ON SOME PROBLEMS RELATED TO PARTITIONS OF EDGES OF A GRAPH

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Abstract : The purpose of this note is to consider various modifications of the following notions: Ramsey graph, Folkman graph, canonical partition, selective graph etc. We survey the recent research in this area. We indicate proofs of some results only. The details will appear elsewhere.

1. Unrestricted partitions

The notions of chromatic number and of Ramsey graph are typical examples of notions abstracted from the study of partitions of vertices (edges respectively) into a bounded number of classes. Following the idea of Erdős and Rado [2] which represents the well known generalization of Ramsey theorem we proposed to study partitions of graphs and set systems into arbitrarily many classes and tried to develop results and theory analogous to that for restricted partitions (see [4], [5], [3]). In this section we state just a few results which complement [3] . First we state the following:

Definition 1.1. Let (X , $\mathcal{M}$) be a k-graph (i.e. $\mathcal{M} \subseteq [X]^k$). Let $c: \mathcal{M} \longrightarrow \mathcal{M}$ be a colouring. We say that c is canonical on $\mathcal{M}$ if there exists an ordering $\leq$ of X and a set $\omega \subseteq \{1, \dots, k\}$ such that

$$c(M) = c(M') \quad \text{iff} \quad M/_\omega = M'/_\omega .$$

Here, for $M = \{m_1, \dots, m_k\}$, $m_1 < m_2 < \dots < m_k$, we put $M/_\omega = \{m_i \,;\, i \in \omega\}$.

The following result extends the "Erdős-Rado canonization lemma" to the general set systems.

Theorem 1.2

For every k-graph $(X, \mathfrak{M})$ there exists a k-graph $(Y, \mathfrak{N})$ which is edge selective for $(X, \mathfrak{M})$. This means the following: for every colouring $c : \mathfrak{N} \to \mathfrak{N}$ there exists an induced subgraph $(X', \mathfrak{M}')$ of $(Y, \mathfrak{N})$, $(X', \mathfrak{M}')$ isomorphic to $(X, \mathfrak{M})$, such that $c|\mathfrak{M}'$ is canonical on $\mathfrak{M}'$. The proof uses the partition property of set systems which was proved in [6].

For the partitions of vertices much stronger results can be proved. This is analogous to the study of Folkman graphs (for the restricted partitions c.f. [7,8]).

Definition 1.3:

Let (V, E) be a graph. We say that a graph (W, F) is selective for (V, E) if for any colouring $c : W \to W$ there exists an induced subgraph (V', E') of (W, F), $(V', E') \simeq (V, E)$, such that c restricted to V' is either constant or 1-1 mapping.

The following result follows from [3]:

Theorem 1.4:

Let G be a graph, $\ell \geq 3$ an integer. Then there exists a graph H with the following properties:

1. H is selective for G ;
2. $\chi(H) = (\chi(G) - 1)(|V(G)| - 1) + 1$;

3. If the edges $e_1, \dots, e_q$ form a cycle of length at most ℓ in H, then $\{e_1, \dots, e_q\} \subseteq E(G')$ for a subgraph G' of H which is isomorphic to G.

Particularly, G and H have the same girth.

(Observe that $\chi(H) \geq (\chi(G)-1)(|V(G)|-1)+1$ for every graph H selective for G.)

Given a graph G, denote by $s(G)$ the minimal number of vertices of a graph H which is selective for G. Put $s(n) = \max s(G)$ where the maximum is taken over all n-vertex graphs. It was noted without proof in [3] that

$$s(n) \leq (2+\varepsilon)^n$$

for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$.

Here we strengthen this fact and prove the following:

Theorem 1.5:

$$s(n+1) \leq n^3 2^n (4 \ln ne + 2n \ln 2).$$

Before we prove this theorem we introduce the following:

Definition 1.6: Let G, H be graphs. The lexicographical product of G and H is a graph $G \bowtie H$ defined by:

$$V(G \bowtie H) = V(G) \times V(H);$$

$$E(G \bowtie H) = \Big\{\{(v,w),(v',w')\};\ \text{either } \{v,v'\} \in E(G) \text{ or } v = v' \text{ and } \{w,w'\} \in E(H)\Big\}.$$

For $v \in V(G)$ we denote by H_v the subgraph of $G \bowtie H$ induced on the set $\{(v,w);\ w \in V(H)\}$. Clearly $H_v \simeq H$.

The proof of 1.5 follows immediately from the following two lemmas:

Lemma 1.7:

Let G be a graph with $n+1$ vertices and let $H \longrightarrow (G)^1_n$. Then $G \bowtie H$ is selective for G.

Here the symbol $H \longrightarrow (G)^1_n$ has the following meaning:
For every partition $V(H) = a_1 \cup \ldots \cup a_n$ there exists an induced subgraph G' of H, $G' \simeq G$, such that $V(G') \subseteq a_i$ for an $i \in \{1, \ldots, n\}$.

Proof: Suppose $G \bowtie H$ fails to be selective and consider a colouring of vertices of $G \bowtie H$ such that no copy of G is coloured either 1-1 or constantly. Then, for every $v \in V(G)$ the graph $H_v \simeq H$ is coloured by at least n colours. Thus for every $v \in V(G)$ there is a vertex $w_v \in V(H)$ such that the subgraph of $G \bowtie H$ induced on set $\{(v, w_v);\ v \in V(G)\}$ is coloured 1-1 and it is clearly isomorphic to G.

Lemma 1.8: Let G be a graph with $n+1$ vertices. Then there exists a graph H with at most $n^2 2^n (4 \ln ne + 2n \ln 2)$ vertices such that $H \longrightarrow (G)^1_n$.

Proof: Let $\mathbf{H}$ be a random graph with $k = n^2 2^n (4 \ln ne + 2n \ln 2)$ vertices $v_1, \ldots, v_k$, where the edges are chosen independently each with the probability $\frac{1}{2}$. Consider a partition $\{v_1, \ldots, v_k\} = W_1 \cup \ldots \cup W_n$. Then there exists $i \leq n$ such that the cardinality of W_i is at least $\frac{k}{n}$. Take the subset $W \subset W_i$ with cardinality $\frac{k}{n}$ and denote by $\mathbf{A}_W$ the following event:

for any choice of $t = \frac{x}{2m^2}$ pairwise disjoint sets $A_1, \dots, A_t \subseteq W$ and for any system of subsets $B_i \subseteq A_i$, $i = 1, \dots, t$, there exists a vertex $v \in W - \bigcup_{i=1}^{t} A_i$ and $i_0 = 1, 2, \dots, t$ such that

$$\{v, b\} \in E(\mathbf{H}) \qquad \text{for every } b \in B_{i_0}$$

$$\{v, b\} \notin E(\mathbf{H}) \qquad \text{for every } b \in A_{i_0} - B_{i_0}.$$

Clearly, $\mathbf{A}_W$ implies that the subgraph of $\mathbf{H}$ induced on a set W contains any graph with $m+1$ vertices as an induced subgraph.

We have

$$1 - \mathrm{Prob}(\mathbf{A}_W) \le \binom{\frac{x}{m}}{m}\binom{\frac{x}{m}-m}{m}\cdots\binom{\frac{x}{m}-(t-1)m}{m} 2^{mt}\left(1-\frac{1}{2^m}\right)^{\frac{x^2}{4m^3}} \le$$

$$\le \frac{\left(\frac{x}{m}\right)!\, 2^{mt}\left(1-\frac{1}{2^m}\right)^{x^2/4m^3}}{(m!)^t\left(\frac{x}{m}-tm\right)!} < \left(\frac{x}{m}\right)^{tm}\left(1-\frac{1}{2^m}\right)^{\frac{x^2}{4m^3}}$$

Thus the probability P that $\mathbf{A}_W$ fails for some t-element subset W of $\{v_1, \dots, v_x\}$ is bounded from above by

$$P < \binom{x}{\frac{x}{m}}\left(\frac{x}{m}\right)^{tm}\left(1-\frac{1}{2^m}\right)^{\frac{x^2}{4m^3}} < (me)^{\frac{x}{m}}\left(\frac{x}{m}\right)^{\frac{x}{2m}}\left(1-\frac{1}{2^m}\right)^{\frac{x^2}{4m^3}} <$$

$$< \exp\left[\frac{x}{m}\ln me + \frac{x}{2m}\ln\frac{x}{m} - \frac{x^2}{4m^3}\,\frac{1}{2^m}\right] < 1.$$

The validity of $\mathbf{A}_W$ for any $\frac{x}{m}$-element subset of $\{v_1, \dots, v_x\}$ implies the fact that there exists $H \in \mathbf{H}$ with $H \longrightarrow (G)^1_n$ for any graph G with n vertices.

2. Some remarks on restricted partitions.

Many notions of Ramsey theory are derived from partitions (colourings) of edges into a bounded number of colour classes. Here we shall investigate colourings which are restricted by some other condition.

First we shall deal with colourings which use each of colours at most k times (where k is a fixed integer).

Towards this end we introduce the following definitions:

Let G be a graph. We say that a colouring $c: E(G) \longrightarrow \{1,2,\dots,m\}$ is a <u>b_k-colouring</u> iff $|c^{-1}(i)| \leq k$ for every $i = 1,2,\dots,m$.

A graph H is said to be <u>b_k-Ramsey</u> (<u>induced b_k-Ramsey</u>, resp.) for G iff for every b_k-colouring c of $E(H)$ there exists a subgraph G' of H (an induced subgraph G' of H respectively), $G' \simeq G$, such that c restricted to the set $E(G')$ is one-to-one.

The following holds:

<u>Theorem 2.1</u> For every graph G and for every positive integer there exists an induced b_k-Ramsey graph H. Moreover, if G has n vertices and m edges then H can be chosen so that the number of its vertices is at most $((k-1)m+1)^3 n$.

Denote by $b_k(n)$ the smallest number of vertices of a b_k-Ramsey graph for the complete graph K_n with n vertices.

We can prove the following:

<u>Theorem 2.2.</u>

$$c\, k \frac{n^2}{\ln n} \leq b_k(n) \leq c' k n^3$$

where c and c' are constants which are independent on k and n.

Note that the lower bound in Theorem 2.2 was obtained by a probabilistic method and the upper bound by a greedy type argument.

The determination of the numbers $b_k(G)$ presents many nice problems and for sparse graphs G the numbers $b_k(G)$ seem to be equal to $|V(G)|$ (for $n \gg k$).
Particularly the following is a restatement of a result due to Erdős and Stein

$$b_k(nK_2) = 2n$$

for all $n > n_0(k)$ (nK_2 is the matching of size n).
Related to this subject is the following function:

<u>Definition 2.3.</u> Let $f(n,k,\ell)$ be the maximal number a with the following property: for every colouring of the edges of K_n by means of a colours there exists a subgraph K of K_n, K isomorphic to K_k, such that the edges of K are coloured by at most ℓ colours.
Clearly $f(n,k,1) \leq a$ iff $r_a(k) \geq n$
where $r_a(k)$ is the Ramsey function.
It is clear that

$$f(2n,3,2) = 2n-2 \qquad \text{and } f(2n+1,3,2) = 2n.$$

P. Erdős observed $f(n,4,2) \geq c\,\dfrac{\ln n}{\ln\ln n}$, where c is a constant independent on n.
This can be generalized to
<u>Theorem 2.4.</u>

$$f(n,k,k-2) \geq c_k\,\frac{\ln n}{\ln\ln n}\ .$$

Some other results related to the function $f(n,k,\ell)$ are mentioned in [1].

At the end of this section we shall consider colourings of edges which are proper (i.e. chromatic) colourings of the underlying graph:

Definition 2.5. Let G be a graph. Graph H is said to be χ-Ramsey for G if for every colouring $c: E(H) \to E(H)$ which satisfies

$c(e) \neq c(e')$ for every pair of distinct edges e and e' with $e \cap e' \neq \emptyset$

there exists an induced subgraph G' of H, $G' \simeq G$ such that c restricted to $E(G')$ is 1-1.

Denote by $r_\chi(G)$ the minimal number of vertices of a χ-Ramsey graph for G.

Then the following is true:

Theorem 2.6. $r_\chi(G) \leq |V(G)|^4$ for any graph G.

L. Babai investigated the case $G = K_n$. His results (which are going to appear in Combinatorica) imply the following:

Theorem 2.7.

$$r_\chi(K_n) = n^{3(1+o(1))}.$$

The simpler part of Theorem 2.7 can be easily generalized. For any graph G set $\rho(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|}$.

Then the following holds:

<u>Theorem 2.8</u>

Let G be a graph. Put $\delta = 2\varrho(G)$. Then for every proper colouring of the edges of K_m with $m \geq \delta(|E(G)|-\delta)+|V(G)|$ there exists a copy of G which is coloured 1 - 1.

<u>Proof</u>: We assume that the statement of 2.8 holds for all proper subgraphs of G, $|E(G)| \geq 4$. Fix $v \in V(G)$ with degree $k \leq \delta$ and put $H = G - v$. Consider a proper colouring of K_m.

By the assumption there exists a copy H' of H with edges coloured 1 - 1. It is $k(|E(G)|-k) \leq \delta(|E(G)|-\delta)$ (by $k \leq \delta \leq \frac{|E(G)|}{2}$). As we consider proper colourings there exist at most $k(|E(G)|-k)$ vertices x such that on the set $V(H') \cup \{x\}$ there fails to be a copy of G. However $|V(H')| + k(|E(G)|-k) < m$.

(For graphs with at most 3 edges is the statement trivial.)

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