpha(\beta)$'s $> C$. Then we require that $X \leqslant Y$ mod κ for no Y in $C \cup U$ and that $Y \leqslant X$ mod κ for no Y in $C \cup V$. There are three cases to consider.

Case 1. $Y_0 \leqslant A$ mod κ for some Y_0 in V. Then we must choose $X \leqslant A$ mod κ. Let X be a subset of A which κ-splits the sets $A \cap Y$ and $A \backslash Y$, Y in $C \cup U \cup V$. Suppose that $X \leqslant Y$ mod κ where Y is in $C \cup U$. Then also $A \leqslant Y$ mod κ ($|A\backslash Y| \geqslant \kappa$ implies $|X \backslash Y| = |(A\backslash Y) \cap X| \geqslant \kappa$). From $Y_0 \leqslant A \leqslant Y$ mod κ, $Y_0 \in C \cup V$, and $Y \in C \cup U$, it follows that $Y = Y_0 \in C$ and $Y_0 \equiv A$ mod κ, so that C meets $S_\kappa(A, B)$ contrary to the choice of $S_\kappa(A, B)$. Suppose that $Y \leqslant X$ mod κ where Y is in $C \cup V$. Then $|(A \cap Y)\backslash X| < \kappa$ implies $|A \cap Y| < \kappa$ which with $|Y \backslash A| < \kappa$ gives $|Y| < \kappa$, and Y must be in C. Because $Y \leqslant A$ mod κ, C meets $S_\kappa(A, B)$ again.

Case 2. $B \leqslant Y_0$ mod κ for some Y_0 in U. Then we must choose $X \geqslant B$ mod κ. This case is dual to the first and may be derived from it by passing to complements in κ.

Case 3. Otherwise. Here we may choose X so that $A \leqslant X \leqslant B$ mod κ. Let X_0 be a subset of $B\backslash A$ which κ-splits the sets $(B\backslash A) \cap Y$ and $(B\backslash A)\backslash Y$, Y in $C \cup U \cup V$, and put $X = A \cup X_0$. Suppose that $X \leqslant Y$ mod κ where Y is in $C \cup U$. Then $|((B\backslash A)\backslash Y) \cap X_0| \leqslant |X \backslash Y| < \kappa$ implies $|(B\backslash A)\backslash Y| < \kappa$ which with $|A \backslash Y| < \kappa$ gives $|B\backslash Y| < \kappa$ so that $B \leqslant Y$ mod κ and we are in case 2. Suppose that $Y \leqslant X$ mod κ where Y is in $C \cup V$. Then $|((B\backslash A) \cap Y)\backslash X_0| \leqslant |Y \backslash X| < \kappa$ implies $|(B\backslash A) \cap Y| < \kappa$ which with $|Y\backslash B| < \kappa$ gives $|Y\backslash A| < \kappa$ so that $Y \leqslant A$ and we are in Case 1.

This completes the definition of $C_\alpha(\beta+1)$. For β a limit ordinal, we put $C_\alpha(\beta) = \bigcup_{\gamma<\beta} C_\alpha(\gamma)$ for each α.

Having defined $C_\alpha(\beta)$ for all α and β, we set $C_\alpha = \bigcup_{\beta<\kappa^+} C_\alpha(\beta)$ for each α. Then $\mathscr{G} = \{C_\alpha : \alpha < \kappa^+\}$ is a mod κ grading of $P(\kappa)$ (every C_α meets every $S_\kappa(A, B)$ because every α is recycled κ^+ times during the construction, and every subset of κ is $\equiv$ mod κ to some member of $\cup\mathscr{G}$ because at each stage β of the construction, κ^+ of the $C_\alpha(\beta)$'s are still empty).

LEMMA 3. *Let E be an infinite set. Then each* mod ω *grading $\mathscr{H}$ of $P(E)$ gives rise to a grading $\overline{\mathscr{H}}$ of $P(E)$.*

Proof. For X, Y in $P(E)$ write $X \sim Y$ if $X \equiv Y$ mod ω and $|X\backslash Y| = |Y\backslash X|$. Then $\sim$ is an equivalence relation on $P(E)$ and for every mod ω equivalence class $\mathscr{A} \neq [\phi]$ or $[E]$, the mod $\sim$ equivalence classes contained in $\mathscr{A}$ form a grading $\mathscr{G}_\mathscr{A}$ of $\mathscr{A}$ of order type $\omega^* + \omega$; for each such $\mathscr{A}$ fix an order isomorphism $\theta_\mathscr{A} : \omega^* + \omega \to \mathscr{G}_\mathscr{A}$. Then for each nontrivial D in $\mathscr{H}$ and each n in $\omega^* + \omega$, $\bar{D}(n) = \cup\{\theta_\mathscr{A}(n)$: $\mathscr{A}$ meets $D\}$ is a cross-cut of $P(E)$. Define $\overline{\mathscr{H}}$ to consist of all these cross-cuts together with the cross-cuts in the unique gradings of $[\phi]$ and $[E]$; then $\overline{\mathscr{H}}$ is a grading of $P(E)$. The detailed verification of the above statements is straightforward.

It can be shown in exactly the same way that each mod ω grading $\mathscr{H}$ of $P_\kappa(E)$ gives rise to a grading $\overline{\mathscr{H}}$ of $P_\kappa(E)$ such that if the mod ω cross-cuts in $\mathscr{H}$ are actually mod ω cross-cuts of $P(E)$ then the cross-cuts in $\overline{\mathscr{H}}$ are cross-cuts of $P(E)$.

Taking κ and E in Lemmas 2 and 3 respectively to be ω, we obtain

THEOREM 1. *Assume* CH (or MA). *Then there exists a grading of* $P(\omega)$.

In Theorem 3 below we assert the existence of a cross-cut of $P(\omega_1)$ consisting of uncountable sets whose complements are also uncountable. To construct such a cross-cut we extend a nontrivial mod ω_1 cross-cut of $P(\omega_1)$ using a suitable grading of $P_{\omega_1}(\omega_1)$ (shown to exist in Theorem 2) in a manner similar to that in which we used the unique grading of $P_\omega(E)$ in the proof of Lemma 3. It is convenient to describe this procedure here.

LEMMA 4. *Let E be an infinite set, κ an infinite cardinal, $\mathscr{G}$ a grading of $P_\kappa(E)$ in which the cross-cuts are cross-cuts of $P(E)$, and D a* mod κ *cross-cut of $P(E)$. Define $\bar{D}$ to consist of the sets $(X\backslash Y_1)\cup Y_2$ where X is in D, Y_1 and Y_2 are in $P_\kappa(E)$, $Y_1 \subseteq X$, $X \cap Y_2 = \phi$, and $C(Y_1) = C(Y_2)$ ($C(Y)$ denotes the unique cross-cut in $\mathscr{G}$ containing Y). Then $\bar{D}$ is a cross-cut of $P(E)$.*

Proof. To see that $\bar{D}$ is unordered, suppose that $(X\backslash Y_1)\cup Y_2 \subseteq (X'\backslash Y_1')\cup Y_2'$ where $X, X', Y_1, Y_1', Y_2, Y_2'$ are as in the definition of $\bar{D}$. Then $X = X'$ since D is unordered mod κ and hence $Y_1' \subseteq Y_1$ and $Y_2 \subseteq Y_2'$. By the acyclicity of $\mathscr{G}$, we must have $C(Y_1) = C(Y_1')$ from which it follows that $Y_1 = Y_1'$ and $Y_2 = Y_2'$. To see that $\bar{D}$ is a cross-cut, let $A \subseteq B$ in $P(E)$ and fix X in $D \cap S_\kappa(A, B)$. There are four possibilities, as follows:

(i) Either $X < A$ mod κ, or $A \equiv X \leqslant B$ mod κ and $C(X\backslash A) < C(A\backslash X)$. Let $Y_1 = X\backslash A$ and let $Y_2 \subseteq A\backslash X$ be such that $C(Y_1) = C(Y_2)$. Then $(X\backslash Y_1)\cup Y_2$ is in $\bar{D}$ and is $\subset A$.

(ii) Either $B < X$ mod κ, or $A \leqslant X \equiv B$ mod κ and $C(B\backslash X) < C(X\backslash B)$. This is similar to (i) but with $(X\backslash Y_1)\cup Y_2 \supset B$.

(iii) $A \leqslant X \leqslant B$ mod κ, $C(A\backslash X) \leqslant C(X\backslash B)$, and either $X < B$ mod κ or $X \equiv B$ mod κ and $C(X\backslash B) \leqslant C(B\backslash X)$. Let $Y_1 = X\backslash B$ and let Y_2 be such that $A\backslash X \subseteq Y_2 \subseteq B\backslash X$, $C(Y_1) = C(Y_2)$. Then $(X\backslash Y_1)\cup Y_2$ is in $\bar{D}$ and lies between A and B.

(iv) $A \leqslant X \leqslant B$ mod κ, $C(X\backslash B) < C(A\backslash X)$, and either $A < X$ mod κ or $A \equiv X$ mod κ and $C(A\backslash X) \leqslant C(X\backslash A)$. Let $Y_2 = A\backslash X$ and let Y_1 be such that $X\backslash B \subseteq Y_1 \subseteq X\backslash A$, $C(Y_1) = C(Y_2)$. Then again $(X\backslash Y_1)\cup Y_2$ is in $\bar{D}$ and lies between A and B.

It is natural to ask whether there is a common generalization of Lemmas 3 and 4 in which from a mod κ grading $\mathscr{H}$ of $P(E)$ one constructs a grading $\overline{\mathscr{H}}$ of $P(E)$ via a suitable grading $\mathscr{G}$ of $P_\kappa(E)$. Suppose that $\mathscr{G}$ satisfies not only the condition (a) that the cross-cuts in it are cross-cuts of $P(E)$ but also the following additivity condition (b): $C(X\cup Z) = C(Y\cup Z)$ whenever X, Y, Z are pairwise disjoint sets in $P_\kappa(E)$ for which $C(X) = C(Y)$. Then such a common generalization can be proved by essentially the same argument as outlined for Lemma 3 (the details are similar to those for Lemma 4). However we do not know if there exist any gradings of $P_\kappa(E)$ satisfying both (a) and (b) (other than in the trivial case $\kappa = \omega$). We can construct a grading of $P_{\omega_1}(\omega_1)$ satisfying (a) and the proof is similar to that of Lemma 2 except that we need to keep control of the order type of the sets in our cross-cuts in order to secure (a) (an idea used by Hajnal for a similar purpose).

In what follows, tp S denotes the order type of a well-ordered set S, ω^δ denotes ordinal exponentiation, and cf δ is the cofinality of δ.

LEMMA 5. *If S is any well-ordered set of type $\geqslant \omega^\delta$ and $\mathcal{Y}$ is a countable set of subsets Y of S, each of type $< \omega^\delta$, then S has a subset X of type ω such that $X \cap Y$ is finite for all Y in $\mathcal{Y}$. If* tp S *ends in ω^δ and* cf $\delta \leqslant \omega$ *then X may be chosen to be cofinal in S.*

Proof. Write $\mathcal{Y} = \{Y_n : n \in \omega\}$ and let $X = \{x_n : n \in \omega\}$ where the x_n's are chosen inductively so that $x_n > x_{n-1}$ and $x_n \notin Y_0 \cup \ldots \cup Y_n$; if tp S ends in ω^δ and cf $\delta \leqslant \omega$ let $\{s_n : n \in \omega\}$ be cofinal in S and take $x_n \geqslant s_n$ also (note that $Y_0 \cup \ldots \cup Y_n$ has type $< \omega^\delta$ and thus its complement will be cofinal in S).

For $\gamma < \omega_1$, let $Q(\gamma)$ be the set of all subsets A of ω_1 such that $\omega^\gamma \leqslant \text{tp}\, A < \omega^{\gamma+1}$.

LEMMA 6. *Assume* CH (*or* MA). *Then there exists a grading of $Q(\gamma)$ consisting of cross-cuts of $P(\omega_1)$.*

Proof. By Lemma 3 and the remark following it, we need only prove this mod ω; also, on account of the upper bound $\omega^{\gamma+1}$ on the order type of the sets involved, it is enough that the cross-cuts we produce are cross-cuts of $P_{\omega_1}(\omega_1)$. The proof is similar to that of Lemma 2 and we just indicate the modifications required. Since we are working mod ω throughout, we will write $\leqslant$ for $\leqslant$ mod ω, splits for ω-splits, etc.

We arrange the sets in $Q(\gamma)$ in a list of type ω_1, likewise the subsets $S(A, B)$ (understood in the mod ω sense) of $P_{\omega_1}(\omega_1)$ and the ordinals $\alpha < \omega_1$ (each repeated ω_1 times). The $C_\alpha(\beta)$'s are countable mod ω unordered subsets of $Q(\gamma)$, at most countably many of them are nonempty, and they form a mod ω acyclic set. Again we wish to find X in $S(A, B)$ by which to extend $C = C_{\alpha_0}(\beta)$ but now also require X to be in $Q(\gamma)$. Defining U and V as before, we require specifically that $X \leqslant Y$ for no Y in $C \cup U$, that $Y \leqslant X$ for no Y in $C \cup V$, and that X is in $Q(\gamma)$. Again we have three cases to consider.

Case 1. $Y_0 \leqslant A$ for some Y_0 in V, or tp $A \geqslant \omega^{\gamma+1}$. Then we must choose $X \leqslant A$.

First suppose that $Y_0 \leqslant A$ where Y_0 is in V and that tp $A < \omega^{\gamma+1}$. Let X be a subset of A which splits the sets $A \cap Y$ and $A \backslash Y$, Y in $C \cup U \cup V$. The argument given for Case 1 in the proof of Lemma 3 shows that X is as desired, except that tp X may be $< \omega^\gamma$. To avoid this, let $\cup_{\xi < \delta} A_\xi$ be a subset of A in $Q(\gamma)$, where each A_ξ is of type ω and $\eta < \eta'$ for $\eta \in X_\xi$, $\eta' \in X_{\xi'}$, $\xi < \xi'$, and choose X to split the A_ξ's also.

Now suppose that tp $A \geqslant \omega^{\gamma+1}$ and let A_0 consist of the first ω^γ elements of A. If $Y_0 \leqslant A_0$ for some Y_0 in V then we proceed as above with A_0 in place of A. If not, we use Lemma 5 to find a subset X_0 of A of type ω such that $X_0 \cap Y$ is finite for all Y in $C \cup U \cup V$ and put $X = A_0 \cup X_0$. Then $X \leqslant Y$ for no Y in $C \cup U$ since this is already true for X_0, and $Y \leqslant X$ for no Y in $C \cup V$ since otherwise $Y \leqslant A_0$ – just ruled out for Y in V and impossible for Y in C because C does not meet $S(A, B)$.

Case 2. $B \leqslant Y_0$ for some Y_0 in U, or tp $B < \omega^\gamma$. We have to choose $X \geqslant B$. By Lemma 5, there is a set $X_0 \subseteq \omega_1$ of type ω such that $X_0 \cap Y$ is finite for all Y in $C \cup U \cup V$. If tp $B \geqslant \omega^\gamma$ let $X = B \cup X_0$. Then $X \leqslant Y$ for no Y in $C \cup U$ since this is already true for X_0, and $Y \leqslant X$ for no Y in $C \cup V$ since otherwise $Y \leqslant B \leqslant Y_0$ which is contrary to the way U and V were defined. So suppose tp $B < \omega^\gamma$. Fix B_1 in $Q(\gamma)$ and let X_1 be a subset of $B_1 \backslash B$ splitting the sets $(B_1 \backslash B) \cap Y$, Y in $C \cup V$; since $B_1 \backslash B$ is in $Q(\gamma)$, we may also choose X_1 to be in $Q(\gamma)$ by the device used in Case 1. Now let $X = B \cup X_0 \cup X_1$. Then X is in $Q(\gamma)$ and as before $X \leqslant Y$ for no Y in $C \cup U$. Suppose that $Y \leqslant X$ where Y is in $C \cup V$. Then $Y \leqslant B \cup X_1$ so that $((B_1 \backslash B) \cap Y) \backslash X_1$ is finite and hence so also is $(B_1 \backslash B) \cap Y$. But this is impossible since $Y \leqslant B \cup ((B_1 \backslash B) \cap Y)$, tp $Y \geqslant \omega^\gamma$.

Case 3. Otherwise – then $Y \leqslant A$ for no Y in $C \cup V$, $B \leqslant Y$ for no Y in $C \cup U$, and tp $A < \omega^{\gamma+1}$, tp $B \geqslant \omega^\gamma$. If tp $B < \omega^{\gamma+1}$ we construct X in exactly the same way as for Case 3 in the proof of Lemma 3. If tp $B \geqslant \omega^{\gamma+1}$ let X_0 be a subset of B of type ω such that $X_0 \cap Y$ is finite for all Y in $C \cup U \cup V$ and put $X = A \cup X_0$; the argument used in the last part of Case 1 shows that X is as required.

As an immediate consequence of Lemma 6 we have:

THEOREM 2. *Assume* CH (*or* MA). *Then there exists a grading of* $P_{\omega_1}(\omega_1)$ *consisting of cross-cuts of* $P(\omega_1)$.

Together with Lemmas 2 and 4, this theorem gives:

THEOREM 3. *Assume* CH (*or* MA) *and* $2^{\omega_1} = \omega_2$. *Then there exists a cross-cut of* $P(\omega_1)$ *consisting of uncountable sets whose complements are also uncountable.*

We now come to the one result we have for ω_2.

THEOREM 4. *Assume* CH (*or* MA). *Then there exists a cross-cut of* $P(\omega_2)$ *consisting of countably infinite sets.*

Proof. The argument is similar to that used for Lemma 2 and again for Lemma 6. We construct, by induction on $\alpha < \omega_2$ with cf $\alpha = \omega$, families F_α of cofinal subsets X of α of type $< \omega^2$ such that for each α, $\cup\{F_\beta : \beta \leqslant \alpha$, cf $\beta = \omega\}$ is a mod ω cross-cut of $P_{\omega_1}(\alpha)$. Then $\cup\{F_\alpha : \alpha < \omega_2$, cf $\alpha = \omega\}$ will be a mod ω cross-cut of $P_{\omega_1}(\omega_2)$ and will give rise, as before, to a true cross-cut of $P_{\omega_1}(\omega_2)$ and, indeed, of $P(\omega_2)$ by virtue of the tp $X < \omega^2$ requirement. As in the proof of Lemma 6, we write $\leqslant$ for $\leqslant$ mod ω, etc.

Suppose that F_β has been defined for all $\beta < \alpha$, cf $\beta = \omega$, where $\alpha < \omega_2$ and cf $\alpha = \omega$. The construction of F_α is by an induction over ω_1: we first list all the $S(A, B)$'s with B a countable cofinal subset of α (if B is not cofinal in α then because we are working mod ω we will have handled $S(A, B)$ at an earlier stage) and then define progressively longer countable pieces F of F_α by adjoining to the current F a set X in the first $S(A, B)$ not meeting $F \cup \bigcup_{\beta < \alpha} F_\beta$. (As stated, this is the same approach as used before except that now we are only constructing a single cross-cut so there is no recycling of cross-cuts, and the sets U and V in the proofs of Lemmas 2 and 6 do not arise.)

In choosing X, we again consider three cases (not quite analogous to those considered earlier however).

Case 1. tp $A \geqslant \omega^2$. Then we must choose $X < A$. If the sup of the first ω^2 elements of A is β and $\beta < \alpha$ then by the inductive hypothesis there exists Y in F_β such that $Y < A$, contrary to the choice of $S(A, B)$. Thus $\beta = \alpha$ and we take X to be a cofinal subset of A

of type ω such that $X \cap Y$ is finite for all Y in F (such an X exists by Lemma 5).

We suppose from now on that tp $A < \omega^2$ and find that we can then always choose X so that $A \leqslant X \leqslant B$. Since tp B is a limit ordinal, it ends in ω^δ for some $\delta \geqslant 1$.

Case 2. $\delta \geqslant 2$. By Lemma 5, there exists a cofinal subset X_0 of B of type ω such that $X_0 \cap Y$ is finite for all Y in F and we put $X = A \cup X_0$ (note that if Y in $F \cup \bigcup_{\beta<\alpha} F_\beta$ is $\leqslant X$ then $Y \leqslant A$).

Case 3. $\delta = 1$. Let $\alpha_0 < \alpha$ be such that cf $\alpha_0 = \omega$ and tp $(B \backslash \alpha_0) = \omega$. By the inductive hypothesis, $B \cap \alpha_0$ is comparable with some element Z_0 of $\bigcup_{\beta<\alpha} F_\beta$.

Suppose first that $B \cap \alpha_0 \leqslant Z_0$. Let $X_0 \subseteq B \backslash A$ split the sets $(B \backslash A) \cap Y$ and $(B \backslash A) \backslash Y$, Y in $F \cup \{\alpha_0\}$, and put $X = A \cup X_0$. Then, as before, X is incomparable with all the sets in F. Moreover, X is cofinal in α: this is clearly the case if A is cofinal in α, and if $B \backslash A$ is cofinal in α then X_0 is cofinal in α since it splits $(B \backslash A) \backslash \alpha_0$ (which will be of type ω here). To see that X is incomparable with all the sets Z in $\bigcup_{\beta<\alpha} F_\beta$, note first that the cofinality of X in α makes $X \leqslant Z$ impossible. On the other hand if $Z < X$ where Z is in F_β, $\beta < \alpha$, then $Z < B$ and hence $Z < B \cap \alpha_0$ (clearly $Z \leqslant B \cap \alpha_0$ and if $Z \equiv B \cap \alpha_0$ then $B \cap \alpha_0 \leqslant X$ whence $Z \equiv B \cap \alpha_0 \leqslant A$ since X_0 splits $(B \backslash A) \cap \alpha_0$). This contradicts $B \cap \alpha_0 \leqslant Z_0$.

Finally suppose that $Z_0 < B \cap \alpha_0$. Now $S(A \cap \alpha_0, B \cap \alpha_0)$ contains an element Z_1 of $\bigcup_{\beta<\alpha} F_\beta$ and because $Z_0 < B \cap \alpha_0$ we must have $A \cap \alpha_0 < Z_1 < B \cap \alpha_0$. Let $B_1 = Z_1 \cup (B \backslash \alpha_0)$ and consider $S(A, B_1)$ instead of $S(A, B)$. Since $S(A, B)$ does not meet $F \cup \bigcup_{\beta<\alpha} F_\beta$, the same also holds for $S(A, B_1)$ (since $A < B_1 \leqslant B$, we need only check that $B_1 \leqslant Y \in F \cup \bigcup_{\beta<\alpha} F_\beta$ cannot occur, and $Z_1 < B_1$ gives this). Also tp $(B_1 \backslash \alpha_0) = \omega$ and $B_1 \cap \alpha_0 \leqslant Z_1$ so we are in the situation already dealt with. Since $A \leqslant X \leqslant B_1$ implies $A \leqslant X \leqslant B$, the proof is complete.

References

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