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ON TURÁN—RAMSEY TYPE THEOREMS II

by

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This paper is a continuation of our papers [5], [10]. We investigated the following problem:

Let the edges of K_n be coloured by r colours, G_i, $1 \leqq i \leqq r$ be the graph formed by the i'th colour. Let $f(n;\ k_1, \ldots, k_r)$ be the largest integer for which there is an r-colouring of K_n such that

$$K_{k_i} \nsubseteq G_i, \quad 1 \leqq i \leqq r$$

and

$$\sum_{i=1}^{r-1} e(G_i) = f(n;\ k_1, \ldots, k_r). \tag{1}$$

(Here $e(G)$ denotes the number of edges of G.)

Due to Ramsey's theorem for fixed $k_1, \ldots, k_r$, $n > N(k_1, \ldots, k_r)$ such a graph does not exist. Therefore the problem makes sense only in the case when at least one of the $k_i \to \infty$ with $n \to \infty$.

It is trivial that $f(n;\ 3, l) \leqq \frac{1}{2} nl$. We proved in [2] that if $l = o(n)$ then

$$f(n;\ 2k+1, l) = \frac{1}{2}\left(1 - \frac{1}{k}\right) n^2 + o(n^2). \tag{2}$$

Bollobás—Erdős [1] and Szemerédi [11] proved that $f(n;\ 4, l) = \frac{n^2}{8} + o(n^2)$ for $l = o(n)$. No asymptotic formula is known for $f(n;\ 2k, l)$ when $l = o(n)$ and $k > 2$.

Here we start to investigate $f(n;\ k_1, \ldots, k_r)$ for $r = 3$.

Notation. $G_n(V; E)$ is a graph with $|V| = n$, $e(G_n) = |E|$, $K(k_1, \ldots, k_r)$ is a complete r-partite graph with k_i vertices in the i'th class, K_n is the complete graph on n vertices.

Let V be the vertex set of the complete graph K_n. If we consider an r-colouring of the edges of K_n, let E_i be the set of edges of K_n having the ith colour for $1 \leqq i \leqq r$. Put $G_i = G(V; E_i)$ and

$$V_i(x) = \{y\colon (x, y) \in E_i\}, \quad d_i = |V_i(x)|,$$
$$V_i(x; U) = \{y\colon (x, y) \in E_i,\ y \in V - U\},$$
$$d_i(x; U) = |V_i(x; U)|.$$

For the case $r = 3$ we prove the following theorems:

THEOREM 1.

$$f(n;\,3,\,3,\,\varepsilon n) < \frac{n^2}{4} + c_2 \varepsilon n^2 \tag{3}$$

and for $n > n_0(\varepsilon)$

$$\frac{n^2}{4} + c_1 \varepsilon n^2 < f(n;\,3,\,3,\,\varepsilon n),$$

where $c_1 > 0,\ c_2 > 0$ *are absolute constants.*

THEOREM 2. *Let* $G_i(V;\,E_i)$, $1 \leqq i \leqq 3$ *be graphs belonging to a* 3*-colouring of* K_n *with the property*

$$K_3 \not\subset G_i \qquad i = 1, 2, \tag{4}$$

$$K_{\varepsilon n} \not\subset G_3 \tag{5}$$

and

$$|E_1| \geqq |E_2| > cn^2. \tag{6}$$

Then

$$|E_1 \cup E_2| < n^2\left(\frac{1}{4} - \sqrt{c} + 2c\right) + \eta n^2 \tag{7}$$

where $\eta \to 0$ *with* $\varepsilon \to 0$.

REMARK. We obtain the lower bound in Theorem 1 by a colouring in which G_1 is the complete bipartite graph $K\left(\left[\frac{n}{2}\right], \left[\frac{n+1}{2}\right]\right)$ and G_2 formed by two copies of a trianglefree graph with maximum independent set of size $o(n)$ and $|E_2| = o(n^2)$. Theorem 2 shows that this extremum is sharp; by the condition (6) we have the stronger inequality (7) instead of (3).

PROOF of Theorem 1.

(a) *The upper bound.*

We shall use the simple observation that

$$K_3 \not\subset G_i \qquad i = 1, 2$$

$$K_{\varepsilon n} \not\subset G_3$$

implies

$$|V_1(x) \cap V_2(y)| < \varepsilon n \tag{8}$$

for any $x \neq y,\ x, y \in V$.

Assume $|E_1| \geqq |E_2|$. Let x_0 be a vertex for which $d_1(x)$ is maximal. Let

$$d_1(y_0) = \max_{y \in V_1(x_0)} d_1(y), \quad y_0 \in V_1(x_0).$$

Since $K_3 \not\subset G_1$

$$V_1(x) \cap V_1(y) = \emptyset.$$

Let $U = V - (V_1(x_0) \cup V_1(y_0))$. Put

$$E_2^* = \{(x, y)\colon\ (x, y) \in E_2,\ x \notin U \text{ or } y \notin U\}.$$

First we prove

$$|E_2^*| < \sqrt{2}\varepsilon n^2. \tag{9}$$

By (8), obviously, any point $z \in V$ can be joined in G_2 to at most $2\varepsilon n$ points of $V_1(x_0) \cup V_1(y_0)$. This gives (9). Thus we only have to consider the set of edges

$$E_2^{**} = \{(x, y)\colon\ (x, y) \in E_2,\ x \in U,\ y \in U\}.$$

Put

$$|U| = \delta n$$

and

$$\delta^* n = \max_{x \in U} d_2(x; V-U) = d_2(x^*; V-U) \qquad (x^* \in U).$$

As before, by (8) we get that the number of edges in G_1 incident to a vertex in $V_2(x^*)$ is at most εn^2. Since $K_3 \not\subset G_1$, the number of the remaining edges of G_1 is less than $\frac{n^2}{4}(1-\delta^*)^2$. By all of these we obtain

$$|E_1 \cup E_2| < \frac{n^2}{4}(1-\delta^*)^2 + \delta\delta^* \frac{n^2}{2} + 3\varepsilon n^2 \tag{11}$$

If $\delta < \frac{2}{3}$ $\left(\text{and consequently } \delta^* < \frac{2}{3}\right)$ then (11) gives

$$|E_1 \cup E_2| < \frac{n^2}{4} + 3\varepsilon n^2.$$

So all we have to show is $\delta < \frac{2}{3}$.

We assumed $|E_1| \geqq |E_2|$, thus we may suppose

$$|E_1| > \frac{n^2}{8}, \quad |V_1(x_0)| > \frac{n}{4}. \tag{12}$$

Put $|V_1(x_0)| = \frac{n}{4} + t$. If $|V_1(y_0)| > \frac{n}{12} - t$ then

$$|V_1(x_0) \cup V_1(y_0)| > \frac{n}{3},$$

i.e., $\delta < \frac{2}{3}$.

If $|V_1(y_0)| \leqq \frac{n}{12} - t$, then

$$d_1(x) \leqq \frac{n}{12} - t \quad \text{for} \quad x \in V_1(x_0).$$

This gives

$$|E_1| \leqq \frac{1}{2}\left(\frac{3n}{4} - t\right)\left(\frac{n}{4} + t\right) + \left(\frac{n}{4} + t\right)\left(\frac{n}{12} - t\right) =$$

$$= \frac{1}{2}\left(\frac{n}{4} + t\right)\left(\frac{5}{6}n - 2t\right) \leqq \frac{1}{2}\left(\frac{5}{24}n^2 + \frac{2}{3}nt - 2t^2\right) < \frac{n^2}{8},$$

which contradicts to (12).

This completes the proof of the upper bound of (3).

(b) *The lower bound* in (3) follows by the adaptation of a construction in P. ERDŐS [2]:

Let l be an integer which will be determined later, let the vertices of G be the $0-1$ sequences of length $3l+1$. Two vertices of G are joined by an edge in G if the Hamming-distance of the corresponding two sequences is at least $2l+1$ (i.e., if the sequences differ in at least $2l+1$ places). This graph has no triangle and it follows from a theorem of KLEITMAN [9] that the size of the maximum independent set equals the common degree of the vertices.

Now from this graph G we construct the graph G^* as follows: we replace each vertex by a set of vertices of size $\left[\frac{m}{2^{3l+1}}\right]$, where l is the smallest integer for which

$$\sum_{i=0}^{l+1}\binom{3l+1}{i}\frac{m}{2^{3l+1}} < \varepsilon m.$$

It is easy to see, that this graph has no triangles and the maximum independent set has $<\varepsilon m$ vertices. The number of edges in G^* is $>c\varepsilon m^2$ where $c>0$ is an absolute constant.

Now we consider the following three-colouring of K_{2m}:

Let $V=V_1\cup V_2$ with $|V_1|=|V_2|=m$. Let $G^*(V_1)$, $G^{**}(V_2)$ be two graphs isomorphic to the above constructed G^* and

$$E_2 = E\big(G^*(V_1)\big)\cup E\big(G^{**}(V_2)\big),$$

$G_1(V)$ be the complete bipartite graph $K(V_1, V_2)$.

This construction gives the proof of the lower bound in (3).

REMARK. Very likely the following stronger result holds: There is an absolute constant c such that $(\varepsilon\to 0)$

$$f(n;\,3,\,3,\,\varepsilon n) = \frac{n^2}{4}+\big(c+o(1)\big)\,\varepsilon n^2$$

but at the moment we do not know how to prove this.

PROOF of Theorem 2.

Now we construct a sequence of points $x_1, \dots, x_k$ and a corresponding sequence of indices $i_1, \dots, i_k$ where $i_\nu\in\{1, 2\}$, with the following property: for $\lambda=\sqrt{\varepsilon}$ let

$$\lambda_{i_1}(x_1) > \lambda n,$$

$$\lambda_{i_\nu}(x_\nu; U_\nu) > \lambda n \quad \text{if} \quad \nu > 1$$

where for $\nu>1$

$$U_\nu = V-\bigcup_{l=1}^{\nu} V_{i_l}(x_l).$$

Let $x_1, \ldots, x_k$ be maximal in the sense that for any $x \in V - \{x_1, \ldots, x_k\}$

$$d_i(x; U_k) < \lambda n.$$

Obviously, $k < \frac{1}{\lambda}$. Put

$$V_1 = \bigcup_{\substack{1 \le l \le k \\ i_l = 1}} V_{i_l}(x_l; U_l), \quad V_2 = \bigcup_{\substack{1 \le l \le k \\ i_l = 2}} V_{i_l}(x_l; U_l)$$

$$(V_1 \cap V_2 = \emptyset) \quad \text{and} \quad V_3 = V - (V_1 \cup V_2),$$

$n_i = |V_i|$, $1 \le i \le 3$.

Consider now the edges in $E_1 \cup E_2$ of the following type:

$$F_{jl,i} = \{(x, y)\colon\ x \in V_j,\ y \in V_l,\ (x, y) \in E_i\},$$

$1 \le j \le 3$, $1 \le l \le 3$, $i = 1, 2$.

(a) $$|F^1_{1,1}| \le \frac{1}{4} n_1^2, \quad |F^2_{2,2}| \le \frac{1}{4} n_2^2$$

since G_1 and G_2 are triangle-free.

(b) $$|F^1_{1,1}| < \lambda n^2, \quad |F^2_{1,2}| < \lambda n^2$$
$$|F^2_{1,1}| < \lambda n^2, \quad |F^1_{2,2}| < \lambda n^2.$$

Otherwise we would have two points x_ν and x_μ with

$$|V_1(x_\nu) \cap V_2(x_\mu)| > \frac{\lambda}{k} n > \lambda \lambda n = \varepsilon n$$

which contradicts (8).

(c) $$|F^i_{j,3}| < \lambda n^2 \quad \text{for} \quad j = 1, 2, 3, \quad i = 1, 2.$$

Otherwise we would have an $x \in V$ with

$$\max_{i=1,2} d_i(x; V_3) \ge \lambda n.$$

But since

$$d_i(x; V_3) = d_i(x; U_k)$$

this would contradict the maximality of the sequence $x_1, \ldots, x_k$.

By (a)—(c) we obtain

$$|E_1| \le \frac{1}{4} n_1^2 + 10\lambda n^2 \tag{13}$$

$$|E_2| \le \frac{1}{4} n_2^2 + 10\lambda n^2. \tag{14}$$

Now by the assumption

$$|E_1| \ge |E_2| \ge cn^2$$

we get

$$n_i \ge 2n\sqrt{c - 10\lambda} \qquad i = 1, 2.$$

Hence by (13) and (14)

$$|E_1|+|E_2| \leqq n^2\left(\frac{1}{4}-\sqrt{c}+2c\right)+\eta(\varepsilon)n^2$$

where, as a simple computation shows, $\eta(\varepsilon)\to 0$ with $\varepsilon\to 0$.

REMARK. If $c>\frac{1}{16}$, there does not exist a three colouring of K_n, for which

$$K_3 \nsubseteq G_i, \qquad i=1,2$$

$$K_{\varepsilon n} \nsubseteq G_3$$

and

$$|E_1| \geqq |E_2| \geqq cn^2.$$

REMARK. First observe that the constant $\frac{1}{4}-\sqrt{c}+2c$ in Theorem 2 is best possible. To see this, let

$$V=V_1\cup V_2, \quad V_1\cap V_2=\emptyset,$$

$$|V_1|=[2\sqrt{c}n], \quad |V_2|=[(1-2\sqrt{c})n],$$

$$V_i=A_i\cup B_i, \quad |A_i|=\left[\frac{1}{2}|V_i|\right], \quad |B_i|=\left[\frac{1}{2}|V_i|+1\right], \quad i=1,2.$$

Join every vertex of A_i to every vertex of B_i in G_i (for $i=1,2$). Let the further edges of G_1, resp. G_2 form a graph on A_2 and on B_2, resp. on A_1 and on B_1 which has no triangle and the number of independent points is $o(n)$. (It is well-known, that such a graph exists and in fact we used this method in P. ERDŐS—V. T. SÓS [1] or in (6) of the proof of Theorem 1. Obviously this colouring has the required properties.

To get the exact result for $f(n;\,3,3,\varepsilon n)$ is rather hopeless because of its close connection with the Ramsey-numbers. This close connection is shown already by the following

PROPOSITION 1. *Let* $\varepsilon(n)\to 0$ *with* $n\to\infty$. *Then*

$$R(3,\varepsilon(n)n)=o(n) \tag{15}$$

implies

$$f(n;\,3,3,\varepsilon(n)n)=o(n^2). \tag{16}$$

(Here $R(k,l)$ is the Ramsey-number.)

PROOF. (a) Suppose $R(3,\varepsilon(n)n)=o(n)$ and that with a constant $c>0$

$$f(n;\,3,3,\varepsilon(n)n)>cn^2$$

holds. This means, that we have a three-colouring of K_n, for which

$$K_3 \nsubseteq G_i \qquad i=1,2$$

$$K_{\varepsilon(n)n} \nsubseteq G_3$$

and, e.g., $|E_1|>\frac{c}{2}n^2$. Thus we have a vertex x with $d_1(x)>cn$. Since $K_3\not\subset G_1$, in $V_1(x)$ we have only edges of E_2 and E_3.

But this means, that we have a two-colouring of the edges of K_{cn}, where in the first colour class there is no K_3 and in the second there is no $K_{\varepsilon(n)n}$. This contradicts (16).

The converse statement, that (16) implies (15) is probably true, too, but we could only prove the following weaker result:

Assume that

$$R(3, \varepsilon(n)n) > cn.$$

Then

$$f\left(n;\, 3,\, 3,\, \frac{\varepsilon(n)n}{2c}\right) > cn^2.$$

We hope to return to this subject later.

Some remarks on the Ramsey-numbers

As it is well-known, ERDŐS and SZEKERES [7] proved

$$R(k, l) \leqq \binom{k+l-2}{k-1}. \tag{17}$$

Probably (17) is not very far from being best possible, in particular

$$c_2\frac{n^2}{(\log n)^2} < R(3, n) < c_1\frac{n^2\log\log n}{\log n}.$$

It seems certain that

$$R(4, n) > n^{3-\varepsilon}. \tag{18}$$

The probability method surely must give (18) but so far technical difficulties prevented success.

GREENWOOD and GLEASON [8] proved

$$R_1(k_1+1, \ldots, k_r+1) \leqq \frac{(k_1+\ldots+k_r)!}{k_1!\ldots k_r!}.$$

This gives for example

$$R_3(3, 3, n) \leqq cn^4$$

and more generally

$$R_r(\underset{1}{3}, 3, \ldots, \underset{r}{3}, n) \leqq c_r n^{2r}.$$

A simple observation leads to the following improvement:

PROPOSITION 2.

$$R(3, 3, n) = o(n^3) \tag{19}$$

and more generally

(20) $$R_r(\underbrace{3, 3, \ldots, 3}_{r}, n) \leqq rnR_{r-1}(\underbrace{3, \ldots, 3}_{r-1}, n) = o(n^{r+1}).$$

PROOF. Let us consider a "good" r-colouring of K_m for $k_1=\ldots=k_{r-1}=3$, $k_r=n$. Let G_i, $1\leqq i\leqq r$ the graph formed by the edges of the ith colour-class. Put

$$V_i(x) = \{y\colon (x, y)\in E_i\}, \quad 1 \leqq i \leqq r.$$

Let $U=\{x_1, \ldots, x_v\}$ be the vertex-set of a maximal-sized complete graph in G_r. We have $v\leqq n-1$. By the maximality of $|U|$ we have

$$\bigcup_{j=1}^{v} \bigcup_{i=1}^{r-1} V_i(x_j) = V-U.$$

Since G_i, $1\leqq i\leqq r-1$ is triangle-free,

$$|V_i(x_j)| < R_{r-1}(3, \ldots, 3, n) \quad \text{for} \quad j = 1, \ldots, v.$$

Now taking into consideration $R(3, n)=o(n^2)$, this proves (20).

REMARK. We have no nontrivial lower bound for $R(3, 3, n)$. It is trivially true, that

$$R(3, 3, n) \geqq 2R(3, n).$$

We expect that

$$R(3, 3, n)/R(3, n) \to \infty$$

$$R(3, 3, n)n^{-2} \to \infty$$

or even more,

$$R(3, 3, n) > n^{3-\varepsilon}.$$

Some remarks on the two-colourings of K_n

The following problem belongs to the questions we considered in [5]. Let $f(n; G)$ be the smallest integer for which every graph of n vertices and of $f(n; G)$ edges contains a subgraph isomorphic to G and $f(n; G, \varepsilon n)$ be the smallest integer for which every graph of n vertices and $f(n; G, \varepsilon n)$ edges either contains a subgraph isomorphic to G or has an independent set of size εn.

First we investigate conditions which imply

(21) $$f(n; G, \varepsilon n) \leqq \eta n^2$$

where $\eta\to 0$ with $\varepsilon\to 0$ or

(22) $$f(n; G, \varepsilon n) < f(n; G)(1-c)$$

with a $c>0$.

We prove some preliminary results about (21) and (22) and state without proof a few more results.

PROPOSITION 1. (21) *holds for* $G\sim K(1, r, r)$.

PROOF. We need the following result of Erdős:

For every l there exists a constant $c_l>0$ such that if $n>n_0$ and $e(G_n)>cn^2$ then G_n contains a $K(l, c_l, n)$.

Using this it is easy to show that if for G_n $e(G_n)=cn^2$ and the largest independent set in G_n has size less than $\varepsilon(c)n$, then G_n contains a $K(1, r, r)$.

PROPOSITION.

$$f(n;\ K(3,3,3), \varepsilon n) = \frac{n^2}{4}(1+\eta)$$

where $\eta\to 0$ *with* $\varepsilon\to 0$.

PROOF. The stronger

$$f(n;\ K(3,3,3)) \leqq \frac{n^2}{4}(1+\eta)$$

follows from ERDŐS—STONE [6].

We can prove the lower bound as follows:

Let $|V_1|=\left[\frac{n}{2}\right]$, $|V_2|=\left[\frac{n+1}{2}\right]$. We join every vertex of V_1 to every vertex of V_2. Additionally on V_1 resp. on V_2 we consider a graph whose largest independent set has size εn and which contains no circuit C_r with $3\leqq r\leqq 5$. (We know the existence of such a graph from [3], [4].) This graph contains no $K(3,3,3)$ since the vertex set of $K(3,3,3)$ cannot be decomposed into two sets neither of which spans a graph without a circuit.

In a forthcoming paper we prove the more general

THEOREM A. *Let G be a graph which is k-chromatic and the vertex-set can be decomposed into* $k-1$ *sets which span graphs without circuits. Then there is a* $c>0$ *such that*

$$f(n;\ G, \varepsilon n) \leqq \frac{n^2}{2}\left(1-\frac{1}{k-1}-c\right)$$

for $\varepsilon<\varepsilon_0, n>n_0$.

As to (22) we prove

THEOREM B. *Let G be a graph which is k-chromatic and the vertex-set of G cannot be decomposed into* $k-1$ *sets such that the subgraphs spanned by these sets have no circuit. Then for every* $\eta>0$

$$f(n;\ G, \varepsilon n) \geqq \frac{n^2}{2}\left(1-\frac{1}{k-1}-\eta\right)$$

if $\varepsilon<\varepsilon_0(\eta), n>n_0(\eta)$.

Added in proof (December, 1981). We proved with A. Hajnal and E. Szemerédi that

$$f(n;\ 2k, l) = \frac{1}{2}\left(\frac{3k-5}{3k-2}\right)n^2+o(n^2) \quad \text{for} \quad k\geqq 2$$

when $l=o(n)$. The proof will appear in a quadruple paper in *Combinatorica.*

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