ON MAXIMUM CHORDAL SUBGRAPH*

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1. Let G(n) denote an undirected graph, with n vertices and V(G) denote the vertex set, E(G) denote the edge set. Let e(G) denote the number of edges of G. It will be assumed that G(n) has no loops or multiple edges. A graph G(n) is chordal (triangulated, rigid circuit) if every cycle of length > 3 has a chord: namely an edge joining two nonconsecutive vertices on the cycle.

The class of chordal graph include trees, k-trees, complete graphs and interval graphs. Chordal graphs have application in facility location [2], scheduling problems [8], and in the solution of sparse systems of linear equation [10]. Such graphs are also known to be perfect. Certain problems that are known to be NP-hard for general graphs can be solved in polynomial time for chordal graphs [6]. As a result, chordal graphs have been studied by many, e.g. [1], [5].

If a graph is not chordal, the following questions are quite appropriate to ask:

1) What is the minimal set of new edges to be added to the graph to make it chordal?
2) What is the minimal set of edges to be deleted from the graph such that the resulting graph is a maximum chordal subgraph?

Rose, Tarjan and Lueker have answered 1) algorithmically [9]. In answer to 2) recently Dearing, Shier, and Warner [3] have developed a polynomial algorithm to generate a maximal chordal subgraph. It may be pointed out that their algorithm does not generate a maximum chordal subgraph.

In answer to 1) Erdos [4] showed that for some positive $\varepsilon > 0$

$$f(n) > \frac{n^2}{2} - n^{2-\varepsilon}$$

*This paper is in final form and will not appear elsewhere.

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Perhaps the method will give that for every $\varepsilon > 0$

$$f(n) > \frac{n^2}{2} - n^{3/2+\varepsilon}.$$

There is no non-trivial upper bound for f(n) and not even

$$f(n) < \frac{n^2}{2} - cn.$$

seems to be known, where f(n) is the smallest integer so that for every graph with n vertices one can add $\leq f(n)$ new edges so that the resulting graph would be chordal.

This note determines asymptotically the minimum number of edges to be deleted from the graph such that the resulting graph is a maximum chordal subgraph.

2. Denote by f(n) the smallest integer such that every graph G(n) with n vertices can be made chordal be deleting $\leq f(n)$ edges of G(n).

Theorem 1. $f(n) \leq \frac{n^2}{2} - (1+o(1))\sqrt{2}\, n^{3/2}$, where $o(1) \to o$ as $n \to \infty$.

Proof: Let T(n,t) be the Turan graph [11], i.e. a complete t-partite graph with n vertices with approximately $\frac{n}{t}$ vertices in each color class. Let $v_1, v_2, \ldots, v_t$ denote the color classes. The number of edges of T(n,t)

$$\begin{aligned} &e(T,(n,t)) \\ &\sim \binom{n}{2} - t\binom{\frac{n}{t}}{2} \\ &\sim \frac{n^2}{2}\left(1 - \frac{1}{t}\right) \end{aligned} \tag{1}$$

Let G^1 be a spanning chordal subgraph of T(n,t) having maximum number of edges. Clearly G^1 cannot contain any cycle whose vertices are in two different color classes. Thus, the induced subgraph of any two color classes of G^1 must be a tree. Hence,

$$e(G^1) \sim \binom{t}{2}\left(\frac{2n}{t} - 1\right) \tag{2}$$

Let H be a spanning subgraph of T(n,t) described as below:

Consider k_t: $\{x_1, x_2, \ldots, x_t\}$, $x_i \in V_i$, i=1, 2, ..., n. Join each x_i to all vertices of V_j, $i \neq j$, j=1, 2, ..., t.
Clearly H is chordal being a split graph [7]. Also $e(H) \sim \binom{t}{2}(\frac{2n}{t} - 1)$.
Thus by (2) H is a spanning chordal subgraph of T(n,t) of maximum size.
The number of edges to be deleted from T(n,t) to obtain H is

$$\frac{n^2}{2}(1-\frac{1}{t}) - \binom{t}{2}(\frac{2n}{t} - 1)$$

$$= \frac{n^2}{2} - \frac{n^2}{2t} - nt + n + \frac{t^2}{2} - \frac{t}{2}$$

Now take $t = \frac{n}{2}$. Then (3) becomes $\frac{n^2}{2} - (1+o(1))\sqrt{2}\, n^{3/2}$ where $o(1) \to 0$ as $n \to \infty$. Thus,

$$f(n) \geq \frac{n^2}{2} - (1+o(1))\sqrt{2}\, n^{3/2}$$

Theorem 2.

$$f(n) < \frac{n^2}{2} - (1 - \varepsilon)\sqrt{2}\, n^{3/2}$$

for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$.

Proof: We can assume that G(n) has at least $\frac{n^2}{2} - (1 - \varepsilon)\sqrt{2}\, n^{3/2}$ edges. Suppose G(n) has $\binom{n}{2} - e_1$ edges and that the largest chordal subgraph of G(n) has e_2 edges. It suffices to prove

$$e_1 + e_2 > (1 - \varepsilon)\sqrt{2}\, n^{3/2} \tag{4}$$

Let $\eta > 0$ be small, much smaller than ε. We construct a subgraph G(m) consisting of m vertices each of which has degree $> n(1-\eta)$. The construction is as follows: delete from G(n) a vertex x_1 (if any) of degree $\leq n(1-\eta)$ and continue this process as long as possible. Suppose after k such steps we obtain $G(m) = G(x_{k+1}, x_{k+2}, \ldots, x_n)$ each vertex of which has degree $> n(1-\eta)$. Claim

$$k < C_{\varepsilon,\eta} \sqrt{n} \tag{5}$$

where $C_{\varepsilon,\eta}$ depends on ε and η and not on n. To prove the claim observe that if $k = [C_{\varepsilon,\eta}\sqrt{n}]$

$$
\begin{aligned}
e(G(n)) &< \binom{m}{2} + kn(1-\eta) \\
&< \frac{(n-k)^2}{2} + kn(1-\eta) \qquad (6)\\
&< \frac{n^2}{2} + \frac{k^2}{2} - \eta kn \\
&< \binom{n}{2} - 2n^{3/2}
\end{aligned}
$$

if $C_{\varepsilon,\eta}$ is sufficiently large. Thus by (4) we have nothing to prove.

Consider G(m). Assume that t is the largest positive integer such that G(m) contains a K_t. Then by Turan's theorem G(m) has at most

$$\frac{m^2}{2}\left(1 - \frac{1}{t}\right) \text{ edges} \qquad (7)$$

Let $K_t = \{y_1, y_2, \ldots, y_t\}$. Now $\deg_{G(m)} y_i > n(1-\eta)$, for each i, i=1, 2, ..., t. Adjoin all the edges joining to each y_i to vertices outside of K_t. This produces a chordal subgraph with at least

$$
\begin{aligned}
&\binom{t}{2} + t[n(1-\eta) - (t-1)] \text{ edges} \\
&> tn(1-\eta) - t^2 \qquad (8)
\end{aligned}
$$

Now if $t = [4\sqrt{n}]$, then (8) becomes

$$
\begin{aligned}
&4n^{3/2}(1-\eta) - 16n \text{ which is} \\
&> 2n^{3/2}.
\end{aligned}
$$

Thus, we get a chordal subgraph whose number of edges $> 2n^{3/2}$. (9)

This proves (4). Thus

$$t < 4\sqrt{n} \qquad (10)$$

Now from (7) and (8)

$$
\begin{aligned}
e_1 + e_2 &> \frac{m^2}{2t} + tn(1-\eta) - t^2 \\
&> \frac{m^2}{2t} + tn(1-\eta) - 16n \\
&> \frac{(n-k)^2}{2t} + tn(1-\eta) - 16n \\
&> \frac{n^2}{2t} - \frac{nk}{t} + \frac{k^2}{2t} + tn(1-\eta) - 16n \\
&> \frac{n^2}{2t} - \frac{C_{\varepsilon,n} n^{3/2}}{t} + tn(1-\eta) - 16n \\
&> (1-\varepsilon)\sqrt{2}\, n^{3/2},
\end{aligned}
$$

since $tn + \frac{n^2}{2t}$ is minimum if $t = \sqrt{2n}$ and other terms can be absorbed into $\varepsilon n^{3/2}$.

Thus combining theorems 1 and 2 we can state

<u>Theorem 3</u>. $f(n) = \frac{n^2}{2} - (1 + o(1))\sqrt{2}\, n^{3/2}$.

Perhaps the following problem is of some interest and deserves some study. Let $f(n;t)$ be the smallest integer for which every $G(n;f(n;t))$ contains a chordal subgraph of t edges. At the moment we only know that

$$f(n,n) = [\frac{n^2}{4}] + 1 \qquad (11)$$

We do not give the details of the proof of (11) but only indicate some of the necessary steps. We hope to return to this problem in the future.

Observe that the complete bipartite graph of $\frac{n}{2}$ white and $\frac{n+1}{2}$ black vertices immediately show that $f(n,n) \geq \frac{n^2}{4}$. To prove the upper bound in (11) we only remark that this immediately follows if our $G(n,[\frac{n^2}{4}]+1)$ is assumed to be connected. For then by Turan's theorem our graph contains a triangle and by a simple argument this triangle can be extended to a chordal graph of n edges. If the graph is not connected somewhat more complicated methods are needed, which as we stated we hope to discuss at another occasion. Just one more remark. It is well known that every graph of $G(n;[\frac{n^2}{4}]+1)$ contains an edge (x_1,x_2) and on further vertices x_i joined to both x_1 and x_2. This implies in fact that $f(n,n(1+\varepsilon)) = [\frac{n^2}{4}] + 1$ for sufficiently small $\varepsilon > 0$.

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