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SOME PROBLEMS ON ADDITIVE NUMBER THEORY

P. ERDŐS

Hungarian Academy of Sciences, Budapest, Hungary

Dedicated to Professor A. Kotzig on the occasion of his sixtieth birthday

Denote by $f(n)$ the largest integer k for which there is a sequence $1 \leqslant a_1 < \cdots < a_k \leqslant n$ so that all the sums $a_i + a_j$ are distinct. Turán and I conjectured about 40 years ago [5] that

$$f(n) = n^{1/2} + O(1). \tag{1}$$

The conjecture seems to be very deep and I offered long ago a prize of 500 dollars for a proof or disproof of (1). The sharpest known results in the direction of (1) state [5]

$$n^{1/2} - n^{1/2-c} < f(n) < n^{1/2} + n^{1/4} + 1. \tag{2}$$

In several papers Abrham, Bermond, Brouwer, Farhi, Germa, Kotzig, Laufer, Rogers and Turgeon considered the following somewhat related problem.

Let $m, n_1, \ldots, n_m, c$ be positive integers. Let $A = \{A_1, \ldots, A_m\}$ be a system of sequences of integers

$$A_i = \{a_{i,1} < \cdots < a_{i,n_i}\}, \quad i = 1, \ldots, m \tag{3}$$

and let

$$D_i = \{a_{i,j} - a_{i,k} \mid 1 \leqslant k < j \leqslant n_i\} \tag{4}$$

be the difference set of A_i. The system

$$S = \{D_i, \ldots, D_m\}$$

is called *perfect* for c if the set $D = \bigcup_{i=1}^m D_i$ consists of the integers

$$c \leqslant t \leqslant c - 1 + \sum_{i=1}^{m} \binom{n_i}{2}.$$

Clearly, for a perfect system, the representation of t in the form (4) must be unique.

The authors proved several interesting results on these sequences [1, 2, 3, 4, 7, 8, 9, 10, 12], but many interesting unsolved problems remain.

Put

$$N = \sum_{i=1}^{m} \binom{n_i}{2}.$$

J. Abrham proved in [1] that, for every perfect system, $m > \alpha N$, where $\alpha > 0$ is an absolute constant. The best value of α is not yet known, though Kotzig has some plausible conjectures.

I noticed some time ago that the method that Turán and myself used to get an upper bound for $f(n)$ can make some useful contribution to the study of perfect systems. In particular I prove the following:

Theorem. *Assume that the integers* (4) *are all distinct and are all in* $[1, N]$, *and that* $D_{i_1} \cap D_{i_2} = \emptyset$ *for all* $1 \leqslant i_1 < i_2 \leqslant m$. *Then, to every* $\varepsilon > 0$, *there is an* $\eta > 0$ *so that, for* $N > N_0(\varepsilon, \eta)$, *if* $|D| > (1+\varepsilon)N/2$ *then* $m > \eta N$.

Let me first explain the relation of this result to our old result with Turán. Our result with Turán states that if $m = 1$, then $|D| < (1 + o(1))N/2$, and our theorem states that if $|D| > (1+\varepsilon)N/2$ then $m > \eta N$, i.e. the number of sequences must be large. Perhaps the following problem is of some interest.

Let $1 \leqslant a_1 < \cdots < a_{k_1} \leqslant n$; $1 \leqslant b_1 < \cdots < b_{k_2} \leqslant n$ and assume that the differences

$$a_i - a_j, \quad b_u - b_v, \qquad 1 \leqslant j < i \leqslant k_1, \ 1 \leqslant v < u \leqslant k_2$$

are all distinct. Determine or estimate

$$\max\left(\binom{k_1}{2} + \binom{k_2}{2}\right) = g(N)$$

as accurately as possible.

Our theorem gives $g(N) < (1 + o(1))N/2$ and trivially

$$g(N) \geqslant \binom{f(N)}{2}.$$

Is it true that

$$g(N) < \binom{f(N)}{2} + O(1)? \tag{5}$$

Perhaps (5) is too optimistic. It might be of some interest to investigate $\max(k_1 + k_2)$. Clearly $\max(k_1 + k_2) > f(N)$ but it is not clear if

$$\max(k_1 + k_2 - f(N)) \to \infty.$$

Now we prove our theorem. The proof will be very similar to our old proof with Turán. Let $t = t_0(\varepsilon, m)$ be large but fixed, i.e. t is independent of N. To prove our theorem it clearly suffices to consider the sequences A_i satisfying $|A_i| > t$ (i.e. $n_i > t$). To see this, observe that the contribution of a sequence $|A_i| \leqslant t$ to D is at most $\binom{t}{2}$; thus one needs $\eta_1 N$ of them to significantly change the size of D.

We will only consider the integers $1 \leqslant m \leqslant N/t^{1/2}$. Clearly every such integer m has at most one representation of the form $a_{i,j} - a_{i,k}$, $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant j \leqslant n_i$. Denote by $n_i(x)$ the number of terms of the sequence A_i in the interval

$$I_x = [x, x + N/t^{1/2}].$$

Consider the set of all the differences

$$a_{i,j} - a_{i,k}, \quad x \leqslant a_{i,k} < a_{i,j} \leqslant x + N/t^{1/2}, \qquad 1 \leqslant x \leqslant N, 1 \leqslant i \leqslant m. \tag{6}$$

Since, by our assumption, the $a_{i,j}-a_{i,k}$ $(1\leqslant i\leqslant m,\ 1\leqslant k\leqslant n_i)$ are all different, an integer $m=a_{i,j}-a_{i,k}$ can occur in (6) for at most $Nt^{-1/2}-m$ intervals I_x, i.e. both $a_{i,j}$ and $a_{i,k}$ must be in I_x. Thus m either does not occur in (6), or it occurs $Nt^{-1/2}-m$ times. Observe that

$$\sum_{x=1}^{N} n_i(x)=n_i N t^{-1/2} \tag{7}$$

since each $a_{i,j}\in A_i$ occurs in $Nt^{-1/2}$ intervals I_x. Now the number of differences of the form

$$a_{i,j}-a_{i,k}, \quad x\leqslant a_{i,k}<a_{i,j}\leqslant x+Nt^{-1/2}$$

clearly equals

$$\binom{n_i(x)}{2}.$$

Now

$$\sum_{i=1}^{N}\binom{n_i(x)}{2}$$

is minimal if the $n_i(x)$ are as nearly equal as possible. Thus from (7) we obtain by a simple computation for every fixed δ, if $t>t_0(\delta)$,

$$\sum_{x=1}^{N}\binom{n_i(x)}{2}\geqslant N\binom{[n_i t^{-1/2}]}{2}>(1-\delta)\frac{Nn_i^2}{2t} \tag{8}$$

Thus from (8)

$$\sum_{i=1}^{m}\sum_{x=1}^{N}\binom{n_i(x)}{2}>(1-\delta)\frac{N}{2t}\sum_{i=1}^{m} n_i^2. \tag{9}$$

On the other hand, since m can occur at most $Nt^{-1/2}-m$ times in (6) we obtain

$$\sum_{i=1}^{m}\sum_{x=1}^{N}\binom{n_i(x)}{2}\leqslant\sum_{l=1}^{Nt^{-1/2}-1} l<\frac{N^2}{2t}. \tag{10}$$

Thus from (9) and (10)

$$N>(1-\delta)\sum_{i=1}^{m} n_i^2. \tag{11}$$

Now by our assumption

$$|D|=\sum_{i=1}^{m}\binom{n_i}{2}>(1+\varepsilon)\frac{N}{2},$$

or

$$\sum_{i=1}^{m} n_i^2>(1+\varepsilon)N \tag{12}$$

which contradicts (11) for sufficiently small δ, and hence our theorem is proved.

It is not difficult to deduce the result of Abrham from our theorem. If $c=1$ this is obvious. If $c=o(N)$ then since

$$\sum_{l=1}^{Nt^{-1/2}} l=(1+o(1)) \sum_{l=c}^{c+Nt^{-1/2}} l \tag{13}$$

nothing has to be changed in the proof. If $c>\eta N$ then (13) of course no longer holds. But if $c>\eta N$ everything is trivial. To see this observe that if $a_1<a_2<\cdots<a_k$ and all the differences a_i-a_j, $1\leqslant j\leqslant i<k$, are in $[c, N+c]$, $c>\eta N$, we immediately obtain $k<1+1/\eta$, thus $m>\eta_1 N$ immediately follows.

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