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ON RAMSEY—TURÁN TYPE THEOREMS FOR HYPERGRAPHS

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Dedicated to Tibor Gallai on his seventieth birthday

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Let H^r be an r-uniform hypergraph. Let $g=g(n;H^r)$ be the minimal integer so that any r-uniform hypergraph on n vertices and more than g edges contains a subgraph isomorphic to H^r. Let $e=f(n;H^r,\varepsilon n)$ denote the minimal integer such that every r-uniform hypergraph on n vertices with more than e edges and with no independent set of εn vertices contains a subgraph isomorphic to H^r.

We show that if $r>2$ and H^r is e.g. a complete graph then

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\binom{n}{r}^{-1}f(n;H^r,\varepsilon n)=\lim_{n\to\infty}\binom{n}{r}^{-1}g(n;H^r)$$

while for some H^r with $\lim_{n\to\infty}\binom{n}{r}^{-1}g(n;H^r)\neq 0$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\binom{n}{r}^{-1}f(n;H^r,\varepsilon n)=0.$$

This is in strong contrast with the situation in case $r=2$. Some other theorems and many unsolved problems are stated.

Let $H^r(V;E)$ be an r-uniform hypergraph and $f(n;H^r)$ be the smallest integer for which every r-uniform hypergraph of n vertices and more than $f(n;H^r)$ edges contains a subgraph isomorphic to H^r. A $G_n^r(V;E)$ is called an extremal graph belonging to H^r, if $|V|=n$, $e(G_n^r)=f(n;H^r)$ and G_n^r does not contain a subgraph isomorphic to H^r ($e(\dots)$ denotes the number of hyperedges).

The determination (or estimation) of $f(n;H^r)$ is the fundamental problem of extremal graph theory which was started by Turán [9]. As a generalization of Turán's theorem, the well-known Erdős—Stone theorem [6] states the following.

For an arbitrary H^2:

$$f(n;H^2)=\left(\frac{1}{2}\frac{\chi(H^2)-2}{\chi(H^2)-1}+o(1)\right)n^2 \quad \text{if} \quad n\to\infty \tag{1}$$

where $\chi(H)$ is the chromatic number of H.

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First of all, we remark that for $r>2$ almost nothing is known about $f(n; H^r)$. E.g. for the simplest graphs K_4^3 (the complete 3-uniform hypergraph on 4 vertices) or $H^3(4; 3)$ (three triples on four vertices) not even the asymptotic value of $f(n; H^3)$ is known. Turán's classical conjecture is, that

$$(2) \qquad f(n;\ K_4^3) \sim \frac{5}{9}\binom{n}{3} \quad \text{if} \quad n \to \infty$$

and it is very probably that

$$(3) \qquad f(n;\ H^3(4;\ 3)) \sim \frac{1}{4}\binom{n}{3} \ \text{if} \ n \to \infty.$$

As to the general case, it is easy to see that for an arbitrary H^r

$$\lim_{n\to\infty} \binom{n}{r}^{-1} f(n;\ H^r) = c(H^r)$$

exists. It is well-known [4] that $c(H^r)=0$ if and only if the vertices of H^r can be split into r classes so that every edge of H^r meets all r classes.

We observed [5], [7] that for $r=2$, $H=K_k$ (where K_k is the complete graph on k vertices), the extremal graph is stable in the following sense: it contains "very large" independent sets and if we put on a condition which decreases the size of the maximal independent set in G_n then the number of edges of the corresponding extremal graphs gets drastically reduced. More precisely, let $f(n; H^r, l)$ be the smallest integer for which every graph of n vertices and more than $f(n; H^r, l)$ edges either contains a subgraph isomorphic to H^r or it contains an independent set of size l.

Due to Ramsey's theorem for fixed H^r and l, $f(n; H^r, l)=0$ if $n>R(H^r, l)$. Therefore, the problem makes sense only in the case when either $|V(H^r)|\to\infty$ or $l\to\infty$. Referring to the case $r=2$ and $H=K_k$, we proved

$$(5) \qquad \lim_{\varepsilon\to 0}\lim_{n\to\infty} \binom{n}{2}^{-1} f(n;\ K_k, \varepsilon n) = \begin{cases} \dfrac{k-3}{(k-1)} & \text{if} \quad k \quad \text{odd} \\[2ex] \dfrac{3k-10}{(3k-4)} & \text{if} \quad k \quad \text{even} \end{cases}$$

while by Turán's theorem

$$(6) \qquad \lim_{n\to\infty} \binom{n}{2}^{-1} f(n; K_k) = \frac{k-2}{(k-1)}$$

(For k odd see [5], for $k=2$ see [1], [8] and for $k>2$, even see [2].)

In this paper, we investigate analogous problems for hypergraphs. The main result of this paper is that surprisingly the situation is quite different for hypergraphs. E.g. for K_k^r (and for a more general class of graphs) the condition that the largest independent set has size $o(n)$ does not change the situation. We prove

Theorem 1. *Let* $r\geqq 3$, *H^r be an r-uniform hypergraph,* $E=\{h_1, \ldots, h_m\}$ *be the edge-set of H^r. Suppose H^r satisfies the condition*

(7) *for every i,* $1 \leqq i \leqq m$ *there exist a* $j \neq i$ *such that* $|h_i \cap h_j| \geqq 2$.

Let $\lim_{n\to\infty}\binom{n}{r}^{-1} f(n; H^r) = c(H^r)$ *and* $\lim_{\varepsilon\to 0}\lim_{n\to\infty}\binom{n}{r}^{-1} f(n; H^r, \varepsilon n) = c^*(H^r)$. *Then*

$$c^*(H^r) = c(H^r). \tag{8}$$

Remark. Condition (7) holds e.g. for K_k^r and also for H^3 (4.3).

Proof. Our idea in the proof is that if there are large independent sets in the extremal graph, we spoil them by adding not too many new edges and then we have to omit some, but not too many, to destroy the possible H's and not to create large independent sets.

Let $\alpha(G^r)$ denote the size of the largest independent set of G^r. We use the following theorem of Erdős—Hajnal [3]:

For arbitrary $\eta>0$ *and* $m>N(\eta)$ *there exists a graph* L_m^r *with the following properties:*

$$(*)\qquad \begin{cases} e(L_m^r) < m^{3/2}, \\ \alpha(L_m^r) < \eta m, \\ \text{if} \quad e_i, e_j \in E(L_m^r), \quad i \neq j, \quad \textit{then} \quad |e_i \cap e_j| \leqq 1. \end{cases}$$

Let $\delta>0$ and $\varepsilon>0$ be arbitrary and (G_n^r) be a graph satisfying

$$H^r \not\subset G_n^r \tag{9}$$

$$\varepsilon n > N(\varepsilon^2)$$

$$e(G_n^r) > (c-\delta)\binom{n}{3} \tag{10}$$

where $c=c(H^r)$. Decompose the vertex set $V=V(G_n^r)$ in the form

$$V = B \cup \bigcup_{i=1}^{k} A_i$$

where $\alpha\big(G^r(B; E(B)\big) < \frac{\varepsilon n}{2}$ and

$$\alpha\big(G^r(A_i; E(A_i)\big) > \frac{\varepsilon n}{2}.$$

Obviously, $k \leqq 2/\varepsilon$. We place into the set A_i a hypergraph $L^r(i)$ with $V\big(L^r(i)\big)=A_i$ and which satisfies (*) with $\eta=\varepsilon^2$. So we added to our G_n^r new edges and the new enlarged hypergraph has clearly no independent set of size $>\varepsilon n$. But this new graph may contain a graph isomorphic to H^r. To avoid this, omit all edges $e\in E(G_n^r)$ which intersect any of our new edges in at least two vertices. So we omitted at most $O\big(n^{r-\frac{1}{2}}\big)$ edges. Observe that this final graph G_n^{*r}

(A) contains no isomorphic copy of H^r due to the condition on H^r.

(B) $\alpha(G_n^{*r})<\varepsilon n$ since we did not omit any edge contained in any of the A_i, $1\leqq i\leqq k$.

(C) $|E(G_n^{*r})| > (c-\delta)\binom{n}{r} + O\big(n^{r-\frac{1}{2}}\big)$.

Since $\delta>0$ was arbitrary, the proof of Theorem 1 is complete. ∎

On the other hand we state

Theorem 2. *Let H^r be a graph for which there is a partition of the vertex set*

$$V(H^r) = \bigcup_{i=1}^{r} A_i$$

so that

$$E(H^r) = E_1 \cup E_2$$

and

$$|h \cap A_i| = 1 \quad \text{for} \quad i = 1, \ldots, r \quad \text{if} \quad h \in E_1, \tag{10}$$

furthermore, for

$$E_2 = \{h_1, \ldots, h_s\} \subseteqq \{h: h \subseteqq A_r\}$$

$$\Big|h_k \cap \bigcup_{1 \le i \le k-1} h_i\Big| \leqq 1 \quad \text{for} \quad k = 2, \ldots, s. \tag{11}$$

Then

$$c^*(H^r) = 0.$$

Proof. We use the following theorem of Erdős [4]. There exists a function $f(t)$ so that if $n > N(t, c)$ and $e(G_n^r) > cn^r$ then

$$K^r\Big(\underset{1}{t}, \ldots, \underset{r-1}{t}, \frac{n}{f(t)}\Big) \subset G_n^r.$$

Let $t = \max_{1 \le i \le r-1} |A_i|$. Suppose there exists an infinite sequence of graphs G_n^r with $e(G_n^r) > cn^r$, for which

$$H^r \not\subset G_n^r \tag{12}$$

and

$$\alpha(G_n^r) = o(n). \tag{13}$$

Let U be the set of vertices of a $K^r\Big(t, \ldots, t, \dfrac{n}{f(t)}\Big)$ contained by G_n^r, and $U_r = \Big\{x_1, \ldots, x_l, l = \dfrac{n}{f(t)}\Big\} \subset U$ be the vertices in the rth class. By (12), the subgraph of G_n^r spanned by U_r cannot contain a subgraph isomorphic to $L(A_\cdot; E_2)$.

Now we prove that by the condition (11) on $L(A_r; E_2)$, G_n^r, more exactly U_r must contain a large independent set.

Let $H(j)$ $(j = 1, \ldots s)$ denote the subgraph of H^r formed by the edges $(h_1, \ldots, h_j)$ and $G_n^r(U_r)$ denote the subgraph of G_n^r spanned by the subset of vertices U_r. Suppose $H(k-1) \subset G_n^r(U_r)$ but $H(k) \not\subset U_n^r(U_r)$ for some $1 < k \leqq s$. Let H_1 be a copy of $H(k-1)$ contained in $G_n^r(U_r)$ and $V_1 = V_1(H_1)$. By (11) there is a vertex $x \in V_1$ so that x is independent of $U_r - V_1$. Note $|U_r - V_1| > n/f(t) - sr$. Now apply the same argument to $G_n^r(U_r - V_1)$. Thus, we obtain a vertex $x_2 \in U_r - V_1$ which is independent of a set $U_r - V_1 - V_2$ where $|U_r - V_1 - V_2| > n/f(t) - 2rs$. We continue this process and obtain and independent set of size $\geqq n/f(t)rs$. Having (13) this contradiction proves the theorem.

Problem 1. *Is condition $|h_i \cap h_j| \geqq 2$ in Theorem 1 necessary for the truth of (8)?*

Problem 2. *Does there exist a graph H_n^r for which*

$$0 < c^*(H^r) < c(H^r)?$$

Problem 3. *Let $V_1=\{x_1\}$, $V_2=\{x_2, x_3, x_4\}$, $V_3=\{x_5, x_6, x_7\}$, and H^3 be the hypergraph with $V(H^3)=\{x_i, 1\leqq i\leqq 7\}$ and*

$$E(H^3) = \big\{\{x_2, x_3, x_4\}, \{x_5, x_6, x_7\} \quad and \quad \{x_i, x_j, x_l\}\colon x_i\in V_1, x_j\in V_2, x_l\in V_3\}\big\}.$$

We know that $c(H^3)>0$. Is $c^*(H^3)=c(H^r)$ or $0<c^*(H^3)<c(H^r)$ or $c^*(H^3)=0$?

Problem 4. *Let $V(H^3)=V_1\cup V_2\cup V_3$, where V_2 and V_3 are independent sets but the graph spanned by V_1 contains a circuit. What can one say on $c(H^3)$?*

Problem 5. *Is condition (11) in Theorem 2 necessary?*

Problem 6. *Find a function $h(n)$ so that*

$$\binom{n}{3}^{-1} f(n;\, K_n^3, h(n)) = O(n^3).$$

Our Theorem 1 gives that there is an $\alpha<1$ for which $f(n; K_4^3, n^\alpha)>c_\alpha n^3$ for a $c_\alpha>0$. Is $\inf\{\alpha\colon \lim n^{-3} f(n; K_4^3, n^\alpha)>0\}>0$?

Graphs of uniform edge density

Remark. We know that for every $\varepsilon>0$ there exists a graph G_n so that $e(G_n)>(1/8-\varepsilon)n^2$, $K_4\not\subset G_n$ and $\alpha(G_n)<\varepsilon n$. On the other hand, it is easy to see that if every subset of V which is "small enough" has a not "too small" edge density then our G_n must contain a K_k. Now we make this vague and heuristic statement more precise in two ways.

Proposition 1. *Let G_{n_i}; $n_1<n_2<\dots$ be a sequence of graphs with the following properties:*

(14) $$e(G_{n_i}) > c_i n_i^2,$$

(15) *If f is an arbitrary function with $\lim_{x\to 0} f(x)=0$ and if $i>i_0(\varepsilon)$ then in G_{n_i} every set of $>\varepsilon n_i$ vertices spans a subgraph of at least $f(\varepsilon)\varepsilon^2 n^2$ edges. Then for i large enough $K_k\subset G_{n_i}$.*

Proposition 2. *Let k be given. For every $\varepsilon>0$ there is an $\eta=\eta(\varepsilon, k)>0$ so that if G_n is a graph which satisfies that every subgraph of more than ηn vertices spans a subgraph of at least $\varepsilon\eta^2 n^2$ edges, then $K_k\subset G_n$.*

The proofs are easy and left to the reader.

Now we consider the analogous question for hypergraphs.

Concerning the case $r\geqq 3$ we have:

Proposition 3. *For every $\eta>0$ there exists an $\varepsilon>0$ and a graph G_n^3 having the following property*

$$K_4^3 \notin G_n^3 \quad (\text{and even more, } H^3(4;3) \not\subset G_n^3)$$

and each spanned subgraph of G_n^3 of more than ηn vertices contains more than $\varepsilon\binom{\eta n}{3}$ edges.

To see this, let $n=3^k$, $V(G_n^3)=\{1,\dots,n\}$, $i=\sum \varepsilon_\nu 3^\nu$ the trinary expansion of i, and

$$E(G_n^3)=\Big\{\{i,j,l\}: i=\sum_{\nu=0}^{k}\varepsilon_\nu^{(1)}3^\nu,\ j=\sum_{\nu=0}^{k}\varepsilon_\nu^{(2)}3^\nu,\ l=\sum_{\nu=0}^{k}\varepsilon_\nu^{(3)}3^\nu,$$

$$\varepsilon_\nu^{(1)}=\varepsilon_\nu^{(2)}=\varepsilon_\nu^{(3)} \quad \text{for} \quad \nu<\nu_0, \quad \{\varepsilon_{\nu_0}^{(1)},\varepsilon_{\nu_0}^{(2)},\varepsilon_{\nu_0}^{(3)}\}=\{0,1,2\}.$$

It is easy to see, that this graph has the above property. At the same time to every $\varepsilon>0$, there is an $\eta>0$ so that there is a spanned subgraph of G_n^3 of more than ηn vertices and contains less than $\varepsilon\binom{\eta n}{3}$ edges. This means that the edge-density is not uniformly positive.

Problem 4. *Assume now that we have an infinite sequence (G_n^3) so that for every spanned subgraph of $m>\eta n$ vertices G_n^3 contains more than $c\binom{m}{3}$ edges. Does it then follow that our graph contains a K_k^3 if $n>n_0(\varepsilon,c,r)$? We do not know the answer even for $k=4$, in fact, we do not even know whether our graph contains a $H^3(4;3)$.*

Perhaps it is clearer to state the problem in a sligthly weaker form.

Problem 5. *Assume that G_n^3 has the property that every spanned subgraph of $m>\frac{n}{\log n}$ vertices contains at least $c\binom{m}{3}$ edges. Does our graph then contain a $H^3(4;3)$ or a K_4^3 or K_k^3?*

The role of $\frac{n}{\log n}$ could be replaced by any $f(n)$ with $\frac{f(n)}{n}\to 0$.

Problem 6. *Assume that there is a c_1 so that for every $x,y\in V(G_n^3)$*

$$\big|\{z:\{x,y,z\}\in E(G_n^3)\}\big|>cn.$$

Is it then true that $H^3(4;3)\subset G_n^3$?

Problem 7. *We define a sequence of graphs $G_{n_i}^3$ $(i=1,2,\dots)$ to be uniformly distributed it for every $\eta>0$ there is a $c(\eta)$ so that for every $i>i_0(\varepsilon)$ every spanned subgraph of $m>\eta n$ vertices has $(c(\eta)+O(1))\binom{m}{3}$ edges. Is there a graph H^3 so that there is an extremal graph belonging to H^3 which is uniformly distributed?*

We expect that such a graph does not exist.

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