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Graphs with Certain Families of Spanning Trees

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Sufficient conditions are given in terms of $\delta(G)$ and $\Delta(T)$, for a graph G with n vertices to contain a tree T with n vertices. One of these sufficient conditions is used to calculate some of the Ramsey numbers for the pair tree-star. Also necessary conditions are given, in terms of $\delta(G)$, for a graph G with n vertices to contain all trees with n vertices.

1. INTRODUCTION

A graph G is *panarboreal* if G contains all trees T with $|V(T)| = |V(G)|$ (i.e., G has a subgraph isomorphic to T). In [5], conditions in terms of $\delta(G)$ and $\Delta(G)$ were given to ensure that G is panarboreal. The following result was proved.

THEOREM 1 [5]. *If $k \geqslant 3$ and $n \geqslant 3k^2 - 9k + 8$, then every graph G of order n satisfying $\Delta(G) = n - 1$ and $\delta(G) \geqslant n - k$ is panarboreal.*

If $\Delta(G) \neq n - 1$ then of course G is not panarboreal. In the third sction of this paper the condition $\Delta(G) = n - 1$ is dropped to see which trees are contained in G whenever only conditions on $\delta(G)$ are specified. The following two theorems are proved. The first of these theorems is essentially contained in the proof of Theorem 1 [5]. However, it is of value to state and prove the result so as to give it independent status.

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THEOREM 2. *If G is a graph of order n and minimum degree*

$$\delta(G) \geqslant \left\{ \frac{(n-1)\Delta+1}{\Delta+1} \right\}$$

then G contains every tree of order n and maximum degree Δ.

THEOREM 3. *If $k \geqslant 2$ and $n \geqslant 2(3k-2)(2k-3)(k-2)+1$, then every graph G of order n and minimum degree $\delta(G) \geqslant n-k$ contains every tree T of order n and maximum degree $\Delta(T) \leqslant n-2k+2$.*

Theorem 1 will be shown to be a corollary of Theorem 3 for appropriately large n.

In the fourth section, necessary conditions, in terms of $\delta(G)$, for G to contain certain trees are given. Examples verifying the following two theorems are constructed.

THEOREM 4. *There exists a positive constant c and an integer N in terms of which the following statement can be made. For every $n > N$ there exists a graph G of order n and minimum degree $\delta(G) \geqslant [n/2 + c \log n]$ which does not contain tree T of order n and maximum degree $\Delta(T) = 3$.*

THEOREM 5. *Let k be a positive integer. If $n \geqslant 2(k+1)^2(2k+3)$, then there is a graph G with $|V(G)| = n$, $\delta(G) \geqslant [(n+k-1)/2]$, and $\Delta(G) = n-1$ which is not panarboreal.*

In the last section of this paper the Ramsey number $r(T, S)$ will be calculated for some trees T and stars S.

2. TERMINOLOGY AND NOTATION

Terminology and notation will generally conform to that used in [2, 7]. All graphs considered in this paper will be finite, undirected, and without loops or multiple edges. The vertex set and edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. Vertices u and v are said to be adjacent if the edge $uv \in E(G)$. The neighborhood of v in the graph G will be denoted by $N_G(v)$ and the degree by $d_G(v)$. A path in G with vertices $\{v_1, v_2, \ldots, v_n\}$ will be written $(v_1, v_2, \ldots, v_n)$. If in addition $d_G(v_i) = 2$ for $2 \leqslant i \leqslant n-1$, the path will be called a *suspended path*. The statement "G contains H" will mean that there is an isomorphic embedding of H into G. That is, there exists a one-to-one map $\sigma: V(H) \to V(G)$ such that $\sigma(u)\,\sigma(v) \in E(G)$ whenever $uv \in E(H)$.

A vertex of degree 1 is called an *end-vertex* and an edge uv is called a *pendant edge* if either u or v is an end-vertex. A set of mutually non-adjacent pendant edges will be called a *set of hairs*. If a graph G contains a vertex which is adjacent to k end-vertices, then we will say that G has a *talon of degree k*.

For graphs G and H, the *Ramsey number* $r(G, H)$ is the smallest positive integer n, such that for any graph F with $|V(G)| = n$, either F contains G or its complement $\bar{F}$ contains H.

3. Sufficient Conditions

Before giving the proof of Theorem 2, we state without proof a well-known lemma. A proof can be found in [5].

Lemma 6. *Let T be a tree with n vertices, and let T' be any of its subtrees. Let G be a graph which satisfies $\delta(G) \geqslant n-1$ and suppose that $\tau: V(T') \to V(G)$ is an embedding or T' into G. Then τ extends to $\sigma: V(T) \to V(G)$, where σ is an embedding of T into G.*

Proof of Theorem 2. Suppose that the stated result is false. Then there exists a tree T of order $\leqslant n$ such that G contains $T' = T - x$ but not T, where x is an end-vertex of T. Let $\sigma: V(T') \to V(G)$ be an embedding of T' into G. Let z denote the vertex to which x is adjacent in T and define

$$U = \{u \in V(T') \mid \sigma(u)\, \sigma(z) \in E(G)\}.$$

Since, by assumption, there is no embedding of T into G, $\sigma(z)$ must not be adjacent in G to any vertex outside of $\sigma[T']$. It follows that $|U| \geqslant \delta(G)$. Select a vertex $v \in V(G) - \sigma[T']$. Extend σ by setting $\sigma(x) = v$ and define

$$W = \{w \in V(T') \mid \sigma(w)\, \sigma(x) \notin E(G)\}.$$

Define a bipartite graph B with "disjoint" parts U and W (actually disjoint copies of U and W) by making $u \in U$ and $w \in W$ adjacent in B iff they are adjacent in T'. We claim that in B no vertex $u \in U$ is isolated. Suppose, to the contrary, that $u \in U$ is isolated in B. Then for every vertex w which is adjacent to u in T', $\sigma(w)$ is adjacent to $\sigma(x)$ in G. It follows that if we define $\tau(x) = \sigma(u)$, $\tau(u) = \sigma(x)$, and $\tau = \sigma$ otherwise, τ yields an embedding of T into G. Since this is contrary to the assumption that no such embedding exists, the claim is justified and so we conclude that the number of edges in B is at least $|U|$ and so at least $\delta(G)$. On the other hand, the number of edges of B is at most $|W|\,\Delta$ and so at most $(n-1-\delta(G))\,\Delta$. We thus conclude that $\delta(G) \leqslant (n-1-\delta(G))\,\Delta$ and so $(\Delta+1)\,\delta(G) \leqslant (n-1)\,\Delta$. As this is contrary to the hypothesis of the theorem, namely, $(\Delta+1)\,\delta(G) \geqslant (n-1)\,\Delta+1$, the desired contradiction has been reached and the theorem is proved.

The specific relationship between Theorems 1 and 2 is worth noting. In the proof of Theorem 1, it is first proved that every graph G satisfying the conditions of the theorem contains every tree T of order n with $\Delta(G) \geqslant 3k-6$. For trees satisfying $\Delta(T) \leqslant 3k-7$, we, in effect, apply Theorem 2. The needed condition, $(\Delta+1)\,\delta(G) \geqslant (n-1)\,\Delta+1$, now translates to $(3k-6)(n-k) \geqslant (n-1)(3k-7)+1$ or $n \geqslant 3k^2-9k+8$.

Theorem 3 is a direct consequence of the following theorem which we will now prove, and a lemma which we will state later.

THEOREM 7. *Let G be a graph of order n which satisfies $\delta(G) \geqslant n-k$. Let T be a tree of order n. Then G contains T under any one of the following circumstances:*

(i) *T has a suspended path with $3k-1$ vertices,*

(ii) *T has a set of $2(k-1)$ hairs,*

(iii) *$n \geqslant (2k+1)(k-1)$, $k \geqslant 2$, $\Delta(T) \leqslant n-2k+2$ and T has a talon of degree $k-1$.*

Remarks. Part (ii) is proved in Lemma 3 of [5] but we include it here for completeness. In the same lemma a result is proved which is similar to (iii), though the result in the lemma is much more easily proved. There the condition on T, namely, $\Delta(T) \leqslant n-2k+2$, is replaced by the strong condition on G, $\Delta(G)=n-1$.

Proof of Theorem 7. (i) Let T'' be the tree obtained from T by shortening the suspended path by $k-1$ vertices. Thus $|V(T'')| = n-k+1$, and T'' is contained in G by Lemma 6. Let T' be the tree contained in G in which this suspended path has been lengthened as much as possible. Let $\sigma\colon V(T') \to V(G)$ be an embedding of T'. Assume $T' \neq T$, and let P be the suspended path in T'.

By assumption, there is a vertex $v \in V(G-\sigma(T'))$. The maximality of T' implies that v is not adjacent in G to two consecutive vertices of $\sigma(P)$. Since P has at least $2k$ vertices v is not adjacent to at least k vertices in G. This implies $d_G(v) \leqslant n-k-1$, a contradiction.

(ii) Delete the $2k-2$ end-vertices of these hairs, to obtain a tree T' with $n-2k+2$ vertices. By Lemma 6, there is an embedding σ of T' into G. Let R be the set of $2k-2$ vertices of T' adjacent to the end-vertices of T deleted, and let $S = V(G-\sigma(T'))$. Thus $|R| = |S| = 2k-2$.

If there is a matching in G between $\sigma(R)$ and S which saturates R, then G contains the tree T. Assume not. Then by Hall's theorem [6], there is a nonempty subset $R' \subseteq \sigma(R)$, such that $S' = N_G(R') \cap S$ satisfies $|S'| < |R|$. Each vertex of $\sigma(R)$ is adjacent to at least $k-1$ vertices of S, since $\delta(G) \geqslant n-k$. Thus $|R'| \geqslant k$. But any vertex of $S-S'$ is not adjacent to any vertex of R', and hence has degree at most $n-k-1$. This gives a contradiction.

(iii) Let v be the vertex which is the center of the talon, and let T' be the tree obtained from T by deleting $k-1$ vertices of degree 1 adjacent to v. Since $\Delta(T) \leqslant n-2k+2$, there are at least $2k-3$ vertices of T' not in $N_{T'}(v)$. Let $O_{T'}$ be the vertices of T' which are an odd distance of at least 3 from v, and let $E_{T'}$ be the vertices an even positive distance from v. Thus either $|O_{T'}| \geqq k-1$ or $|E_{T'}| \geqq k-1$.

Select a subtree T'' of T' containing v, which has a minimal number of vertices, subject to the condition that either $|O_{T''}| \geqslant k-1$ or $|E_{T''}| \geqslant k-1$. The minimality of the number of vertices of T'' implies that either $|O_{T''}| = k-1$ or $|E_{T''}| = k-1$, $|O_{T''}| + |E_{T''}| \leqslant 2k-3$, and $|N_{T''}(v)| \leqslant k-1$. Thus $|V(T'')| \leqslant 3(k-1)$.

Let w be a fixed vertex of G. We will show that there is an embedding τ or T'' into G, such that $\tau(v) = w$ and $V(\tau(T'')) \supseteq N_{\bar{G}}(w)$. To prove this, we need to consider two subcases.

Subcase (a): $|E_{T''}| = k-1$. The tree T'' is a bipartite graph with parts of order $|E_{T''}| + 1$ and $|O_{T''}| + |N_{T''}(v)|$. Thus T'' is a subgraph of the complete bipartite graph $K_{k,2k-3}$. Select a set $A \subseteq V(G)$ with $|A| = k$ and $A \supseteq N_{\bar{G}}(w) \cup \{w\}$. This can be done since $\delta(G) \geqslant n-k$. Let

$$B = \{x \in V(G-A) \mid xa \in E(G) \text{ for all } a \in A\}.$$

Thus $|B| \geqslant n-(k-1)(k-2) \geqq 2k-3$, and G contains a complete bipartite graph with parts A and B. Hence, there is an embedding $\tau: V(T'') \to V(G)$ with $\tau(v) = w$ and $\tau[E_{T''} \cup \{v\}] = A$.

Subcase (b): $|O_{T''}| = k-1$. The graph $T''-v$ is a bipartite graph with parts of order $|E_{T''}|$ and $|O_{T''}| + |N_{T''}(v)|$. Select a set $A_1 \subseteq V(G-w)$, such that $|A_1| = k-1$ and $A_1 \supseteq N_{\bar{G}}(w)$. Next select a set A_2 of $|N_{T''}(v)|$ vertices of G disjoint from $A_1 \cup \{w\}$, and let $A = A_1 \cup A_2$. Thus $|A| \leqslant 2k-2$. Just as in the previous subcase let

$$B = \{x \in V(G-(A \cup \{w\})) \mid xa \in E(G) \text{ for all } a \in A\}.$$

Then, $|B| \geqslant n-1-k(2k-2) \geqslant k-2$. The graph G contains the complete bipartite graph with parts A and B. Hence there is a copy of $T''-v$ in G with $O_{T''}$ corresponding to A_1 and $N_{T''}(v)$ corresponding to A_2. Therefore there is an embedding $\tau: V(T'') \to V(G)$ with $\tau(v) = w$, $\tau[O_{T''}] = A_1$, and $\tau[N_T(v)] = A_2$.

In both subcases we have an embedding τ of T'' into G with $\tau(v) = w$ and $N_{\bar{G}}(w) \subseteq \tau[V(T'')]$. Since $|V(T')| = n-k+1$ and $\delta(G) \geqslant n-k$, Lemma 6 implies τ can be extended to an embedding σ of T' into G. The vertices of G

not in $\sigma[V(T')]$ are all adjacent to w, and thus σ can be extended to an embedding of T into G. This completes the proof of the final case.

Before giving the proof of Theorem 3, it should be mentioned that this result is the best possible in the following sense. Any graph G which satisfies the conditions of Theorem 3 does not necessarily contain all trees T with $\Delta(T) \leqslant n-2k+3$. Consider the case when n is divisible by k, and T contains a star S with $n-2k+3$ edges, such that $T-S$ has independence number $k-2$. (Attaching a path onto an end-vertex of the star S would give such a graph.) The tree T is not contained in the graph G, which is the complete (n/k)-partite graph with each part containing k vertices. This is true since $G-S'$, where S' is a star with $n-2k+3$ edges, has independence number at least $k-1$.

The following lemma, which we state without proof, will be used in the proof of Theorem 3. A proof of a slightly stronger version of this lemma can be found in [3].

LEMMA 8. *Let T be a tree with n vertices. If T does not contain any suspended path with more than s vertices, then T has at least $n/(2s)$ end-vertices.*

Proof of Theorem 3. Assume none of the conditions of Theorem 7 are satisfied. Since (i) is not satisfied, Lemma 8 implies that there are at least $n/(2(3k-2))$ vertices of degree 1. On the other hand (ii) and (iii) not true imply that T has at most $(2k-3)(k-2)$ vertices of degree 1. Therefore $n \leqslant 2(3k-2)(2k-3)(k-2)$, a contradiction which completes the proof.

For $n \geqslant 2(3k-2)(2k-3)(k-2)$, Theorem 1 is a corollary of Theorem 3. This can be seen as follows. If $\Delta(T) \leqslant n-2k+2$, then Theorem 3 implies directly that G contains T. If $\Delta(T) > n-2k+2$, let v be the vertex of T of degree $\Delta(T)$ and let w be a vertex of G of degree $n-1$. Then let $T' = T-v$ and $G' = G-w$. Again Theorem 3 implies there is an embedding of T' into G'. Clearly this can be extended to an embedding of T into G by mapping v onto w.

4. NECESSARY CONDITIONS

By a well-known theorem of Dirac [4], any graph G, with n vertices and $\delta(G) \geqslant (n-1)/2$, has a Hamiltonian path. However, by Theorem 4 this degree condition is not sufficient to ensure that G contains all trees on n vertices; in fact, it will not even ensure all trees of bounded degree are contained in G.

Theorem 4 is essentially a restatement of a result of Bollobás *et al.* [2], which we now state.

THEOREM 9 [1]. *There is a constant $c > 0$, such that if n is sufficiently large, then there is a ternary tree T on n vertices such that $K_{a,a} \not\subseteq \bar{T}$ for $a = \lfloor (n - c \log n)/2 \rfloor$.*

The proof of Theorem 4 which follows uses Theorem 9 and was communicated to the authors by F. Chung.

Proof of Theorem 4. Let $r = \lfloor c \log n \rfloor$ and $s = (n - r)/2$. Consider the graph

$$G = K_r + (K_{\{s\}} \cup K_{\lfloor s \rfloor}).$$

Clearly $\bar{G}$ contains $K_{\lfloor s \rfloor, \{s\}}$. Therefore Theorem 9 implies that G does not contain some ternary tree T on n vertices. Also $\delta(G) = \lfloor c \log n \rfloor + \lfloor (n - \lfloor c \log n \rfloor)/2 \rfloor - 1 \geqslant \lfloor (n + c' \log n)/2 \rfloor$ for an appropriate c'. If we let $d = c'/2$, the theorem follows.

For a graph G to be panarboreal it is clear that $\Delta(G) = n - 1$. The minimal value of $\delta(G)$ which ensures that G is panarboreal is not known. However, Theorem 5 gives a necessary lower bound.

Proof of Theorem 5. We will first define a graph G which satisfies the conditions of the theorem, and then we will describe a tree on n vertices which is not contained in G.

The graph G is defined as follows. The vertices $V(G)$ of G are partitioned into two sets A and B, with $|A| = \{(n - k)/2\}$ and $|B| = \lfloor (n + k)/2 \rfloor$. Each vertex in A is adjacent in G to each vertex in B, and no two vertices of A are adjacent. There is a fixed vertex b_0 in B, which is adjacent to every vertex of B. The remaining vertices of B form a disjoint union of complete graphs with either k or $k - 1$ vertices. Thus $\Delta(G) = n - 1$, $\delta(G) \geqslant \min(\lfloor (n + k)/2 \rfloor, \{(n - k)/2\} + k - 1) = \lfloor (n + k - 1)/2 \rfloor$.

The tree T is defined as follows. There is a vertex x (which is the center of the tree) adjacent to precisely $2k + 1$ vertices $x_1, x_2, \ldots, x_{2k+1}$ (which we will call the subcenters of the tree). Also $(n - 2k - 2) - \lfloor (n - 2k - 2)/(2k + 1) \rfloor (2k + 1)$ of the subcenters will have degree $\{(n - 2k - 2)/(2k + 1)\} + 1$ and the remaining subcenters will have degree $\lfloor (n - 2k - 2)/(2k + 1) \rfloor + 1$. Therefore the tree T has $n - 2k - 2$ vertices of degree 1.

We need to verify that T is not contained in G. We will assume that there is an embedding σ of T into G, and show that this leads to a contradiction. Three cases will be considered.

Case 1: $\sigma(x) \in A$. In this case, all of the subcenters $\{\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_{2k+1})\} \subseteq B$. Each vertex in B, with the exception of b_0, is adjacent to at most k other vertices of B. This implies that at least each of $2k$ subcenters of $\sigma(T)$ are adjacent to at least $\lfloor (n - 2k - 2)/(2k + 1) \rfloor - k$ vertices of $\sigma(T)$ in

A. Therefore, $\{(n-k)/2\} = |A| > 2k([(n-2k-2)/(2k+1)] - k)$. Direct calculation verifies that this implies $(2k-1)n < 4k(k+1)(2k+3)$, a contradiction.

Case 2: $\sigma(x) \in B - \{b_0\}$. Since every vertex of $B - \{b_0\}$ is adjacent to at most k vertices of B, at least $k+1$ of the subcenters of $\sigma(T)$ must be in A. Each of these subcenters of $\sigma(T)$ in A is adjacent in $\sigma(T)$ to at least $[(n-2k-2)/(2k+1)]$ vertices in B. Hence $[(n+k)/2] = |B| > (k+1)[(n-2k-2)/(2k+1)]$. This leads to a contradiction, just as in Case 1.

Case 3: $\sigma(x) = b_0$. In this case, either at least $k+1$ of the vertices of $\sigma(T)$ are in $B - \{b_0\}$, or at least $k+1$ are in A. In the latter case, a contradiction is reached just as in Case 2. In the former case, the same reasoning used in Case 1 implies $\{(n-k)/2\} = |A| > (k+1)([(n-2k-2)/(2k+1)] - k$. This gives, by direct calculation, that $n < 2(k+1)^2(2k+3)$, a contradiction. This completes the proof of Theorem 5.

5. Ramsey Result

Before we can state the theorem of this section, some additional notation must be given. A star with k edges will be denoted by S_k. The independence number of a graph G will be written as $\alpha(G)$. If T is a tree, then

$$\alpha'(T) = \min\{\alpha(T - V(S)) \mid S \text{ is a star contained in } T\}.$$

Thus, $\alpha'(T)$ is a measure of how small the independence number of the non-neighborhood of a vertex of the tree can be. The parameter α' will play a role in the following theorem only if T has a vertex of large degree. This is true since if T has no vertices of large degree, then $\alpha'(T)$ will be large.

Theorem 10. *Let k be an integer $\geqslant 2$, and $n \geqslant 2(3k-2)(2k-3)(k-2)+1$. If T is a tree with $|V(T)| = n$, then* $\max\{n, n+k-1-\alpha'+\beta\} \leqslant r(T, S_k) \leqslant \max\{n, n+k-1-\alpha'\}$, *where $\alpha' = \alpha'(T)$, and $\beta = 0$ if $n+k-2-\alpha'$ is divisible by k and $\beta = 1$ otherwise.*

Proof of Theorem 10. We first verify the lower bounds. The graph K_{n-1} implies $r(T, S_k) \geqslant n$. Let H be the graph on $n+k-2-\alpha'+\beta \geqslant n$ vertices, whose complement is the disjoint union of complete graphs K_k if $\beta = 0$, and is the disjoint union of complete graphs K_k and K_{k-1} if $\beta - 1$. Clearly $\bar{H}$ does not contain S_k. Also H does not contain T. To see this, assume σ is an embedding of T into H. If v is a vertex of T, then $\sigma(v)$ must be in some component of $\bar{H}$ with at least $k-\beta$ vertices. This same component of $\bar{H}$ must contain at least $k-\beta-1-(k-2-\alpha'-\delta) = \alpha'+1$ other vertices of

$\sigma(T)$. Since this is true for any vertex v of T, this implies $\alpha'(T) \geqslant \alpha' + 1$, a contradiction.

We now consider the upper bounds. Let G be a graph with $t = \max\{n, n+k-1-\alpha'\}$ vertices, whose complement does not contain an S_k. Thus $\delta(G) \geqslant t-k$. We will show that G contains T.

If $\Delta(T) \leqslant n-2k+2$, then $\alpha'(T) \geqslant k-1$ and $t=n$. Since $\delta(G) \geqslant n-k$, Theorem 3 implies that G contains T. Thus we assume that $\Delta(T) \geqslant n-2k+3$.

Let v be a vertex of T with $d_T(v) = \Delta(T)$, and let T' be the tree obtained from T by deleting the end-vertices of T which are adjacent to v. Since v is not adjacent to at most $2k-4$ vertices of $T-\{v\}$, $|V(T')| \leqslant 2(2k-4)+1$. Also T' has α' independent vertices, which are not adjacent to v. Hence T' is a subgraph of $\bar{K}_{\alpha'+1} + \bar{K}_{|V(T')|-\alpha'-1}$. We will show that T' can be embedded in G by using the graph $\bar{K}_{\alpha'+1} + \bar{K}_{|V(T')|-\alpha'-1}$.

Select a vertex w in G, and let W be a set of $\alpha'+1$ vertices, which contains w and a maximum number of vertices of $N_{\bar{G}}(w)$. Thus w is adjacent in $\bar{G}$ to at most $k-1-\alpha'$ vertices of $\bar{G}$ not in W. Since $\delta(G) \geqslant t-k$, we can get a copy of $\bar{K}_{\alpha'+1} + \bar{K}_{|V(T')|-\alpha'-1}$ in G, with W corresponding to the $\alpha'+1$ independent vertices. This is done by merely deleting the vertices which are not adjacent to each vertex of W. The only condition that must be saisfied is that $t \geqslant k|V(T')|$. This is certainly true, since $|V(T')| \leqslant 4k-7$ and $t \geqslant n > k(4k-7)$.

Hence, there is an embedding σ of T' into G with $\sigma[V(T')] \supseteq W$ and $\sigma(v) = w$. Since w is not adjacent in G to at most $k-1-\alpha'$ vertices of G not in W, σ can be extended to an embedding of T into G. This completes the proof.

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