Miscellaneous problems in number theory

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In this note I discuss some somewhat unconventional problems on consecutive integers and on additive number theory and on prime factors. Some of the work is joint with J. L. Selfridge. I will try to give as exact references as my memory permits.

I

Put

$$(1) \qquad n! = \prod_{p_i} p_i^{\alpha_i(n)}, \qquad \alpha_i(n) = \sum_{k=1}^{\infty} \left[\frac{n}{p_i^k}\right] < \frac{n}{p_i - 1} .$$

Recently Selfridge and I proved the following

Theorem 1. Denote by $h(n)$ the number of different exponents $\alpha_i(n)$. There are absolute constants c_1 and c_2 for which

$$(2) \qquad c_1 \left(\frac{n}{\log n}\right)^{1/2} < h(n) < c_2 \left(\frac{n}{\log n}\right)^{1/2}$$

The upper bound of (2) is trivial. If $p > (n\log n)^{1/2}$

then by (1) there are at most $(\frac{n}{\log n})^{1/2}$ possible choices of $\alpha_i(n)$ and if $p \le (n \log n)^{1/2}$ there are $(2+o(1))(\frac{n}{\log n})^{1/2}$ such primes. Thus the upper bound of (2) holds for every $c_2 > 3$, $n > n_o(c_2)$.

The proof of the lower bound is slightly less trivial. We will not try to give as large a value of c_1 as possible since it will be clear from the proof that our method will never give the best possible value. We will leave some of the details of the proof to the reader. Let $\varepsilon > 0$ be fixed and sufficiently small. Let

$$(3) \qquad n^{1/2} < p_{i_1} < \dots < p_{i_r} < \varepsilon^2 (n \log n)^{1/2}, \; p_{i_{j+1}} - p_{i_j} > \varepsilon \log n$$

be a maximal subsequence of the primes satisfying (3) (i.e. $p_{i_{j+1}}$ is the smallest prime greater than p_{i_j} satisfying (3)). It follows from (1) and (3) by a simple computation that $(\alpha_{i_j}(n) = [\frac{n}{p_{i_j}}])$ the $\alpha_{i_j}(n)$, $1 \le j \le r$ form a strictly decreasing sequence. Thus $h(n) \ge r$. Further it easily follows from Brun's method that for sufficiently small ε, $r > \varepsilon^4 (\frac{n}{\log n})^{1/2}$ which completes the proof of Theorem 1. Just a few words of explanation about Brun's method. It is well known and easily follows from Brun's method that there is an absolute constant c so that for sufficiently small $\delta > 0$ the number of solutions of

$$p_i < x, \quad 0 < p_i - p_j < \delta \log x$$

is less than $c\,\delta \frac{x}{\log x}$. This easily implies $r > \varepsilon^4 \frac{x}{\log x}$

for sufficiently small $\varepsilon>0$.

There is no doubt that there is a constant $c>0$ for which

$$(4) \qquad h(n) = (c+o(1))\left(\frac{n}{\log n}\right)^{1/2}.$$

The proof of (4) seems to present very serious difficulties since we do not know enough about the difference of consecutive primes.

Put

$$(5) \qquad \prod_{i=1}^{n}(x+i) = \Pi_1\, p_i^{\alpha_i(x,n)}\; \Pi_2\, q_j^{\beta_j(x,n)}$$

where in Π_1 the p_i run through all the primes not exceeding n and the primes q_j are the primes greater than n. It is easy to see that for every n there are infinitely many values of x for which all the $\alpha_i(x,n)$ are all distinct. In fact surely the $\alpha_i(x,n)$ are subject only to the condition $\alpha_i(x,n) \geq \alpha_i(n)$. (I did not carry out the details, but do not expect any difficulties.) On the other hand I do not believe that for large n all the exponents

$$(6) \qquad \{\alpha_i(x,n),\quad \beta_j(x,n)\}$$

can be different. In fact I would conjecture that for every k there is an n_o so that for $n > n_o$ at least k of the exponents (6) must be 1. This conjecture is no doubt unattainable at present. Several years ago Selfridge and I proved that the product (5) is never a power and we conjectured that at least one of the exponents (6) is 1. This conjecture seemed to us hopeless then and it still is hopeless and probably will stay so for some time.

For small n of course all the exponents (6) can be distinct, but I can not even prove that for $n=2$ there are infinitely many values of n for which the exponents (6) are all distinct. No doubt there are infinitely many primes p for which $8p^2+1$ is a prime, thus $\{3, 2, 1\}$ occurs infinitely often for $n=2$.

Let $f(n)=p_j$ where

$$\prod_{i=1}^{j-1} p_i^{\alpha_i(n)} < (n!)^{1/2} < \prod_{i=1}^{j} p_i^{\alpha_i(n)}$$

In the American Math Monthly Selfridge and I proved the following problem (advanced problem 6339). Prove that

$$\lim_{n\to\infty} f(n)/n = c \quad \text{and that if } f(n)=p_j \text{ then}$$

for $m>n$ $f(m)\geq p_{j-1}$. Further $f(n+1)\leq p_{j+1}$.

The proof of these statements is not difficult. We could not prove that there are infinitely many values of n for which $f(n)=p_j$, $f(n+1)=p_{j-1}$ (i.e. $f(n)$ is not monotonic). In fact we do not know any such value of n. This question remains open.

Let $n=\prod p_i^{\alpha_i}$. Put $f(n)=p_j$ where

$$\prod_{i=1}^{j-1} p_i^{\alpha_i} \leq n^{1/2} < \prod_{i=1}^{j} p_i^{\alpha_i}.$$

It is not difficult to prove that the density of integers n for which $f(n)<n^{\alpha}$ exists. Denote this density by $g(\alpha)$. $g(0)=0$, $g(1)=1$, $g(\alpha)$ is continuous and strictly increasing. We

leave the simple proof to the reader. The following problem is perhaps interesting and certainly seems difficult:

Is it true that for every $\varepsilon>0$ there is a K_ε so that for every $k>K_\varepsilon$ the density of integers n for which

$$(7) \qquad n^{k(\frac{1}{2}-\varepsilon)} < f\Big(\prod_{i=1}^{k}(n+i)\Big) < n^{k(\frac{1}{2}+\varepsilon)}$$

is greater than $1-\varepsilon$?

I have not been able to prove or disprove this conjecture. Perhaps the right and left side of (7) can be replaced by $n^{\frac{k}{2}-\varepsilon}$ and $n^{\frac{k}{2}+\varepsilon}$.

Denote by $P(m)$ the greatest prime factor of m. Selfridge and I considered the following question. Consider

$$(8) \qquad \frac{a_1\, a_2\, \ldots\, a_k}{n!} = I_n\ , \quad P(I_n) \le n\ .$$

In other words I_n is an integer all prime factors of which are not exceeding n. Put

$$(9) \qquad L(n) = \min\,(a_k - a_1)$$

where the minimum in (9) is to be extended over all $k>1$ and all sequences $a_1<\ldots<a_k$ satisfying (8).

We hope to prove that for all n $L(n)\le n-3$ and that this is best possible for infinitely many n. We hope to return to this problem in the future.

We could not prove that $L(n)\to\infty$ as $n\to\infty$. In fact we could not even prove that $L(n)>2$ for all $n>n_o$. In other words we could not prove that if $\frac{x(x+1)}{n!}$ is an integer then $P(\frac{x(x+1)}{n!}) > n$. In fact we could not even prove that $L(n) > 2$ for infinitely many n.

It is very likely that for $n > n_o$,

$$n! = (x+1) \dots (x+k),$$

has no solutions for $x \geq 2$, $k \geq 2$. Spiro pointed it out to me that $n! = x(x+1)$ is unsolvable for infinitely many n.

It is not difficult to prove that

$$(10) \qquad L(n) < n - c_1 \log n$$

holds for almost all n if c_1 is sufficiently small. To prove (10) observe that for almost all n

$$(11) \qquad \prod_{1 \leq i \leq c_1 \log n} (n+i) \mid \binom{2n}{n}$$

(11) immediately implies (10). The proof of (11) is not difficult and I do not give it here. I expect that apart from the value of c_1 (10) is best possible i.e. for almost all n

$$(12) \qquad n - c_2 \log n < L(n) < n - c_1 \log n$$

I am very far from being able to prove (12), but hope to be able to convince the reader that (12) is well motivated. If the lower bound of (12) would fail then it is easy to see that we could assume $a_1 > Cn$ for every C if $n > n_o(C)$. Now we prove

Theorem 2. Assume $a_1 > (1+\varepsilon)n$ and

(13) $$\frac{a_1 a_2 \dots a_k}{n!} = I_n \ , \ a_k - a_1 < n, \ I_n \text{ integer } \ P(I_n) \leq n.$$

Then

$$a_1 > 2^{n - c_3 n \log\log n / \log n}$$

In other words if (13) holds then a_1 must be enormously large. I hope that the following much stronger conjecture holds: (13) implies $k=1$, $a_1 \equiv 0 (\text{mod } n!)$. The reason that I believe this conjecture is that I think that for $x > (1+\varepsilon)^n$ $u<n$, $n>n_0(\varepsilon)$ we have $P(x(x+u)) > n$. Unfortunately the tools at our disposal are far too weak to attack this conjecture.

We prove Theorem 2 in several steps. First we prove

Lemma 1. For every prime p there is an integer m, $x < m \leq x+n$ so that

$$p^{\alpha_p(n)+1} \nmid \frac{1}{m} \prod_{i=1}^{n} (x+i) .$$

In other words $\frac{1}{m} \prod_{i=1}^{n} (x+i)$ is not divisible by a higher power of p than $n!$.

The Lemma is well known and the proof is immediate. It suffices to choose m so that m divides p to the highest power amongst all the integers $x+1, \dots, x+n$.

The Lemma immediately implies that $\prod_{i=1}^{n} (x+i)$ divides p to a power at most

(14) $$\alpha_2(n) + \frac{\log m}{\log 2} < \alpha_2(n) + 2 \log x$$

(14) will be our main tool. Denote by $\Pi'(x+i)$ the product of the integers $x+i$, $1 \le i \le n$, $P(x+i) \le n$. To prove Theorem 2 it suffices to show that if

$$n(1+\varepsilon) < x < 2^{n-c_3 n \operatorname{loglog} n/\log n}$$

then 2 divides $n!$ to a higher power than $\Pi'(x+i)$.

Assume first

(15) $$n(1+\varepsilon) < x \le 2n$$

Observe that if $n < p < n(1+\varepsilon/2)$ then $x < 2p \le x+n$. By the prime number theorem the number of these primes is $(1+o(1)) \frac{\varepsilon}{2} \frac{x}{\log x}$. Thus by (14) if (15) holds then $\Pi'(x+i)$ divides 2 to an exponent less than

$$\alpha_2(n)+2 \log 2n - (1+o(1)) \frac{\varepsilon x}{2 \log x} < \alpha_2(n)$$

which proves Theorem 2 for $x \le 2n$.

Assume next

(16) $$2n < x \le n^{1+\delta}$$

where $\delta>0$ is a sufficiently small absolute constant.

It follows from the prime number theorem of Hoheisel (see e.g. K. Prachar, Primzahlverteilung, Springer Verlag 1956) that if (16) is satisfied then

$$\Pi(\frac{x+n}{2}) - \Pi(\frac{x}{2}) > c_4 n/\log n$$

Thus there are at least $c_4 n/\log n$ even integers m, $x < m < x+n$ with $P(m) > \frac{x}{2} > n$. Thus $\Pi'(x+i)$ divides 2 to an exponent less than

$$\alpha_2(n) + 2 \log x - c_4 n/\log n < \alpha_2(n) ,$$

which proves Theorem 2 for $x \le n^{1+\delta}$.

Now we need

Lemma 2. Let $x > n^{1+\delta}$. Then there are at least $c_5(\delta)n$ even integers m, $x<m\le x+n$, satisfying $P(m) > n$.

To see this consider

$$\Pi'_1 ([\frac{x}{2}] + i), \quad 1 \le i \le \frac{n}{2}$$

where in Π'_1 we omit all the $[\frac{x}{2}]+i$ with $P([\frac{x}{2}] + i) > n$ and for every $p \le n$ we omit the integer $[\frac{x}{2}] + i$ which divides p to the highest power amongst all the $[\frac{x}{2}] + i$, $1 \le i \le [\frac{n}{2}]$. By Lemma 1

$$(17) \qquad \Pi'_1 ([\frac{x}{2}] + i) \Big| [\frac{n}{2}] !$$

If Lemma 2 would be false then the product (17) would have at least

$$\frac{n}{2} - 2 \frac{n}{\log n} - c_5(\delta)\, n > (\frac{1}{2} - 2\, c_5(\delta))\, n$$

factors $[\frac{x}{2}]+i$ with $p([\frac{x}{2}]+i) \le \frac{n}{2}$. This contradicts (17). To see this observe that $x > n^{1+\delta}$ and thus for sufficiently small $c_5(\delta)$ a simple computation shows that the left side of (17) is larger than the right one. This contradiction proves Lemma 2.

From Lemma 2 and (14) we obtain that for $x > n^{1+\delta}$ the highest power of 2 which divides $\Pi'(x+i)$ is less than

$$\alpha_2(n) + 2 \log x - c_5(\delta)\, n < \alpha_2(n)$$

for $x < 2^{c_5(\delta)n/2}$, which proves Theorem 2 for $x < 2^{c_5(\delta)n/2}$.

<u>Lemma 3. Let</u> $x > 2^{cn}$. <u>Then the number of integers</u> $x < m \le x+n$, $P(m) \le n$ <u>is at most</u> $(1+o(1))\frac{n}{\log n}$.

It follows from Lemma 1 that

(18) $$\Pi_1'(x+i) \mid n!$$

where in (18) $1 \le i \le n$ and the $x+i$ with $P(x+i) > n$ are omitted and we further omit for every $p \le n$ an integer $x+i_p$ which is divisible by a power of p not lower than any other $x+i$, $1 \le i \le p$. From (18) and $x > 2^{cn}$ it follows that the product on the left side of (18) has at most $c' \log n$ terms, which immediately gives Lemma 3.

Now we complete the proof of Theorem 2. Denote by $x+t$ the integer $(1 \le t \le n)$ which divides 2 to a higher power than the other integers $x+i$, $1 \le i \le n$. Clearly if $i \ne t$ then the largest exponent to which 2 can divide $x+i$ is $[\frac{\log n}{\log 2}]$ and there are at most 2^r integers $x+i$, $1 \le i \le n$ which divide 2 to an exponent $[\frac{\log n}{\log 2}]-r$. Now by Lemma 3 $\Pi'(x+i)$ has at most $(1+o(1))\frac{n}{\log n}$ factors and thus

$\frac{1}{x+j} \Pi'(x+i)$ divides 2 to an exponent less than $c\, n \log\log n/\log n$. But by our assumption $n! \mid \Pi'(x+i)$ thus $x+j$ divides 2 to an exponent greater than $\alpha_2(n) - \frac{c\, n \log\log n}{\log n}$, which completes the proof of Theorem 2 (since $\alpha_2(n) > n - \frac{\log n}{\log 2}$).

I am sure that Theorem 2 is not best possible but I could not improve it.

Perhaps the following remarks are not without interest. Theorem 2 shows that it is difficult for $\Pi'(x+i)$ to be a multiple of $n!$ since usually 2 divides $n!$ to a higher exponent than $\Pi'(x+i)$. In fact "usually" more is true. There is an absolute constant C so that for every $\varepsilon>0$ and $n>n_o(\varepsilon)$

(19) $\Pi'(x+i) > n!$ for $x < (C-\varepsilon)n$

and for $(C+\varepsilon)n < x < nf(n)$

(20) $\Pi'(x+i) < n!$

where $f(n)$ tends to infinity sufficiently slowly.

The proof of (19) and (20) is not difficult and can be left to the reader. The condition $x < nf(n)$ is probably not needed and can no doubt be replaced by the following much stronger

Conjecture: If $n > n_o(\varepsilon)$ and $x > (C+\varepsilon)n$ then $\Pi'(x+i) > n!$ holds only if there is an m $x < m \leq x+n,\ m > n!$ $P(m) \leq n$.

Very likely if $x > n!$ (or even if only x tends to infinity exponentially) there is at most one m $x < m \leq x+n$ satisfying $P(m) \leq n$.

The proof of the conjecture (if true) will no doubt be very difficult. The reason that I can not replace in (20) $nf(n)$ by n^{1+c} is the following curious difficulty: It is well known that the number of integers $m < n^{\alpha}$ $(\alpha > 1)$ satisfying $\rho(m) \leq n$ is $(\rho(\alpha)+o(1))n^{\alpha}$ where $\rho(\alpha)$ is the Dykman function. It is known that $\rho(\alpha) < \frac{1}{\alpha}$. We would need that the number of integers $n^{\alpha} < m \leq n^{\alpha}+n$ satisfying $P(m) \leq n$ is also $(P(\alpha)+o(1))n$. The proof of this seems to present considerable difficulties, but perhaps I am overlooking a simple idea.

Denote by $1 = u_1 < u_2 < \dots$ the sequence of integers satisfying $P(u_i) \leq n$. I thought that for every integer n and m

$$(21) \qquad \prod_{i=1}^{n} u_{m+i} \equiv 0 \pmod{n!} .$$

I do not at present believe that (21) is true. In fact I would expect that for infinitely many m there are primes $p \leq n$ with $p \nmid \prod_{i=1}^{n} u_{m+i}$ but I can not prove anything. In fact let $u_{i_1} < u_{i_2} < \dots$ be the u's which are even. Is it true that

$$(22) \qquad \limsup (i_{j+1} - i_j) = \infty \quad ?$$

In other words: are there arbitrarily large consecutive blocks of u's which are odd? It is easy to see that there are arbitrarily large consecutive blocks of u's which are even, since almost all u's are even and in fact if $p(m) \leq n$ then almost all u's are multiples of m.

P. Erdös, Problems and results on number theoretic properties of consecutive integers and related questions, Proc. fifth Manitoba conference 1975, 25-44.

P. Erdös and J. L. Selfridge, The product of consecutive integers is never a power, Illinois J. Math. 19(1975), 292-301.

N. G. de Bruijn, On the number of positive integers $\leq x$ and free of primes $> y$, Indigationes Math. 13(1951), 50-60.

II

Now I discuss a few somewhat unconventional problems in additive number theory. Let A and B be two disjoint sets of positive integers whose union is the set of all integers. Denote by A^+ respectively B^+ the set of integers which are the distinct sum of integers $a_i \in A$ (resp. $b_i \in B$).

It is easy to see that either A^+ or B^+ (or both) must have upper density 1. In fact the following stronger result holds:

Theorem 3. There is an absolute constant c and an infinite sequence $n_1 < n_2 < \dots$ so that for every i every $n_i < m < cn_i^2$ belongs entirely to A^+, (respectively to B^+).

It is easy to see that apart from the value of c this result is best possible.

Let $a_1<a_2<\dots$ be an infinite sequence of integers. Denote by $\bar{d}_\ell(A)$ the upper logarithmic density of our sequence i.e.

$$\bar{d}_\ell(A) = \limsup_{x\to\infty} \frac{1}{\log x} \sum_{a_i \le x} \frac{1}{a_i} .$$

It is easy to see that there are sequences A and B for which

(23) $$\max (\bar{d}_\ell(A^+), \bar{d}_\ell(B^+)) < 1 .$$

To prove (23) let A be the integers satisfying

(24) $$2^{4^{2k}} < m \le 2^{4^{2k+1}}, \quad m = 0, 1, \ldots$$

and B is the complementary set of the integers (24). Then a simple computation shows that the integers (24) satisfy (23).

I am sure that

(25) $$\max(\bar{d}_\ell(A^+), \bar{d}_\ell(B^+)) > \frac{1}{2}$$

It will probably not be difficult to prove (25), but at the moment I do not see how to determine

(26) $$\min_{A,B} \max(\bar{d}_\ell(A^+), \bar{d}_\ell(B^+)) = c$$

and I postpone the proof of Theorem 3 until I can settle (26). The proof of Theorem 3 and probably (25) is routine, perhaps the proof of (26) is not so trivial.

Define a non uniform hypergraph $G(n)$ as follows: The vertices are the integers $1 \le x < n$. The edges are the sets $a_1 < \ldots < a_k$ for which $a_1 + \ldots + a_k = n$. Denote by $r(n)$ the chromatic number of this graph, i.e. $r(n)$ is the smallest integer for which the integers not exceeding n can be divided into $r(n)$ classes so that no subsetsum of the a's of the same class equals n. It would be of some interest to determine or estimate $r(n)$ as accurately as possible. $r(n) < c_1 n^{1/2}$ is trivial and it is not difficult to prove that $r(n) > n^{c_2}$ for some positive constant c_2. It is easy to see that $r(13)=2$ and 13 is the smallest integer with $r(n) > 1$.

I asked: Is there an absolute constant c so that for every $n > n_o$ there are c partitions of n into distinct summands whose chromatic number is ≥ 3 ? This is of course possible if $n \equiv 0 \pmod{13}$, but Joel Spencer proved that the chromatic number is 2 for infinitely many n in fact for all sufficiently large primes p.

Another similar hypergraph can be defined as follows: The vertices are the non-zero residue classes mod n, and the edges are the sets of distinct residue classes $a_1, \ldots, a_k$ satisfying $a_1 + \ldots + a_k \equiv 0 \pmod{n}$. Determine or estimate the chromatic number of this hypergraph as accurately as possible.

Here are two further problems which may be of some interest. Denote by $f_n(k)$ the largest integer so that for every choice of the integers $1 \leq a_1 < \ldots < a_k < n$ there is always a subsequence of at least $f_n(k)$ integers $a_{i_1} < \ldots$ no subsetsum of which equals n. Similarly $F_n(k)$ is the largest integer so that if $a_1, \ldots, a_k$ is any set of distinct non-zero residue classes mod n there always is a subset of $F_n(k)$ residues no subsetsum of which is $\equiv 0 \pmod{n}$. Clearly, $f_n(k) \geq F_n(k)$. Determine or estimate $f_n(k)$ and $F_n(k)$ as accurately as possible. Perhaps this problem will turn out to be easy.

Let $a_1^{(n)} < a_2^{(n)} < \ldots$ be the infinite sequence of integers all whose prime factors are not exceeding n. Denote by $g(n)$ the smallest integer which is not the sum of two a's. I expect that the true order of magnitude of $g(n)$ is about $\exp n^{1/2}$. A counting argument gives that for $n > n_o(\varepsilon)$

$$g(n) < \exp\left(n^{1/2+\varepsilon}\right)$$

and I would expect that for $n > n_o(\varepsilon)$

(27) $g(n)$ exp $n^{\frac{1}{2}-\epsilon}$

I am very far from being able to prove (27) and in fact could not even prove $g(n) > n^{2+\epsilon}$. Balogh informs me that he can prove $g(n) > n^{2+c}$ for a certain $c > 0$.

Denote by $h(n)$ the smallest integer which is not the distinct sum of pairwise relatively prime $\alpha_i^{(n)}$'s. I can prove

(28) $\exp c_1 n < h(n) < \exp c_2 n$.

Very likely there is an absolute constant $c > 0$ for which

$$h(n) = \exp (c + o(1))n.$$

MacMahon defines a sequence 1,2,4,5,8,10,15,... as follows: n occurs in this sequence if and only if n is not the sum of two or more consecutive terms of this sequence. Andrews conjectured

$$m_n = (1 + o(1))n \log\log n/\log n$$

where m_n is the n-th term of this sequence.

The remarks that he can not even prove

$$m_n/n \to \infty, \quad m_n/n^{1+c} \to 0$$

and I also could not prove these attractive conjectures.

Let $1 \le a_1 < a_2 < \dots$ be a sequence of integers for which every sufficiently large n is the sum of one or more consecutive a's. No doubt $\frac{1}{n^2} a_n \to 0$ and perhaps even $\frac{1}{n^{1+c}} a_n \to 0$ for some $c < 1$.

Let $f(x)$ be the smallest integer for which there is a sequence $1 = a_1 <\ldots < a_t \le x_1 (t = f(x))$ so that every integer $1 \le n \le x$ is the sum of one or more consecutive a's. It would be of interest to determine (or estimate) $f(x)$ as accurately as possible. No doubt

$$f(x) = o(x) \text{ but } f(x)/x^{1/2} \to \infty .$$

R. Guy, Unsolved problems in number theory, Problem books in Mathematics Vol. 1 E. 30 p. 120, Springer-Verlag 1981.

III

Finally I state a few problems and results on prime factors of integers. Let $n=\Pi p_i^{\alpha_i}$. Put

$$(30) \qquad F(n) = \max \sum a_i \ , \ (a_i,a_j)=1, \quad a_i \le n$$

where the a_i are entirely composed of the prime factors of n. Let further

$$(31) \qquad f(n) = \sum p_i^{\beta_i}, \quad p_i^{\beta_i} \le n < p^{\beta+1}$$

i.e. in (31) the a_i are assumed to be powers of primes.

Clearly $F(n) \ge f(n)$ holds for all n and it is not hard to see that $F(n)=f(n)$ for infinitely many n which are not powers of a prime and in fact for which $\omega(n)\to\infty$ ($\omega(n)$ is the number of distinct prime factors of n). I expect that $F(n)/f(n)\to\infty$ for almost all n. I have not proved this, but do not expect that the proof will be very difficult.

I further expect that for almost all n

$$F(n) = (1+o(1)) \frac{n \log\log n}{2} .$$

Also probably the logarithmic density $L(C)$ of the integers n satisfying $f(n)/n \ge C$ exists and is a continuous strictly decreasing function of C, $L(0)=1$, $L(\infty)=0$. The ordinary density will not exist and even the mean value $\frac{1}{x}\sum_{n=1}^{x} f(n)/n$ will not exist since $f(n)/n\to\infty$ on a sequence of upper density 1.

Finally

$$\max_{n<x} f(n) = (1+o(1))x \log x/\log\log x$$

but no doubt

$$(32) \qquad \frac{1}{x}\left(\max_{n\le x} F(n) - \max_{n<x} f(n)\right) \to \infty .$$

The maximum possible value of $f(n)$ and $F(n)$ are both less than $n \cdot \omega(n)$ where $\omega(n)$ denotes the number of distinct prime factors of n. It is very likely that $n\,\omega(n) - F(n)$ tends to infinity if n runs through the integers which are not powers of primes. It would perhaps be of interest to try to estimate how fast (32) or

$$\max_{n<x} (n\omega(n)-F(n)) \quad \text{and} \quad \max_{n<x} (n\omega(n)-f(n))$$

tends to infinity. At the moment I have no interesting results.

Put

$$(33) \qquad G(n) = \max \sum a_i , \quad (a_i,a_j)=1, \quad a_i \le n .$$

It would be of interest to determine as accurately as possible the a's which occur in the sum (33). This does not seem to be easy, I am not even sure that the maximum in (33) is assumed for a unique choice of the a's.

n occurs in the sum (33) if and only if $F(n)=n$, and the density of these n is o. Perhaps one can get an asymptotic formula for the number of these integers and also for the integers $n < x$ with $f(n)=F(n)$.

It is not difficult to prove that both $G(n)=\ldots=G(n+k)$ and $G(m)<\ldots<G(m+k)$ hold for arbitrarily large values of k but I have no good estimate for the largest possible values of these k .