RAMSEY NUMBERS FOR BROOMS

by

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ABSTRACT

A broom $B_{k,\ell}$ is a tree obtained by identifying an end-vertex of a path on ℓ vertices with the central vertex of star on k edges. The Ramsey number $r(B_{k,\ell})$ is determined precisely for $\ell \geq 2k$ and relatively sharp bounds are found for $1 \leq \ell < 2k$. For appropriate choices of k and ℓ we show $r(B_{k,\ell}) = \{4(k+\ell)/3-1\}$ which is the smallest possible value of the Ramsey number of any tree on $k+\ell$ vertices.

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I. INTRODUCTION

Finding the Ramsey number of an arbitrary tree on n vertices is a difficult unsolved problem in generalized Ramsey theory. A more tractable problem involves finding the best upper and lower bounds of such numbers. Harary [7] has conjectured that the best upper bound is $2n-2$ $(2n-3)$ when n is even (odd), the value of the Ramsey number for a star on n vertices. We show that the best lower bound is $\{4n/3-1\}$ and demonstrate that this value is obtained for a certain tree called a broom. A broom is a generalization of both a path and a star and is defined precisely below. The lower bound is also obtained for the tree formed by joining two stars (of appropriate size) with a path of length three from their central vertices. This last result was noted by Burr and Erdös in [2].

All graphs will be finite without loops or multiple edges. For G a graph we let $V(G)$ and $E(G)$ denote its vertex and edge set respectively. The <u>Ramsey number</u> $r(G,H)$ of a pair of graphs (G,H) is the smallest positive integer n, such that each red-blue two coloring of the edges of K_n produces a red copy of G or a blue copy of H as a subgraph of K_n. When $G = H$, the notation $r(G,H)$ is shortened to $r(G)$. Red-blue two colorings of the edges of a K_n will be referred to simply as two colorings. We will always let R and B denote the red and blue edges respectively of the two colored K_n. Thus the subgraph induced by the red edges will be denoted by $\langle R\rangle$ and the one by the blue edges by $\langle B\rangle$. The red (blue) degree of a vertex x will be denoted by $d_R(x)(d_B(x))$, while $N_R(x)(N_B(x))$ will denote its red (blue) neighborhood. Further if A and D are disjoint nonempty sets of vertices in K_n, $K(A,D)$ will indicate the complete bipartite subgraph with each vertex of A adjacent to each vertex of D. Other notation will follow that of [1] and [6].

It is easy to establish that the best lower bound for the Ramsey number of a tree on n vertices is $\{4n/3-1\}$. Consider any bipartite graph G whose parts have a and b vertices respectively, $a \leq b$. Observe that a two coloring of $E(K_{2a+b-2})$ with $\langle R\rangle = K_{a-1} \cup K_{a+b-1}$ contains no monochromatic copy of G. Also a two coloring of $E(K_{2b-2})$ with $\langle R\rangle = K_{b-1} \cup K_{b-1}$ contains no monochromatic G. Hence $r(G) \geq \max\{2a+b-1, 2b-1\}$. For a+b fixed this maximum is smallest when $2a=b$. Since each tree T_n on n vertices is bipartite, this shows $r(T_n) \geq \{4n/3-1\}$.

II. BROOMS

A <u>broom</u> $B_{k,\ell}$ is a tree on $k+\ell$ vertices obtained by identifying an endvertex of a path P_ℓ on ℓ vertices with the central vertex of a star $K_{1,k}$ on k edges, these graphs being otherwise disjoint. We refer to the ℓ vertices of the "path part" of the broom as the <u>handle</u> and the k endvertices of the "star part" as the <u>bristles</u>. Clearly a $B_{k,1}$ is a star while a $B_{1,k}$ is a path.

The main results involve finding $r(B_{k,\ell})$ precisely when $\ell \geq 2k$ and finding relatively sharp bounds for $r(B_{k,\ell})$ when

$1 \leq \ell < 2k$. Before we establish our first result we state the following theorem of Jackson used in its proof.

Theorem 2.1. (Jackson) [8]. Let $G(A,D)$ be a bipartite graph with parts A and D such that $d(x) \geq t$ for all $x \in A$ where $2t-2 \geq |D| \geq t$. Then $G(A,D)$ contains as subgraphs all cycles on $2m$ vertices for $1 \leq m \leq \min\{|A|,t\}$.

Theorem 2.2. $r(B_{k,\ell}) = k+\{3\ell/2\}-1$ for $\ell \geq 2k$, $k \geq 1$.

Proof. When $k = 1$ this result agrees with the known value for $r(P_{\ell+1})$ so we assume throughout the proof that $k \geq 2$. Since $B_{k,\ell}$ is bipartite with parts of size $\{\ell/2\}$ and $k+[\ell/2]$, the previously given examples show that $k+\{3\ell/2\}-1 \leq r(B_{k,\ell})$.

To establish $k+\{3\ell/2\}-1$ as an upper bound consider a two colored $K_{k+\{3\ell/2\}-1}$. For notational convenience let G denote this graph. Since $r(C_{2t}) = 3t-1$, when $t \geq 3$, and $r(C_4) = 6$ (see [4]), G contains a monochromatic cycle $C_{2\{\ell/2\}}$. Note this cycle has ℓ vertices when ℓ is even and $\ell+1$ when ℓ is odd. Assume this is a blue cycle and let $D = V(G)-V(C_{2\{\ell/2\}})$ so that $|D| = k+[\ell/2]-1$. If there exists a vertex x of the cycle with $N_B(x) \cap D$ of cardinality at least k for ℓ even or at least $k-1$ for ℓ odd, then G contains a blue $B_{k,\ell}$. Thus we assume the contrary, that $|N_R(x) \cap D| \geq \{\ell/2\}$ for each vertex x of the cycle. But $|D| = k+[\ell/2]-1$ and $\ell \geq 2k$, so that there exists a vertex $u \in D$ such that $|N_R(u) \cap V(C_{2\{\ell/2\}})| \geq k+1$.

Let $$\{a_1,a_2,\ldots,a_k,a_{k+1}\} \subseteq N_R(u) \cap V(C_{2\{\ell/2\}}).$$

Choose any $\{\ell/2\}$ vertices of the cycle including the vertex a_1, but excluding all the vertices $a_2, a_3, \ldots, a_k, a_{k+1}$. Call this set of chosen vertices A. Consider the subgraph $K(A,D)$ of the two edge colored graph G. The red graph $\langle R \cap K(A,D) \rangle$ satisfies the conditions of Theorem 2.1 when $\ell \geq 2k+1$. This follows since for $\ell \geq 2k+1$, $2\{\ell/2\}-2 \geq k+[\ell/2]-1 \geq \{\ell/2\}$. Thus when $\ell \geq 2k+1$ the graph $\langle R \cap K(A,D) \rangle$ contains a cycle C' with $2\{\ell/2\}$ vertices. Since C' contains a_1, avoids $\{a_2, a_3, \ldots, a_k, a_{k+1}\}$, and u is adjacent in red to $\{a_1, a_2, \ldots, a_k, a_{k+1}\}$, G contains a red $B_{k,\ell}$. Thus for $\ell \geq 2k+1$, G contains a monochromatic $B_{k,\ell}$.

When $\ell = 2k$ one can give an argument similar to the one just presented, provided G contains a monochromatic C_{2k+1} with $k \geq 3$. We use this fact below leaving the remaining case when $k = 2$ to the interested reader.

Since $\ell = 2k$, $k \geq 3$, the graph G has $4k-1$ vertices and thus contains a monochromatic C_{2k+2}. We suppose the result is false for this case, i.e. G contains no monochromatic $B_{k,2k}$. Since the argument given above (with a slight modification) works if G contains a monochromatic C_{2k+1}, we have that G contains a monochromatic (say blue) C_{2k+2} and no monochromatic C_{2k+1}. Thus each pair of vertices at distance two on the C_{2k+2} are adjacent in red, giving disjoint red cycles each with $k+1$ vertices, say C' and C''. Also, letting $D = V(G)-V(C_{2k+2})$, we have $|N_R(x) \cap D| \geq k-1$ for each $x \in V(C_{2k+2})$, otherwise G contains a blue $B_{k,2k}$. Since any red edge between C' and C'' gives a red $B_{k,2k}$ we have that $K(C',C'')$ is blue. Also $\langle C' \rangle$

and <C"> are complete red graphs and no vertex of D is simultaneously adjacent in blue to a vertex of C' and one of C", otherwise G contains a blue C_{2k+1}. Thus D is partitioned into sets D' and D" such that K(C',D') and K(C",D") are red. But $|D'| \geq k$ or $|D"| \geq k$ gives a red C_{2k+1}, so we assume $|D'| = k-1$ and $|D"| = k-2$. Since a red edge from a vertex of C' ∪ D' to one of C" ∪ D" gives a red $B_{k,2k}$, the graph K(C' ∪ D',C" ∪ D") is blue. This blue graph contains a blue $B_{k,2k}$, a contradiction. Hence the theorem also holds in case $\ell = 2k$, $k \geq 3$. □

When $2k \leq \ell \leq 2k+2$ the last theorem shows $r(B_{k,\ell}) = \{4(k+\ell)/3-1\}$, giving a specific tree whose Ramsey number is as small as possible.

The remainder of the section is devoted to proving a good upper bound for $r(B_{k,\ell})$ when $1 \leq \ell < 2k$. The canonical examples given in the introduction show $2k+2[\ell/2]-1 \leq r(B_{k,\ell})$ when $\ell < 2k-1$ and $2k+2[\ell/2] \leq r(B_{k,\ell})$ when $\ell = 2k-1$. Thus the upper bound given in the next theorem is close to the best possible. Unfortunately the techniques of the proof prevent further lowering of this upper bound.

Theorem 2.2. $r(B_{k,\ell}) \leq 2k+\ell$ for $5 \leq \ell < 2k$.

Proof. Two color the edges of a $K_{2k+\ell}$ red and blue, so that $E(K_{2k+\ell})$ is partitioned into the classes R and B. Call this graph G and let x be a vertex of G of maximal monochromatic degree. Assume this maximal degree occurs in blue. Set

$s = d_B(x)$, and let $A = N_B(x)$ and $D = N_R(x)$.

We first consider the case where at least one of the following occur.

(1) $s \geq k+\ell-1$.

(2) The graph $\langle D \rangle$ contains a blue path on $\ell-2-[s-(k+1)] = \ell+k-s-1$ vertices.

(3) There exists a blue path on $2(\ell+k-s-1)$ vertices in $K(A,D)$.

Observe that each vertex of D is adjacent in blue to some vertex of A, otherwise some vertex of D has red degree greater than $d_B(x)$. Build the longest blue path in $\langle A \cup D \rangle$ having an endvertex in A and containing at least $\ell+k-s-1$ vertices of D. Note when case (1) occurs this path may lie entirely in A. If this path has at least $\ell-1$ vertices, then $\langle B \rangle$ contains a $B_{k,\ell}$. Thus assume that the maximal blue path in $\langle A \cup D \rangle$, starting at a vertex z in A and ending at a vertex y, has at most $\ell-2$ vertices. This path contains at least $\max\{\ell+k-s-1,0\}$ vertices of D, so that it fails to contain at least $s-[(\ell-2)-(\ell+k-s-1)] = k+1$ vertices of A. The maximality of the path length implies $d_R(y) \geq 2k+\ell-1-(\ell-2) = 2k+1$. Thus $s \geq 2k+1$ and $|D| \leq \ell-2$.

Let A' be a subset of $N_R(y) \cap N_B(x)$ such that $|A'| = k$ and denote the graph $\langle A \cup D \cup \{x\}-A' \rangle$ by H. Note that $|V(H)| = k+\ell$. Since $r(C_{2t}) = 3t-1$, for $t \geq 3$, H contains a monochomatic C_{2t} with $2t \geq 2[(k+\ell+1)/3] \geq \ell$. Now both $N_B(x) \cap V(H)$ and $N_R(y) \cap V(H)$ are of cardinality at least $k+1$, so that both $N_B(x)$ and $N_R(y)$ contain a vertex of the monochromatic cycle. Thus whether or not x (or y) belongs to this cycle. the original two edge colored graph G contains a monochromatic $B_{k,\ell}$. The vertices of the handle of the broom come from the cycle and those of the bristles come from A'.

We next consider the case when none of the three conditions are satisfied. For convenience we define ℓ_1 and ℓ_2 by setting $|A| = k+\ell_1$ and $|D| = k+\ell_2$. Note that this is possible since $(2k+t-1)/2 \leq s \leq k+\ell-2$, $|A| = s$, and $|D| = 2k+\ell-1-s$. Thus $\ell_1+\ell_2 = \ell-1$ with $\ell_1 \geq (\ell-1)/2 \geq \ell_2 \geq 1$. Since neither (2) nor (3) occurs $\langle D\rangle$ contains no blue path on $\ell+k-s-1 = \ell_2$ vertices and $K(A,D)$ contains no blue path on $2\ell_2$ vertices.

Since $\langle D\rangle$ contains no blue path on ℓ_2 vertices, a well known extremal result for paths of Erdös and Gallai [3] implies that $\langle D\rangle$ contains at most $(k+\ell_2)(\ell_2-2)/2$ blue edges. In [5] it is shown that a bipartite graph with parts of size a and b, $a \leq b$, and no path on $2t$ vertices, $2(t-1) \leq a$, contains at most $(t-1)(a+b-2(t-1))$ edges. Hence, since $K(A,D)$ contains no blue path on $2\ell_2$ vertices, it contains at most $(\ell_2-1)(2k+\ell_1-\ell_2+2)$ blue edges. By assumption each vertex of G is at least of degree $k+\ell_2$ in both colors so that $|B \cap E(\langle D\rangle)| \leq (k+\ell_2)(\ell_2-2)/2$ implies that $K(A,D)$ contains at least $(k+\ell_2)^2 - (k+\ell_2)(\ell_2-2) = (k+\ell_2)(k+2)$ blue edges. Furthermore $K(A,D)$ has fewer than $(k+\ell_2)(\ell_1-2)$ blue edges, otherwise $(k+\ell_2)\cdot\max\{k+2,\ell_1-2\} \geq k(k+2)+\ell_2(\ell_1-2) > (\ell_2-1)(2k+\ell_1-\ell_2+2)$, a contradiction. This last inequality follows since

$k^2 \geq 2k\ell_2 - \ell_2^2$, $k > \ell_2$, and $\ell_1 \geq \ell_2$.

We have established, since none of (1), (2), and (3) hold, that $\langle D \rangle$ contains at least

$$\binom{k+\ell_2}{2} - (k+\ell_2)(\ell_2-2)/2 = (k+\ell_2)(k+1)/2$$

red edges and $K(A,D)$ contains at least $(k+\ell_2)(k+\ell_1)-(k+\ell_2)(\ell_1-2) = (k+\ell_2)(k+2)$ red edges. Hence there exists a vertex $z \in D$ with $d_R(z) \geq 2k+4$ (recall $N_R(x) = D$). If $|N_R(z) \cap A| < k+1$ choose a vertex $w \in D$ such that $|N_R(w) \cap A| \geq k+1$. In this case let A' be a subset of $N_R(w) \cap A$ with $|A'| = k+1$. If in addition $(N_R(w) \cup \{w\}) \cap (N_R(z)-\{x\}) = \phi$, we show that there exists a $u \in D \cap N_R(z)$ such that $N_R(u) \cap (N_R(w)-\{x\}) \neq \phi$. To see this first observe, since $K(A,D)$ contains no blue path on $2\ell_2$ vertices, at most ℓ_2-1 vertices of D have their red neighborhoods disjoint from A'. Hence at least $k-1$ of the vertices in $D-\{z,w\}$ have red adjacencies to vertices of A'. At least one of these vertices must belong to $N_R(z)$, since

$$|N_R(z)-\{x\}|+k-1 > |(A-A') \cup (D-\{w,z\})|$$

Thus one of the following possibilities occur. There exists a subset A', $|A'| = k+1$, such that

(i) $A' \subseteq A \cap N_R(z)$, $z \in D$ and $d_R(z) \geq 2k+4$,

(ii) $A' \subseteq A \cap N_R(w)$, $w,z \in D$, $d_R(z) \geq 2k+4$, and $(N_R(w) \cup \{w\}) \cap (N_R(z)-\{x\}) \neq \phi$, or

(iii) $A' \subseteq A \cap N_R(w)$, $w,z \in D$, $d_R(z) \geq 2k+4$, $(N_R(w) \cup \{w\}) \cap (N_R(z)-\{x\}) = \phi$, and there exists a $u \in D \cap N_R(z)$ such that $N_R(u) \cap (N_R(w)-\{x\}) \neq \phi$.

No matter which possibility occurs denote the graph $\langle A \cup D \cup \{x\} - A' \rangle$. which has $k+\ell$ vertices by H. As in the first part of the proof H contains a monochromatic cycle C_{2t} with $2t \geq \ell$. Since $d_R(z) \geq 2k+4$, the choice of x gives $d_B(x) \geq 2k+4$. Hence $|N_B(x) \cap V(H)| \geq k+3$ and $|N_R(z) \cap V(H)| \geq k+2$, so that both $N_B(x)$ and $N_R(z)$ contain a vertex of the monochromatic C_{2t}. It is now easy to check that for each of the above possibilities the original two colored graph G contains a monochromatic $B_{k,\ell}$. This completes the proof of the theorem. □

One can easily adjust the last theorem to include all values of ℓ, $1 \leq \ell < 2k$, by increasing the upper bound from $2k+\ell$ to $2k+\ell+3$. Of course the last result leaves as unsettled the exact value of $r(B_{k,\ell})$ for $1 \leq \ell < 2k$.

These results suggest a general question. If T_n is any tree with parts of size $n/3$ and $2n/3$ is $r(T_n) = \{4n/3-1\}$?

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