On some of my conjectures in number theory and combinatorics

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During my very long life I have made many conjectures in these subjects and have written several papers with similar titles. To avoid repetition as much as possible I will mainly mention problems where significant progress has been made in the last year. I start with an old conjecture of mine on the divisors of numbers.

I

1. Denote by $\tau(n)$ the number of divisors of n. $1 = d_1 < d_2 < \dots < d_{\tau(n)} = n$ denotes the consecutive divisors of n. I conjectured about 45 years ago that for almost all n (i.e., for all n, neglecting a sequence of density 0) we have

$$(1) \qquad \min d_{i+1}/d_i < 2 .$$

My first idea was to attack (1) as follows: Let m be primitive with respect to the property (1) if m satisfies (1) but no proper divisor of m satisfies it. Let $u_1 < u_2 < \dots$ be the sequence of primitive numbers. Clearly the integers satisfying (1) are the multiples of the u's. Thus to prove my conjecture it would only be necessary to prove that the set of multiples of the u's has a density and that this density is 1. This method was successful for the primitive abundant numbers, but the sum of the reciprocals of the primitive abundant numbers converges and the density of the abundant numbers is <1 [1]. Here I could prove that $\sum \frac{1}{u_i} = \infty$ and that the density of the integers satisfying (1) exists [2]. R. R. Hall and

I conjectured [3] that for every $\varepsilon>0$ and almost all n

$$(2) \qquad \min \frac{d_{i+1}}{d_i} < 1 + \frac{1}{(\log n)^{\log 3 - 1 - \varepsilon}}$$

We proved that (2) (if true) is best possible; i.e., it fails if in (2) $-\varepsilon$ is replaced by $+\varepsilon$.

Hall, Tenenbaum and I proved several theorems on the "propinquity of divisors", but (1) seemed elusive [4].

In a recent paper which will soon appear in Inventiones Mathematicae, Meier and Tenenbaum finally proved (1); in fact they proved that for almost all n

$$(3) \qquad \min d_{i+1}/d_i < 1 + (\log n)^{1-\log 3} \exp(\zeta(n)\sqrt{\log\log n})$$

where $\zeta(n) \to \infty$ as slowly as we please. (3) is a strengthening of our conjecture (2) and in fact is fairly close to being best possible, since Hall and I proved that $\zeta(n)$ can not tend to $-\infty$ as fast as $-c\,(\log\log\log\log n)^{1/2}$ [3].

Hooley considered a related problem [5]. Put

$$\Delta_\alpha(n) = \sup_u \text{card} \{d:\ d \mid n \quad u<d<\alpha u\} .$$

In other words $\Delta_\alpha(n)$ is the largest integer for which there is an integer u so that n has $\Delta_\alpha(n)$ divisors d satisfying $u<d<\alpha u$.

Hooley and later Tenenbaum, Hall and I obtained various upper and lower bounds for $\Delta_\alpha(n)$ and for the sum function $\sum_{n=1}^{x} \Delta_\alpha(n)$ [7]. In a letter, Meier states very much stronger results have been proved previously. Meier proved that for

almost all n, $\Delta_\alpha(n)$ behaves like a power of loglog n .
I hope Meier will soon publish detailed proofs.

Hall and I [8] and others investigated

$$f(n) = \max_x \sum_{\substack{d|n \\ d<x}} \mu(d) .$$

Meier proved that for almost all n, $f(n)$ behaves like a power of loglog n .

Put

$$h(n) = \sum_{i=1}^{\tau(n)} \left(\frac{d_{i+1}}{d_i} - 1\right)^2 ,$$

where $\tau(n)$ denotes the number of divisors of n.
I conjectured that there is an absolute constant c so that for infinitely many n

(3) $$h(n) < c .$$

This conjecture was proved by Vose, his paper will appear very soon in the Journal of Number Theory. I further conjectured that (3) holds if $n=k!$ or if n is the product of consecutive primes. This conjecture was proved by Tenenbaum.

I observed that (3) implies $\tau(n) > c(\log n)^2$ i.e. if $\tau(n) = o(\log n)^2$ then $h(n) \to \infty$. Is this best possible? Vose proved that (3) can hold if $\tau(n)$ is comparatively small; i.e., (3) can hold if $\tau(n) < (\log n)^c$ but it is not clear if c can be chosen to be arbitrarily close to 2. Perhaps if $\tau(n_h) <$ $< C(\log n_h)^2$ then $h(n) \to \infty$.

I observed that every $u<n!$ is the sum of at most n distinct divisors of $n!$. I conjectured that n can be replaced

by $(\log n)^c$. This, if true, is easily seen to be best possible. More generally denote by $F(n)$ the smallest integer, if it exists, for which every integer $u<n$ is the sum of at most $F(n)$ distinct divisors of n . Srinivasan called these numbers practical numbers, and I proved that the density of practical numbers is 0. It is easy to see that $F(n) > c \log\log n$. I conjectured that for infinitely many n

$$(4) \qquad F(n) < (\log\log n)^c .$$

It would be of some interest to decide about (4) and to decide whether

$$(5) \qquad F(n)\Big/(\log\log n) \to \infty .$$

I was led to these problems by the study of "Egyptian fractions". As in [7], let

$$\frac{a}{b} = \frac{1}{x_1} + \dots + \frac{1}{x_k}, \quad 0<a<b \quad , 1<x_1<\dots<x_k .$$

Denote by $G(a,b)$ the smallest value of k . An old conjecture of Straus and myself states $G(4,b) \le 3$. In memory of Straus I offer a reward of 500 dolllars for a proof or disproof of this conjecture.

Schinzel and Sierpinski conjectured that for every a there is a $b_o(a)$ so that for every $b > b_o(a)$, $G(a,b) = 3$. Put

$$\max_{1 \le a \le b} G(a,b) = G(b) .$$

I proved $G(b) < c \log b\Big/\log\log b$ [7], and (4) would imply

$G(b) < (\log\log b)^c$. Vose proved $F(b) < (\log n)^\varepsilon$ for every $\varepsilon>0$ which implies $G(n) < (\log n)^\varepsilon$ for every $\varepsilon>0$.

2. Let $1=d_1<d_2<\ldots<d_{\tau(n)}=n$ be the consecutive divisors of n. Montgomery conjectured at the Durham meeting in 1979 that the density of integers for which d_{i+1}/d_i is an integer for $c\cdot\tau(n)$ values of i is positive. I was convinced that this conjecture is incorrect but Montgomery was right; in fact, Tenenbaum and I proved Montgomery's conjecture [4].

Some more perhaps difficult and fruitful questions can be asked here. It is easy to see that if d_{i+1}/d_i is an integer then it must be a prime number. A prime number $p|n$ is "bad" if it cannot be written in the form d_{i+1}/d_i . How many bad primes are there for almost all n ? How often can the same prime $p|n$ be written in the form d_{i+1}/d_i ?

Also a prime $p|n$ could be called "bad" if it does not divide $d_i d_{i+1}$ where $d_i<d_{i+1}<2d_i$, $(d_i,d_{i+1})=1$. Perhaps almost all n have many bad prime factors, this may very well follow by the method of Meier-Tenenbaum. I asked: denote by $f(n)$ the maximum number of pairs $(d_i,d_{i+1})=1$, $d_1<d_{i+1}<2d_i$ where every prime factor p of n can occur in only one $d_i d_{i+1}$. Tenenbaum informs me that for almost all n, $f(n)$ is of the order of magnitude $\log\log\log n$. This also follows from Meier-Tenenbaum.

What can one say about the distribution function of $\max d_{i+1}/d_i$? It is not hard to see that $\frac{\log d_{i+1}/d_i}{\log n}$ has a continuous distribution function. This is probably contained in the old results of Tenenbaum. I do not see how to solve the same problem for the largest p which is of the form d_{i+1}/d_i.

Put

$$r(n) = \sum_{(d_i, d_{i+1})=1} d_i/d_{i+1} .$$

I conjectured that for almost all n, $r(n) \to \infty$. This follows from the results of Meier and Tenenbaum. It would be of interest to estimate $\sum_{n=1}^{x} r(n)$ and $\max_{n<x} r(n) = R(x)$.

Denote

$$f(n) = \sum_{(d_i, d_{i+1})=1} 1 .$$

Hall, Tenenbaum and I proved that [9]

$$\sum_{n=1}^{x} f(n) > c\, x \log\log x \tag{7}$$

but we could never prove that (7) holds for every c. Tenenbaum and I proved that for infinitely many n

$$f(n) > \exp\left(c \log n/(\log\log n)^2\right). \tag{8}$$

We do not know the exact order of magnitude of $f(n)$. Perhaps (8) is best possible, but we can not even prove that for $c < \log 2$ and $n > n_0(c)$

$$f(n) < \exp\left(c \log n/\log\log n\right).$$

Our proof of (8) will be published soon.

3. In a paper with Graham, Ruzsa and Szemerédi [12] we conjectured that for $n>4$ $\binom{2n}{n}$ is never squarefree, and I later conjectured that if $n > n_0(\alpha)$, then there is always an odd prime p for which $p^{\alpha} \| \binom{2n}{n}$. A. Sárközy proved our conjecture for $n > n_0$ in a paper which will soon appear in the Journal of Number Theory.

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Now I discuss some problems in combinatorics. First I discuss the so called jumping constant conjecture.

1. First of all I define the density of a sequence of r-graphs. Let $G_i^r(n_i\,;e_i)$, $i=1,2,\dots,$ $n_i \to \infty$, be a sequence of r-graphs of n_i vertices and e_i edges. The density of this sequence is α , $0 \le \alpha \le 1$, if α is the largest real number for which there is a subgraph $G_i^{(r)}(m_i\,;e_i')$ of $G_i^r(n_i\,;e_i)$ for which $m_i \to \infty$ and $\overline{\lim}\, e_i' / \binom{m_i}{r} = \alpha$.

Slightly imprecisely we can say that the density of $G^{(r)}(n,e)$ is α if n is large and α is the largest real number for which $G^{(r)}(n;e)$ has a subgraph $G^{(r)}(m;e')$ with m large and $e' = (1+o(1))\,\alpha\,\binom{m}{r}$.

An old theorem of Stone, Simonovits and myself [1] states that for $r=2$ the only possible values of the density are 0 , 1 and $1-\frac{1}{t}$ $1<t<\infty$.

For $r>2$ I proved [2] that if the density is positive then it is at least $\frac{r!}{r^r}$. In fact, I showed that if $n>n_0(t\,\varepsilon)$ every $G^{(r)}(n;\varepsilon n^r)$ contains r disjoint sets, $\{A_i\}$, with $|A_i| = t$ and all the t^r r-tuples which meet each A_i in exactly one point.

The original jumping constant conjecture stated that there is a constant c_r so that if a sequence of r -graphs has density $> \frac{r!}{r^r} + \varepsilon$ then in fact its density is $\geq \frac{r!}{r^r} + c_r$. This attractive conjecture is still open. Then I went on to conjecture that for every r the set of possible densities forms a well ordered set. This conjecture was disproved a few weeks ago for every $r>2$ by P. Frankl and V. Rödl. They did not at the moment of this writing settle my original "jumping constant conjecture" and also did not determine the possible values of the densities for $r>2$.

I hope Frankl and Rödl will publish their result soon.

Let me explain to the (I hope) interested reader why I partially reneged on a promise of mine. I stated in a previous paper that I offer 1000 dollars for a proof or disproof of the conjecture that the set of possible values of the densities is well ordered. I was sure that the conjecture is true. Now that the conjecture has been disproved the problem of the densities is far from being completely settled. I repeat my offer of 1000 dollars for the complete clearing up of the problem: What are the possible values of the densities of a sequence of r -graphs? I offer 500 dollars for the original jumping constant conjecture.

I was so sure that the densities form a well ordered sequence that I stated my offer carelessly. I should have said 1000 dollars for determining the possible values of the densities.

Thus I only paid 500 dollars to Frankl and Rödl for their beautiful result. I of course realize that if these offers were legal documents I would be obligated to pay the full offer.

W. Brown, M. Simonovits and I have several papers where we study the possible values of the densities of digraphs and multigraphs (most of our results are not yet published, but many unsolved problems remain which I do not discuss here).

The problem of the possible values of the density can be asked not only for the subgraphs of $K(n)$ or more generally for the subgraphs of $K^{(r)}(n)$; e.g., it is easy to prove that for the subgraphs of $K(n,n)$, the only possible values of the density are 0 and 1. Denote by $K^r(n_1,\dots,n_\ell)$ $(n_1 \to \infty)$ the r-graph whose vertices are disjoint sets $|A_i|=n_i$, $1 \le i \le \ell$ and whose edges are the r-tuples which have at most one vertex in each A_i.

It is not hard to prove that there are only a finite number of possible values for the densities of the subgraphs of $K^r(n_1,\dots,n_\ell)$ and it would be easy to determine all these values. Perhaps the following problem is not quite easy: Denote by $G^{(3)}_{1,2}(n;n)$ the three-graph whose vertices are the disjoint sets $|A|=|B|=n$ and whose edges are the $n\binom{n}{2}$ triples which contain one vertex from A and two vertices from B. What are the possible values of the densities of the subgraphs of $G^{(3)}_{1,2}(n;n)$? If I live I hope to return to these problems, but I hope that one of the interested readers (if any) will clear matters up before me.

2. At the Poznan meeting on random graphs (1983 Aug. 25-27) Nesetril and I conjectured that for every $\varepsilon>0$ there is a $G(n;e)$ which contains no $K(4)$ but every subgraph of it which has more than $(\frac{1}{2}+\varepsilon)e$ edges contains a $K(3)$. This conjecture was proved a few weeks ago by Frankl and Rödl using probabilistic methods. Their graph has $n^{\frac{3}{2}-\varepsilon}$ edges and we believe that our conjecture fails if we also assume that $e > c\,n^2$.

Many further problems and generalizations are possible, I hope to discuss them (if I live) next year.

Observe that it is well known and easy to see that every graph of e edges contains a bipartite subgraph of more than $\frac{e}{2} + c\, e^{1/2}$ edges. This explains the constant $\frac{1}{2}$ in the problem.

3. Bollobás, Simonovits, Szemerédi and I proved a few years ago the following theorem [3]. For every $c>0$ there is an $\ell(c)$ so that if $G(n)$ does not contain an odd circuit C_{2r+1}, $1 \le r \le \ell(c)$ then $G(n)$ can be made bipartite by the omission of $\le cn^2$ edges. We conjectured the following extension for r-chromatic graphs: For every $c>0$ there is an ℓ and a set of r-chromatic graphs $G_1, G_2, \dots, G_\ell$, so that if $G(n)$ does not contain any of the graphs $G_1, \dots, G_\ell$, then $G(n)$ can be made $r-1$ chromatic by omitting at most $c\, n^2$ edges. This conjecture was recently proved by Rödl.

Simonovits and I proved a few months ago that if H is any graph of m vertices and $G(n)$, $n>n_0(m;t)$, is a graph for which if we omit from it $\varepsilon\, n^2$ arbitrary edges the remaining graph will always contain H as a subgraph, then the vertex set of G contains m disjoint sets $|A_i|=t$, $1 \le i \le m$, so that if A_i is joined to A_j in H, then in $G(n)$ every vertex of A_i is joined to every vertex of A_j.

4. Here is an old conjecture of Hanani and myself [4].

Let E be a set of n elements and $\ell<k<n$ be given positive integers. $M(k,\ell,n)$ is a minimal system of k-tuples so that every ℓ-tuple is contained in at least one k tuple

of our system. Denote by $\bar{M}(k,\ell,n)$ the number of k-tuples in our system. Hanani and I conjectured

$$\bar{M}(k,\ell,n) = (1+o(1))\binom{n}{\ell}\binom{k}{\ell}^{-1} .$$

This conjecture was recently proved by (guess who) Rödl.

The exact determination of $\bar{M}(k,\ell,n)$ is a very beautiful and difficult problem, unfortunately I have no contribution to make to this question. The most striking question is: For which values of n does

$$(1) \qquad \bar{M}(k,\ell,n) = \binom{n}{\ell}\binom{k}{\ell}^{-1}$$

hold? Hanani and Wilson made perhaps the most important progress here [5].

5. Let H be a graph. Denote by $f(n;H)$ the smallest integer for which every graph $G(n;\, f(n;H))$ contains H as a subgraph ($G(n;e)$ denotes a graph of n vertices and e edges). Recently an excellent book appeared on these extremal problems by B. Bollobás (<u>Extremal Graph Theory</u>, London Math. Soc. Monographs No. 11. Academic Press 1978). Here we deal with a special problem. Let $H=C_4$ i.e. a circuit of 4 vertices. Brown, Rényi, V. T. Sós and I proved [6]

$$f(n;C_4) = (\tfrac{1}{2} + o(1))\ n^{3/2}$$

We further showed that if p is a power of a prime then

$$f(p^2+p+1,\ C_4) \geq \tfrac{1}{2}\, p(p+1)^2 .$$

We conjectured that there is equality here and this conjecture was recently proved by Z. Füredi [7] if q is a power of 2 and a few months ago he proved our general conjecture.

There may not be a simple formula for $f(n;\, C_4)$. I once conjectured

(1) $$f(n;\ C_4) = \frac{1}{2}\, n^{3/2} + \frac{n}{4} + o(n)\ .$$

We are very far from being able to prove (1).

Kövári, V. T. Sós and P. Turán [8] and independently I proved that ($K(r,r)$ denotes the complete bipartite graph of r black and r white vertices)

$$f(n;\ K(r,r)) < c_r\, n^{2-\frac{1}{r}}\ .$$

Weall conjectured that this result is best possible apart from the value of the constant c_r . Brown [6] proved this for $r=3$. The general case is still open, but a few months ago P. Frankl achieved a breakthrough; he proved, $f(n;K(r;2^r)) < c\, n^{2-\frac{1}{r}}$.

6. Gallai and I conjectured that the edges of every $G(n)$ can be covered by at most $n-1$ circuits and edges of $G(n)$. We also conjectured that there is an absolute constant c so that the edges can be covered by at most $c\,n$ edge disjoint circuits and edges of $G(n)$. An example of $G(n)$ shows that $c \geq \frac{3}{2}$. Our first conjecture was proved a few months ago by L. Pyber, using a result of Lovász [9]. The second conjecture remains open and perhaps will need new ideas.

7. V. T. Sós and I [10] investigated the following question which V. T. Sós called Ramsey-Turán type theorems: Let H^r be an r -uniform hypergraph. Let $g = g(n;\ H^r)$ be the smallest integer so that any r -uniform hypergraph on n vertices and more than g edges contains a subgraph isomorphic to H^r . Let $e = f(n;\ H^r, \varepsilon n)$ denote the smallest integer such that every r-uniform hypergraph on n vertices with more than e edges and with no independent set of εn vertices contains a subgraph isomorphic to H^r .

We show that if $r>2$ and H^r is e.g. a complete graph then

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \binom{n}{r}^{-1} f(n;H^r,\varepsilon n) = \lim_{n\to\infty} \binom{n}{r}^{-1} g(n;H^{(r)}) \tag{2}$$

while for some $H^{(r)}$ with $g(n;H^r) > c\binom{n}{r}$ for all n

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \binom{n}{r}^{-1} f(n;\, H^r,\, \varepsilon n) = 0\ . \tag{3}$$

This is in strong contrast with the situation in case $r=2$ [11]. We could not find a graph H^r for which the limits in (2) are different but the limit in (3) is not 0 . Rödl found such an H^r for every $r>2$. In [10] many other open problems are stated, several of which were solved in the mean time by us and others, e.g., problems 4, 5 and 6 p. 299, but these results are not yet in their final form and I postpone their discussion.

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III.

Finally I discuss a geometric problem. I have many unsolved geometric problems,and in the last few years significant progress has been made by Beck, F. Chung, Spencer, Trotter and others. I have written about this already, and here I only want to discuss a problem of mine which was partially solved by G. ELEKES.

I conjectured (Some problems on elementary geometry, Australian Math. Soc. Gazette 2 (1975), 2-3) that if $f(n)$ is the largest integer for which there are n points in the plane so that there are $f(n)$ distinct circles of radius 1 which pass through at least three of the x_i, then

$$\frac{f(n)}{n} \to \infty, \quad \frac{f(n)}{n^2} \to 0 .$$

Elekes gave a very nice proof of $f(n) > c\, n^{3/2}$ which proves the first inequality. Here is his simple but ingenious proof: Let $z_1, z_2, \ldots, z_n$ be n unit vectors all starting from the origin. Assume that all the sums $\sum_{i=1}^{n} \varepsilon_i z_i$, $\varepsilon_i = 0$ or 1, are distinct. The $\binom{n}{2}$ points $z_i + z_j$ clearly determine at least $\binom{n}{3}$ distinct unit circles of radius 1 at center $z_i + z_j + z_k$.

At the moment we have no non-trivial upper bound for $f(n)$.

To end the paper, I just state a somewhat unconventional problem in number theory which occurred to me recently. Put

$$f(n) = \sum_{\substack{p \mid n \\ p^{\alpha} < n \leq p^{\alpha+1}}} p^{\alpha} .$$

Are there any integers n with $f(n)=n$? Is the number of solutions infinite?

Define n to be good if it can be written in the form

$$n = \sum_{\substack{p \mid n \\ p^{\alpha} < n}} p^{\alpha} ,$$

e.g. 30 and 42 are good integers $30=5^2+2+3$, $42=2^5+3+7$. Are there infinitely many good integers? Is the density of good integers 0 ? I am sure that the answer to the second question is affirmative.

Define n to be "not bad" if it can be written in the form

$$n = \sum a_i \quad \text{where} \quad a_i < n , \quad (a_i, a_j) = 1$$

and all prime factors of the a 's divide n , e.g. 210 is not bad, $210 = (5^2 \cdot 7)+2^3+3^3$, but by trial and error it is easy to see that 210 is not good.

Are there infinitely many integers which are not bad? Is their density 0 ? Are there infinitely many integers which are not bad and not good? What happens to $n!$ or $2,3,\ldots p_k^2$.

My paper on these and related problems will appear in the jubilee issue of the Calcutta Math. Soc.