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RECIPROCALS OF CERTAIN LARGE ADDITIVE FUNCTIONS

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1. **Introduction and statement of results**

Let $\beta(n)=\sum_{p\mid n} p$ and $B(n)=\sum_{p^\alpha\| n} \alpha p$ denote the sum of distinct prime divisors of n and the sum of all prime divisors of n respectively. Both $\beta(n)$ and $B(n)$ are additive functions which are in a certain sense large (the average order of $B(n)$ is $\pi^2 n/(6\log n)$, [1]). For a fixed integer m the number of solutions of $B(n)=m$, is the number of partitions of m into primes, while the number of solutions of $\beta(n)=m$, $\mu^2(n)=1$ is the number of partitions of m into distinct primes. There is a certain analogy between the relation of $\beta(n)$ to $B(n)$ and the relation of the well-known additive functions $\omega(n)=\sum_{p\mid n} 1$ and $\Omega(n)=\sum_{p^\alpha\| n} \alpha$. Asymptotic estimates of $B(n)$ were investigated in [1], revealing the connection between $B(n)$ and large prime factors of n. In this paper we turn our attention to sums involving reciprocals of $\beta(n)$ and $B(n)$. We shall prove the following theorems:

THEOREM 1. *For any $\varepsilon>0$ and $x\ge x_0(\varepsilon)$,*

$$(1)\quad x\exp(-(2+\varepsilon)(\log x\cdot\log\log x)^{1/2})\le \sum_{2\le n\le x} 1/B(n)$$
$$\le \sum_{2\le n\le x} 1/\beta(n)\le x\exp(-(\tfrac{1}{2}-\varepsilon)(\log x\cdot\log\log x)^{1/2}).$$

THEOREM 2. *There exist positive constants $C_1, C_2>0$ such that*

$$(2)\qquad \sum_{2\le n\le x} B(n)/\beta(n)=x+O(x\exp(-C_1(\log x\cdot\log\log x)^{1/2})),$$

$$(3)\qquad \sum_{2\le n\le x} \beta(n)/B(n)=x+O(x\exp(-C_2(\log x\cdot\log\log x)^{1/2})).$$

THEOREM 3.

$$(4)\qquad \sum_{n\le x}{}' 1/(B(n)-\beta(n))=Cx+O(x^{1/2}\log x),$$

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where

$$(5)\quad C=\int_0^1 (F(t)-6\pi^{-2})t^{-1}\,dt,\qquad F(t)=\prod_p \Big(1+\sum_{k=2}^{\infty}(t^{p(k-1)}-t^{p(k-2)})p^{-k}\Big),$$

and $\sum'$ *denotes summation over* $n\le x$ *such that* $B(n)\ne\beta(n)$.

2. Proofs

We first prove the lower bound in (1). Let

$$A_k=\{n \mid (n\le x)\wedge(\mu^2(n)=1)\wedge(p(n)\le x^{1/k})\},$$

where we shall use $p(n)$ to denote the largest prime factor of n, x will be sufficiently large and $k=(\log x/\log\log x)^{1/2}$. If n is a product of k different primes each not exceeding $x^{1/k}$, then $n\in A_k$. There at least $U=3kx^{1/k}/(4\log x)$ primes not exceeding $x^{1/k}$, which means

$$(6)\qquad \sum_{n\in A_k} 1\ge\binom{U}{k}=\frac{U(U-1)\cdots(U-k+1)}{k!}\ge(\tfrac{2}{3}U)^k/k!,$$

since $U-k+1\ge 2U/3$ for x sufficiently large. From Stirling's formula or by induction it is seen that $(k/2)^k>k!$ for $k\ge 6$, which when combined with (6) gives

$$(7)\qquad \sum_{n\in A_k} 1\ge x\log^{-k} x.$$

Now for $n\in A_k$ we have $B(n)=\beta(n)\le p(n)\omega(n)\ll\dfrac{x^{1/k}\log x}{\log\log x}$, hence

$$(8)\qquad \begin{aligned}\sum_{2\le n\in A_k} 1/B(n)&=\sum_{2\le n\in A_k} 1/\beta(n)\gg x^{-1/k}\log^{-1}x\sum_{n\in A_k}1\\ &\ge x^{1-1/k}\log^{-k-1}x=x\exp(-2(\log x\cdot\log\log x)^{1/2})\log^{-1}x,\end{aligned}$$

which proves the lower bound in (1). To prove the upper bound in (1) write

$$(9)\qquad \begin{aligned}\sum_{2\le n\le x} 1/\beta(n)&=\sum_{2\le n\le x,\,p(n)\le y} 1/\beta(n)+\sum_{n\le x,\,p(n)>y} 1/\beta(n)\\ &\le\sum_{2\le n\le x,\,p(n)\le y} 1+y^{-1}\sum_{n\le x,\,p(n)>y} 1\le\psi(x,y)+xy^{-1}\end{aligned}$$

where $y=y(x)>2$ will be suitably chosen in a moment. For the function

$$\psi(x,y)=\sum_{n\le x,\,p(n)\le y} 1$$

we use the following estimate of [2]:

$$(10)\qquad \psi(x,y)<c_3x\log^2 y\cdot\exp(-\alpha(\log\alpha+\log\log\alpha-c_4)),$$

where c_3 and c_4 are some positive, absolute constants, $\lim_{x\to\infty} y=\infty$ and

$$(11)\qquad 3<\alpha=\log x/\log y<4y^{1/2}/(\log y).$$

Now we choose

$$y = \exp((\log x \cdot \log\log x)^{1/2}). \tag{12}$$

Then (11) is satisfied for $x \geq x_0$ and

$$\psi(x, y) \ll_\varepsilon x \exp(-(\tfrac{1}{2}-\varepsilon)(\log x \cdot \log\log x)^{1/2}), \tag{13}$$

where $\ll_\varepsilon$ means that the constant implied by the symbol $\ll$ depends on ε only. Substitution in (9) then gives the right-hand side inequality in (1), finishing the proof of Theorem 1.

To prove Theorem 2 it is enough to prove (2), since trivially

$$\sum_{2 \leq n \leq x} \beta(n)/B(n) \leq x + O(1), \tag{14}$$

and by the Cauchy–Schwarz inequality we have

$$x^2 + O(x) \leq \Big(\sum_{2 \leq n \leq x} 1\Big)^2 \leq \sum_{2 \leq n \leq x} B(n)/\beta(n) \sum_{2 \leq m \leq x} \beta(m)/B(m), \tag{15}$$

so that (2) then implies (3). Let

$$S = \sum_{2 \leq n \leq x} B(n)/\beta(n) = S_1 + S_2, \tag{16}$$

where in S_1 summation is over $2 \leq n \leq x$ such that $B(n) < k\beta(n)$, and in S_2 over $2 \leq n \leq x$ such that $B(n) \geq k\beta(n)$, where $k = k(x)$ is a large number which will be suitably chosen later. Note that if $B(n) \geq r\beta(n)$ for some integer $r \geq 2$, then n must be divisible by p^r for some prime p, so that the number of $n \leq x$ for which p^r divides n for some p is $\ll \sum_p xp^{-r} \ll x2^{-r}$. Then we have

$$S_2 = \sum_{r \geq k} \sum_{2 \leq n \leq x, r \leq B(n)/\beta(n) < r+1} B(n)/\beta(n) \ll \sum_{r \geq k} x(r+1)2^{-r} \ll x\exp(-C_3 k) \tag{17}$$

for some $C_3 > 0$. To estimate S_1 write

$$S_1 = S_1' + S_1''. \tag{18}$$

In S_1'', summation is over $2 \leq n \leq x$ such that $B(n) < k\beta(n)$ and n is divisible by p^2 for some prime $p > L$, where $L = L(x)$ is a large number that will be suitably chosen. Thus we obtain

$$S_1'' \ll k \sum_{n^2 m \leq x, n > L} 1 \ll k \sum_{n > L} xn^{-2} \ll kx/L. \tag{19}$$

If $n = p_1^{a_1} \cdots p_i^{a_i}$ is counted in S_1' then $a_j = 1$ for $p_j > L$ and $j = 1, \ldots, i$, which implies

$$\begin{aligned} B(n) &= (a_1 - 1)p_1 + \cdots + (a_i - 1)p_i + \beta(n) \leq L(a_1 + \cdots + a_i - i) + \beta(n) \\ &\leq L(\Omega(n) - \omega(n)) + \beta(n) \leq L(\log x/\log 2) + \beta(n). \end{aligned} \tag{20}$$

Therefore we have

$$S_1' \le \sum_{n\le x} 1 + L(\log x/\log 2) \sum_{2\le n\le x} 1/\beta(n) \tag{21}$$
$$\le x + O(xL \log x \cdot \exp(-C_4(\log x \cdot \log\log x)^{1/2})),$$

where we have used (1) to estimate $\sum_{2\le n\le x} 1/\beta(n)$. From (16)–(21) we obtain

$$S \le x + O(kx/L) + O(x\exp(-C_3k)) \tag{22}$$
$$+ O(xL \log x \cdot \exp(-C_4(\log x \cdot \log\log x)^{1/2})).$$

Noting that trivially $S \ge x + O(1)$ and choosing

$$k = (\log x \cdot \log\log x)^{1/2}, \tag{23}$$

$$L = \exp(C_5(\log x \cdot \log\log x)^{1/2}), \qquad C_5 = C_4/2, \tag{24}$$

we obtain (2) from (22).

To prove Theorem 3 we employ an analytical method. Let $0 \le t \le 1$ and observe that $t^{B(n)-\beta(n)}$ is a multiplicative function of n satisfying $t^{B(p^k)-\beta(p^k)} = t^{p(k-1)}$ for $k = 1, 2, \ldots$ and every prime p. Therefore for $\operatorname{Re} s > 1$

$$\sum_{n=1}^{\infty} t^{B(n)-\beta(n)} n^{-s} = \prod_p (1 + p^{-s} + t^p p^{-2s} + t^{2p} p^{-3s} + \cdots) \tag{25}$$
$$= \zeta(s) \prod_p (1 + (t^p - 1)p^{-2s} + (t^{2p} - t^p)p^{-3s} + \cdots) = \zeta(s) G(s, t),$$

where $\zeta(s)$ is the Riemann zeta-function and for $\operatorname{Re} s > \frac{1}{2}$

$$G(s, t) = \sum_{n=1}^{\infty} g(n, t) n^{-s}, \tag{26}$$

and $g(n, t)$ is a multiplicative function of n for which $g(p, t) = 0$ and $|g(p^k, t)| \le 1$ for $k \ge 2$. Therefore uniformly for $0 \le t \le 1$ we have

$$\sum_{n\le x} |g(n, t)| \ll x^{1/2}, \tag{27}$$

and by partial summation we subsequently obtain

$$\sum_{n\le x} t^{B(n)-\beta(n)} = \sum_{n\le x} g(n, t)[x/n] = x \sum_{n\le x} g(n, t)/n + O\Big(\sum_{n\le x} |g(n, t)|\Big) \tag{28}$$
$$= xG(1, t) + O(x^{1/2}),$$

where

$$G(1, t) = \prod_p \Big(1 + \sum_{k=2}^{\infty} (t^{p(k-1)} - t^{p(k-2)})p^{-k}\Big) = F(t),$$

and therefore

$$F(0) = \prod_p (1 - p^{-2}) = 6/\pi^2.$$

Note that $B(n)=\beta(n)$ if and only if n is squarefree. Therefore we have uniformly in t

$$\sum_{n\le x}{}' t^{B(n)-\beta(n)-1}=\sum_{n\le x, B(n)\ne\beta(n)} t^{B(n)-\beta(n)-1}$$

$$(29)\qquad =xt^{-1}F(t)+O(x^{1/2}t^{-1})-\sum_{n\le x}\mu^2(n)t^{-1}$$

$$=x(F(t)-6/\pi^2)t^{-1}+O(x^{1/2}t^{-1}).$$

Since $F(0)=6/\pi^2$ the function $(F(t)-6/\pi^2)t^{-1}$ is continuous for $0\le t\le 1$, and we obtain the conclusion of the theorem integrating (29) over t from $\varepsilon(x)=x^{-2/3}$ to 1, since

$$(30)\qquad \int_{\varepsilon(x)}^{1}\sum_{n\le x}{}' t^{B(n)-\beta(n)-1}\,dt=\sum_{n\le x}{}' 1/(B(n)-\beta(n))+O(x^{1/3}),$$

$$(31)\qquad x\int_{0}^{\varepsilon(x)}(F(t)-6/\pi^2)t^{-1}\,dt\ll x\varepsilon(x)=x^{1/3},$$

$$(32)\qquad \int_{\varepsilon(x)}^{1}O(x^{1/2}t^{-1})\,dt\ll x^{1/2}\log 1/\varepsilon(x)\ll x^{1/2}\log x.$$

3. Some remarks

It seems probable that the inequalities (1) may be replaced by asymptotic formulae, viz.

$$(33)\quad \log\sum_{2\le n\le x}1/B(n)\sim\log x-C(\log x\cdot\log\log x)^{1/2},\qquad x\to\infty,\quad C>0$$

(and a similar formula with $\beta(n)$ instead of $B(n)$), but we are unable to prove (33). Our results concerning $B(n)$ and $\beta(n)$ may be compared with corresponding results for "small" additive functions $\Omega(n)$ and $\omega(n)$. Utilizing essentially the method of proof of Theorem 3 it was shown in [3] that

$$(34)\qquad \sum_{2\le n\le x}1/\Omega(n)=x/\log\log x+a_2x/(\log\log x)^2+\cdots+a_{N-1}x/(\log\log x)^{N-1}$$
$$+O(x/(\log\log x)^N),$$

$$(35)\qquad \sum_{2\le n\le x}1/\omega(n)=x/\log\log x+b_2x/(\log\log x)^2+\cdots+b_{N-1}x/(\log\log x)^{N-1}$$
$$+O(x/(\log\log x)^N),$$

where the a_i's and b_i's are computable constants and N is arbitrary, but fixed.

Similarly [4] contains a proof that

$$(36)\qquad \sum_{2\le n\le x}\Omega(n)/\omega(n)=x+c_1x/\log\log x+\cdots+c_{N-1}x/(\log\log x)^{N-1}$$
$$+O(x/(\log\log x)^N),$$

and the formulae (34)–(36) are further sharpened in [5].

The degree of sharpness of the above formulae is not attained in our theorems concerning $\beta(n)$ and $B(n)$, which is to be expected since $\beta(n)$ and $B(n)$ are much larger functions than $\omega(n)$ and $\Omega(n)$, possessing notably wider fluctuations in size.

It is clear that the method of proof of Theorem 2 would yield (2) and (3) with $B(n)$ and $\beta(n)$ replaced by $B^m(n)$ and $\beta^m(n)$ respectively, where m is a fixed positive integer. Our methods also work in the general case of other large additive functions defined by

$$f(n)=\sum_{p\mid n} h(p), \qquad F(n)=\sum_{p^\alpha \| n} \alpha h(p),$$

where for some fixed $K, \gamma>0$ and a fixed real δ we have

$$h(x)=\exp(K\log^\gamma x\cdot(\log\log x)^\delta).$$

For other results and problems concerning $B(n)$ and $\beta(n)$ the reader is referred to [1].

Closely related to $B(n)$ and $\beta(n)$ is the function $B_1(n)=\sum_{p^\alpha\|n} p^\alpha$. From $B_1(n)\ge\beta(n)$ and the fact that $B_1(n)=B(n)=\beta(n)$ if $n\in A_k$ (the set defined at the beginning of §2) we conclude that the bounds of Theorem 1 hold also for

$$\sum_{2\le n\le x} 1/B_1(n).$$

It seems likely that

$$\sum_{2\le n\le x} B_1(n)/\beta(n)=(c_1+o(1))x\log\log x \tag{37}$$

and

$$\sum_{2\le n\le x} B_1(n)/B(n)=(C+o(1))x, \quad C>0. \tag{38}$$

We can rigorously prove at present only

$$\sum_{2\le n\le x} B_1(n)/\beta(n)\ge\tfrac{1}{2}x\log\log x+o(x\log\log x). \tag{39}$$

To see this let $p_1<\cdots<p_k$ be the primes not exceeding x. Suppose $p_i^{l_i}\le x<p_i^{l_i+1}$ $(i\le k)$ and define $t_i\ge1$ by

$$t_ip_i^{l_i}\le x<(t_i+1)p_i^{l_i}, \tag{40}$$

so that $t_i<p_i$. Then we have

$$S=\sum_{2\le n\le x} B_1(n)/\beta(n)>\sum_{i\le k}\sum_{s\le t_i} B_1(sp_i^{l_i})/\beta(sp_i^{l_i}), \tag{41}$$

Now $\beta(sp_i^{l_i}) \leq \beta(s) + \beta(p_i^{l_i}) \leq s + p_i \leq t_i + p_i < 2p_i$ and $B_1(sp_i^{l_i}) \geq p_i^{l_i}$, which gives

$$S > \sum_{i \leq k} \sum_{s \leq t_i} p_i^{l_i}/(2p_i) \geq \sum_{i \leq k} t_i p_i^{l_i}/(2p_i) \geq \frac{1}{2} \sum_{i \leq k} (xp_i^{-l_i} - 1) p_i^{l_i - 1}$$

$$\geq \frac{x}{2} \sum_{i \leq k} 1/p_i + O\left(\sum_{i \leq k} p_i^{l_i - 1}\right) \geq \frac{x}{2} \log \log x + o(x \log \log x),$$

since

$$\sum_{p \leq x} 1/p = \log \log x + O(1) \quad \text{and} \quad \sum_{i \leq k} p_i^{l_i - 1} = o(x \log \log x).$$

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