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PROBLEMS AND RESULTS ON FINITE AND INFINITE COMBINATORIAL ANALYSIS II [1])

by Paul Erdös

During my long life I have published many papers of similar title, at the end of the introduction I give a partial list of my papers on related subjects.

Hajnal and I published two survey papers on problems in set theory. Sufficient progress has been made to make a new paper necessary, but I did not follow all the independence proofs sufficiently to undertake this task alone. Thus I will almost entirely deal with problems and results where I was personally involved and in which I am currently interested.

I hope that the survey paper in question will eventually (soon?) be written and perhaps (if I live) I will be one of the coauthors.

In this paper infinite problems will be discussed much more thoroughly than finite ones.

P. Erdös, Problems and results on finite and infinite combinatorial analysis, Coll. Math. Soc. J. Bolyai, Infinite and finite sets, Keszthely Hungary 1973, 403-424. I will refer to this paper as 1.

P. Erdös and A. Hajnal, Unsolved problems in set theory, Proc. Symp. Pure Math. Vol. 13, Part 1, Amer. Math. Soc., 1971, 17-48 and Unsolved and solved problems in set theory, Vol. 25, Tarski Symposium 269-287.

P. Erdös, Problems and results in combinatorial analysis, Proc. Symp. Pure Math. XIX Amer. Math. Soc., 77-89.

P. Erdös and D. Kleitman, Extremal problems among subsets of a set, Combinatorial Math and Applications, Proc. Chapel Hill Conference 1970, 146-170, see also Discrete Math. 8 (1979), 287-294.

P. Erdös, Some old and new problems in various branches of combinatorics, Southeastern Conference in Combinatorics, Graph Theory, and Computing Vol. 10. This paper contains a fairly complete list of references to my older problem papers.

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[1]) Presented at the *Symposium über Logik und Algorithmik* in honour of Ernst Specker, Zürich, February 1980.

P. Erdös, On the combinatorial problems which I would most like to see solved, *Combinatorica 1* (1981), 25-42.

1. First of all I discuss a problem where E. Specker had a very important role to play.

First I define a special case of the arrow symbol of Rado and myself. Let $\alpha, \beta_1, ..., \beta_n$ be ordinal numbers. Then $\alpha \to (\beta_1, ..., \beta_n)^r_n$ means that if we divide the r-tuples of a set of order type α into n classes then for some i there is a set of type β_i all whose r-tuples are in the i-th class. Rado and I wondered whether $\omega^\alpha \to (\omega^\alpha, n)^2_2$ holds for every $\alpha < \omega_1$ and $n < \omega$. We could not even prove $\omega^2 \to (\omega^2, 3)^2_2$. In November 1954 I was on the way from the Amsterdam International Congress to Israel and stopped off for two weeks in Switzerland for a few lectures. I told Specker our problem $\omega^2 \to (\omega^2, 3)^2_2$ and offered 20 dollars if he decides this question. Not long after I arrived in Israel, a letter of Specker arrived which contained the proof of $\omega^2 \to (\omega^2, n)^2_2$. I thought that I could prove by his methods $\omega^k \to (\omega^k, n)^2_2$ but when I tried to tell the proof to Specker in the summer of 1955 I realized that I can only prove a much weaker result. Soon afterwards Specker disproved $\omega^k \to (\omega^k, 3)^2_2$ for every $3 \leqslant k < \omega$. Specker very soon realized that neither the proof nor the counterexample works for $\omega^\omega \to (\omega^\omega, 3)^2_2$. A few years later I offered 250 dollars for a proof or disproof of this conjecture. In 1970 finally Chang proved $\omega^\omega \to (\omega^\omega, 3)^2_2$ and a few months later E. Milner proved $\omega^\omega \to (\omega^\omega, n)^2_2$. Jean Larson obtained a much simpler proof and also various generalizations for higher cardinals.

To help study these partition relations Specker introduced the notion of pinning. Let A and B be well ordered sets. A mapping Π of A into B is called a pinning map if for every set $X \subset A$ which is order isomorphic to A, $\Pi(A)$ is order isomorphic to B. If α and β are ordinals, we say α can be pinned to β, in symbols $\alpha \to \beta$, if there is a pinning map from α into β. (As far as I know the name pinning is due to J. Larson, the symbol $\alpha \to \beta$ to Rotman).

The problem now is to decide which ordinals α and β satisfy $\alpha \to \beta$. Specker observed that if $\alpha \to \beta$ and $\alpha \to (\alpha, n)^2_2$, then $\beta \to (\beta, n)^2_2$. Specker proved $\omega^3 \not\to (\omega^3, 3)^2_2$ and $\omega^m \to \omega^3$ for every $3 \leqslant m < \omega$. Thus he showed $\omega^m \not\to (\omega^m, 3)^2_2$ for every $3 \leqslant m < \omega$. F. Galvin and J. Larson (answering a question of Specker) characterized all countable ordinals α for which $\alpha \to \omega^3$ as those ordinals α of the form ω^γ where γ is a decomposable ordinal with $\gamma \geqslant 3$, if $\alpha = \omega^{\gamma_0} + ... + \omega^{\gamma_n}$, $\gamma_0 \geqslant ... \geqslant \gamma_n$, then $\alpha \to \omega^3$ iff $\omega^{\gamma_i} \to \omega^3$ for some i. Thus for ordinals α of this form $\alpha \not\to (\alpha, 3)^2_2$. The negative partition result for ω^3 and its corollaries through pinning are the only known negative results for the relation $\alpha \nrightarrow (\alpha, 3)_2^2$ where α is an infinite countable indecomposable ordinal.

Thus ω^{ω^2} is the first indecomposable ordinal which can not be pinned to ω^3 and for which $\omega^{\omega^2} \rightarrow (\omega^{\omega^2}, 3)_2^2$ is open.

Specker proved that for every countable $\alpha \geqslant 2$, $\omega^\alpha \rightarrow \omega^2$ and J. Larson characterized all pairs of countable ordinals α, β which satisfy $\alpha \rightarrow \beta$.

She observed that there infinitely many countable ordinals ω^α which can be pinned only to $1, \omega, \omega^2$ and themselves. For such an ordinal β, $\beta \rightarrow (\beta, 3)_2^2$ may be particularly hard to settle. And ω^{ω^ω} is the smallest such ordinal.

Perhaps the most attractive and interesting unsolved problem in the theory of pinning is due to Rotman: Can one ever pin an ordinal α to a larger one? He showed that such an ordinal α, if it exists must be non-denumerable.

J. Larson started a systematic study of $\alpha \rightarrow \beta$ for uncountable ordinals. There are no surprises for $\alpha < \omega_1^{\omega+2}$, but she showed that $\omega_1^{\omega+2} \rightarrow \omega^2$ if $c = \aleph_1$ and it is consistent that $\omega_1^{\omega+2} \nrightarrow \omega^2$.

E. Nosal almost completely determined the truth value of $\omega^n \rightarrow (\omega^m, k)_2^2$. In fact she proved that if $f(m, n)$ is the smallest integer for which $\omega^n \nrightarrow (\omega^m, f(m, n))_2^2$ then for $5 \leqslant m \leqslant n \, f(m, n) = 2^{[n-1/m-1]} + 1$ and $f(3, n) = 2^{n-2} + 1$. She conjectures that $f(4, n) = 2^{[n-1/3]} + 1$.

Haddad and Sabbagh, and independently Galvin and Hajnal, reduced previously the truth value of $\omega^n \rightarrow (\omega^m, k)_2^2$ to a finite combinatorial problem.

I offer 2000 Swiss Francs for a characterization of the countable ordinals α which satisfy $\omega^\alpha \rightarrow (\omega^\alpha, 3)_2^2$ and 500 for a proof or disproof of $\omega^{\omega^2} \rightarrow (\omega^{\omega^2}, 3)_2^2$.

As far as I know, for every $\alpha \geqslant \omega$, $\alpha \rightarrow (\alpha, 3)_2^2$ implies $\alpha \rightarrow (\alpha, n)_2^2$ for every $n < \omega$. It certainly would be interesting to decide if this really holds, and I offer 500 Swiss Francs for a proof or disproof of this conjecture.

P. Erdös and R. Rado. A partition calculus in set theory. *Bull. Amer. Math. Soc. 62* (1956), 427-489.

E. Specker. Teilmengen van Mengen mit Relationen. *Comment. Math. Helv. 31* (1957), 302-214.

C. C. Chang. A partition theorem for the complete graph on ω^ω. *J. Comb. Theory (A) 12* (1972), 396-452.

J. Larson. A short proof of a partition theorem for the ordinal ω^ω, *Ann. Math. Logic 6* (1973), 129-145.

F. Galvin and J. Larson. Pinning countable ordinals. *Fund. Math. 82* (1975), 357-361.

J. LARSON. An independence result for pinning for ordinals. *J. London Math. Soc. (2) 19* (1979), 1-6.
J. LARSON. For pairs of countable ordinals. *Fund. Math.*, to appear.
E. NOSAL. Partition relations for denumerable ordinals. *Journal Comb. Theory*, *27* (B), (1979), 190-198. The paper has extensive references to earlier papers on this subject.

2. Let me now state in an unsystematic way some other problems and results on the partition calculus of ordinal numbers and order types. Baumgartner and Hajnal proved an old conjecture of Rado and myself: For every ordinal $\alpha < \omega_1$ and integer n

$$\omega_1 \to (\alpha)^2_n \tag{1}$$

The proof used Martin's axiom. Later Galvin obtained a generalization of (1) by complicated but purely combinatorial arguments.

Very recently Galvin and Prikry proved the following surprising result: Color the edges of a complete graph $K(\omega_1)$ by n colors. Then there is a color i and a closed cofinal set S so that for every ordinal $\alpha \in S$ there is a set of ordinals $\beta_\xi < \alpha$ of type α so that all the edges joining two of the β's are of the i-th color.

Hajnal and I proved that (C. H. is assumed)

$$\omega_1^2 \to (\omega_1\alpha, 3)^2_2 \text{ for every } \alpha < \omega_1 \text{ and } \omega_1\alpha \nrightarrow (\omega_1\omega, 3)^2_2 .$$

We could not decide

$$\omega_1^2 \to (\omega_1\omega, 4)^2_2 .$$

Also we could not decide

$$\omega_2\omega \to (\omega_2\omega, 3)^2_2 .$$

One of the most striking unsolved problems is the following question of Hajnal, Milner and myself: Let α be a limit ordinal, G a graph whose vertices are the elements of α. Is it true that G either contains an infinite path or an independent set of type α? We proved this for every $\alpha < \omega_1^{\omega+2}$. Our proof breaks down at $\omega_1^{\omega+2}$, but perhaps the theorem holds for all ordinal numbers α. I remind the reader that in chapter 1, J. Larson also had difficulty with $\omega_1^{\omega+2}$. It is not clear to me at the moment if there is any connection between these results and difficulties.

Hajnal, Milner and I proved that if α is a limit ordinal and G a graph whose vertices are the elements of α then either G contains a C_4 (i.e. a circuit of length 4) (or more generally a $K(r, r)$ for every $r < \omega$) or an independent set of type α. Our proof has not been published since it was superseded by the following deep result of R. Laver: (for the definition see the paper of Laver). Let ϕ be of scattered limit type then $\phi \rightarrow (\phi, K(\aleph_0, r))_2^2$.

Assume that $c = \aleph_1$. Hajnal proved $\omega_1^2 \nrightarrow (\omega_1^2, 3)_2^2$, more generally he showed that if K is regular and $2^k = K^+$ then $(K^+)^2 \nrightarrow ((K^+)^2, 3)_2^2$. Baumgartner showed the above result remains true without assuming the regularity of K. Baumgartner further proved that if $V = L$ then $K^2 \rightarrow (K^2, 3)^2$ holds if and only if K is weakly compact. For further problems and results see a forthcoming paper of J. Larson, A counterexample in the partition calculus for an uncountable ordinal, which will appear in the Israel Journal of Math.

Hajnal and I proved $\omega_1^\omega \rightarrow (\omega_1^\omega, C_5)_2^2$ without much difficulty, but we could not decide $\omega_1^{\omega+1} \rightarrow (\omega_1^{\omega+1}, C_5)_2^2$ (in other words: let G be a graph whose vertices form a well-ordered set of type $\omega_1^{\omega+1}$; if G does not contain a pentagon, then there is a set of vertices of type $\omega_1^{\omega+1}$ no two vertices of which are joined in G).

Let λ be the order type of the set of reals. Rado and I proved that for every integer $n < \omega$

$$\lambda \rightarrow (\omega+n, 4)_2^3 . \quad (1)$$

We could not prove $\omega_1 \rightarrow (\omega+n, 4)_2^3$. As far as we know (1) could be strengthened to

$$\lambda \rightarrow (\alpha, n)_2^3 \text{ and } \omega_1 \rightarrow (\alpha, n)_2^3$$

for every ordinal $\alpha < \omega_1$, and integer $n < \omega$. It is surprising that no progress has been made on these question. As far as I know the only nontrivial positive result for splitting the r-tuples ($r \geqslant 4$) of reals is $\lambda \rightarrow (\omega+1)_n^r$ for all r, $n < \omega$. This was proved by Galvin.

J. Baumgartner and A. Hajnal. A proof (involving Martin's axiom) of a partition relation. *Fund. Math. 78* (1973), 193-203.

F. Galvin. On a partition theorem of Baumgartner and Hajnal. *Coll. Math. Soc. J. Bólyai*, Infinite and finite sets, Keszthely Hungary 1973, 711-729.

P. Erdös and A. Hajnal. Ordinary partition relations for ordinal numbers. *Periodica Math. Hung. 1* (1971), 171-185.

A. Hajnal. A negative partition relation. *Proc. Nat. Acad. Sci. USA 68* (1971), 142-144.

J. Baumgartner. Partition relations for uncountable ordinals. *Israel J. Math. 21 (4)* (1975), 296-307.

P. Erdös, A. Hajnal and E. Milner. Set mappings and polarized partition relations. *Coll. Math. Soc. J. Bólyai 4*, Combinatorial theory and its applications 1969, 327-365.

R. Laver. An order type decomposition theorem. *Annals of Math. 98* (1973), 96-119.

P. Erdös and R. Rado. A partition calculus in set theory. *Bull. Amer. Math. Soc. 62* (1965), 427-489.

3. Now I discuss problems on chromatic numbers of graphs and set systems. A graph G is said to have chromatic number m, in symbols $\chi(G) = m$, if its vertices can be coloured by m colors so that two vertices of the same color are not joined but that this can not be done by fewer than m colors. The famous four color theorem is equivalent to the statement that every planar graph has chromatic number not exceeding four, but soon it was realised that chromatic graphs have a great deal of interest of their own. Tutte, Ungar and Zykov were the first to realise that there are graphs of arbitrarily large (finite) chromatic number which contain no triangles. Rado, Hajnal and I proved that for every infinite cardinal m and every integer k there is a graph of m vertices which has chromatic number m and which contains no odd circuit of length less than $2k + 1$.

By probabilistic methods I proved that for every $k < \omega$, $n < \omega$ there is a finite graph which contains no C_l for $l < k$ (C_l is a circuit of size l) and whose chromatic number is $\geqslant n$. Lovász later gave an ingenious construction for such graphs. Hajnal and I proved that every graph of chromatic number $\chi(G) > k \geqslant \omega$ has to contain a complete bipartite graph $K(m, k^+)$ for every $m < \omega$; thus in particular it must contain all C_{2n}. Also we proved that it must contain an infinite path. Hajnal and I showed that if $c = \aleph_1$, there is a graph G with $\chi(G) = \aleph_1$ which does not contain a $K(\omega, \omega)$ and Hajnal further proved that there is such a graph which does not contain a $K(\omega, \omega)$ and a triangle. We could not decide if a G with $\chi(G) \geqslant \aleph_1$ must contain either a C_5 and a $K(\omega, \omega)$.

About 10 years ago Galvin asked whether the chromatic numbers have the Darboux property. More precisely, let $\chi(G) = m > \aleph_1$. Is it true that for every $n < m$, G has a subgraph G' satisfying $\chi(G') = n$? Galvin showed that if $c > \aleph_1$, then there is such a G with $\chi(G) = \aleph_2$, so that G has no induced subgraph G' with $\chi(G') = \aleph_1$. Galvin's problem is open if we assume the generalized continuum hypothesis or if we make no such assumptions but do not insist that the subgraphs should be induced.

The most striking unsolved problem is due to W. Taylor. Let G be a graph of chromatic number $\aleph_1$. Is it true that for every cardinal $n > \aleph_1$, there is a G_n with $\chi(G_n) = n$ so that all finite subraphs of G_n are also subgraphs of G? Hajnal, Shelah and I proved that if $\chi(G) \geqslant \aleph_1$, then there is a k_0 so that G contains all C_k with $k \geqslant k_0$. Our simplest unsolved problem states: Is there an edge e of G so that for every $k > k_0$, G contains a circuit C_k one edge of which is e? Hajnal recently observed that there are infinitely many k for which G has a C containing e.

We call a family F of finite graphs full if for every infinite cardinal m, there is a graph G^m of chromatic number m every finite subgraph of which can be imbedded in some graph of the family F. We tried to characterize F and stated some conjectures which we do not really believe but could not disprove.

In a forthcoming paper (on almost bipartite large chromatic graphs) Hajnal, Szemerédi and I prove that for every $n > \omega$ and $\varepsilon > 0$ there is a G with $\chi(G) \geqslant n$ so that every finite subgraph of G is "almost bipartite" in the following sense: Let $x_1, \ldots, x_k$ be any finite set of vertices of G and $G(x_1, \ldots, x_k)$ is the induced subgraph of G having the vertices $x_1, \ldots, x_k$. Then we can always omit εk of these vertices so that the remaining graph should be bipartite. It is not known if our result remains true if we assume $|G| = \chi(G) = n$ (where $|G|$ denotes the cardinal number of the vertices of G). The following somewhat weaker statement is also open: Is there a G satisfying $|G| = \chi(G) = \aleph_1$ so that for every $k < \omega$ and every set of k vertices $x_1, \ldots, x_k$ there is a subset of ck x_1's no two of which are joined in G i.e., $G(x_1, \ldots, x_k)$ has an independent set of size $> ck$.

The following problem discussed in our paper seems very interesting: Let $\chi(G) \geqslant \omega$. Denote by $f_G^0(k)$ the largest integer for which G has a subgraph $G(k)$ of k vertices satisfying $\chi(G(k)) = f_G^0(k)$. Clearly $f_G^0(k)$ tends to infinity with k, but if $\chi(G) = \omega$ then it is known that it can tend to infinity as slowly as we please. Our problem now states: for which functions $f(k)$ is it true that there is a G satisfying $\chi(G) \geqslant \aleph_1$ and for all sufficiently large $k, f_G^0(k) < f(k)$? We show that $\log_r n$ (the r-times iterated logarithm) is such a function. Is there such an $f(k)$ which tends to infinity slower than $\log_r n$ for every r? Can $f(k)$ tend to infinity as slowly as we please? I offer 500 Swiss Francs for clearing up these problems.

We define $f_G^{(3)}(k)$ as the smallest integer for which every induced subgraph $G(k)$ of G can be made bipartite by the omission of at most $f_G^{(3)}(k)$ edges. We prove that for every cardinal $p \geqslant \omega$ there is a G with $\chi(G) > p$ and $f_G^{(3)}(k) < 2k^{3/2}$. By one of the theorems of Hajnal, Shelah and myself there is an n_0 so that G contains $\aleph_1$ vertex disjoint $C_{2n_0+1}^0$ s. Thus $2n^{3/2}$ can certainly not be replaced by $o(n)$, but perhaps it can be replaced by cn for some $c = c(G)$. This conjecture might be too optimistic, but we hope that $2n^{3/2}$ can be replaced by $n^{1+\varepsilon}$ for every $\varepsilon > 0$.

An example of Gallai-Lovász gives a finite graph G satisfying $\chi(G) \geqslant r + 2$ and $f_G^{(3)}(n) = 0(n^{1-1/r})$. Is this best possible? For all we know, $f_G^{(3)}(n)$ could tend to infinity as slowly as we please. More precisely: Let $\chi(G) = \omega$. Can $f_G^{(3)}(n)$ tend to infinity very slowly? (As slowly as we please)? Can it be $o(\log n)$ or $o(\log_k(n))$? I offer 500 Swiss Francs for clearing up this question. (Added in proof) Tuza proved that there is for every k and arbitrarily large n a k chromatic graph which can be made bipartite by the omission of $\binom{k-1}{2}$ edges. $\binom{k-1}{2}$ is best possible.

Many problems and results on chromatic numbers of hypergraphs are contained in our joint paper with Galvin and Hajnal.

1. P. ERDÖS and R. RADO. A construction of graphs without triangles having pre-assigned order and chromatic number. *J. London Math. Soc. 35* (1960), 444-448.
2. P. ERDÖS and A. HAJNAL. On chromatic number of graphs and set-systems. *Acta Math. Acad. Sci. Hungar. 17* (1966), 61-99. On chromatic number of infinite graphs. *Graph theory symposium held at Tihany Hungary* 1966 (Akad. Kiadó Budapest and Academic Press New York) 83-98.
3. F. GALVIN. Chromatic numbers of subgraphs. *Periodica Math. Hungar. 4* (1973), 117-119.
4. P. ERDÖS, A. HAJNAL and S. SHELAH. On some general properties of chromatic numbers. *Coll. Math. Soc. J. Bolyai 8*. Topics in topology Keszthely (Hungary) 1972, 243-255.
5. P. ERDÖS. Graph theory and probability. *Canad. J. Math. 11* (1959), 34-38. On circuits and subgraphs of chromatic graphs. *Mathematica*, 9 (1962), 170-175.
6. P. ERDÖS, F. GALVIN and A. HAJNAL. On set-systems having large chromatic number and not containing prescribed subsystems. *Coll. Math. Soc. J. Bólyai 10*. Infinite and finite sets, Keszthely Hungary 1973, 425-513.
7. L. LOVÁSZ, On chromatic number of finite set systems. *Acta Math. Acad. Sci. Hungar. 19* (1968), 59-67.

4. In this chapter I give a short report on those problems of I where significant progress has been made since I has been written.

On p. 407 of I, I state the following problem. Let $\aleph_\alpha = |\alpha|$, $\aleph_\alpha$ not inaccessible. Is there a family of $\aleph_{\alpha+1}$ sets A_β, $1 \leqslant \beta < \omega_{\alpha+1}$, $|A_\beta| = \aleph_\alpha$, so that for every $\beta_1 < \beta_2 < \beta$,

$$|A_{\beta_1} \cap A_{\beta_3}| \neq |A_{\beta_2} \cap A_{\beta_3}|.$$

If $\aleph_\alpha$ is inaccessible, the construction of such a family is easy, and such a family clearly can not exist if $|\alpha| < \aleph_\alpha$. I believed that if such a family exists, then $\aleph_\alpha$ must be inaccessible. Ketonen to my great surprise showed that if $|\alpha| = \aleph_\alpha$ then such a family always exists. Unfortunately Ketonen's proof is unpublished.

Let $m(n)$ be the smallest integer for which there is a three chromatic uniform hypergraph having $m(n)$ hyperedges. On p. 410 the inequality

$$2^n \left(1 + \frac{2}{n}\right)^{-1} < m(n) < c\, n^2\, 2^n$$

is given. The upper bound has, as far as I know, never been improved, but Beck very significantly improved the lower bound. He showed

$$m(n) > c\, n^{1/3}\, 2^n .$$

It would of course be very desirable to obtain an asymptotic formula for $m(n)$, but this is nowhere in sight.

Let $|S| = 2n + k$. Define a graph of $\binom{2n+k}{n}$ vertices as follows: The vertices are the $\binom{2n+k}{n}$ n-tuples of S, two of the n-tuples are joined if they are disjoint. Kneser conjectured (p. 424 of I) that the chromatic number of this graph is $k + 2$. Kneser's conjecture has recently been proved first by Lovàsz, then Bàràny obtained a much simpler proof.

On p. 419 I state the following problem: Is there an infinite cardinal m so that if $|S| = m$ and if we divide all subsets of S into two classes then there is a sequence of disjoint subsets $A_k \subset S$, $k = 1, 2, \ldots$ so that all finite or infinite unions of the A_k belong to the same class? I expected the answer to be negative even if we restrict ourselves only to infinite unions and further insist that all the A_k are denumerable. Galvin just informs me that he proved that if $2^{\aleph_1} \leqslant (2^{\aleph_0})^+$ then for every set S we can color the subsets of power $\leqslant \aleph_1$ by $2^{\aleph_0}$ colors in such a way that given any family of $\aleph_0$ disjoint nonempty sets of cardinality $\leqslant \aleph_1$ all the colors occur among their infinite unions.

If we only consider finite subsets of S and allow only finite unions (see p. 418 of I) then we obtain the well known conjecture of Graham and Rothschild which was first proved by Hindman, simpler proofs were later obtained by Baumgartner and Glazer.

On p. 405 of I, I state our old conjecture with Rado: A family $\{A_\alpha\}$ of sets is said to form a strong Δ system if the intersection of any two of them is the same set. It is a weak Δ system if the intersection of any two of them has the same size. Denote by $f_s(k, l)$ (I use the notations of I) the smallest integer for which for any choice of $f_s(k, l)$ sets of size k there are l of them which form a strong Δ system. Rado and I conjectured

$$f_s(k, l) < c^k\, l^k . \tag{1}$$

(1) is open even for $l = 3$! This is one of my favourite finite problems. The best upper bound for $f_s(k, l)$ is due to J. Spencer, he proved that for every $\varepsilon > 0$ there is a $k > k_0(\varepsilon, l)$ for which

$$f_s(k, l) < (1 + \varepsilon)^k\, k!$$

In I (p. 406) I state that Abbott raised the following problem: Denote by $f_w^{(l)}(n)$ (I use the notations of I) the largest integer t for which there are t subsets of a set S of size n so that no l of them form a weak Δ system. Abbott observed that it is not trivial to show that $f_w^{(3)}(n) > cn$ for every c if $n > n_0(c)$. Sharpening a previous result of Szemerédi, Szemerédi and I proved

$$f_w^{(3)}(n) > \exp\left(\frac{c(\log n)^2}{\log\log n}\right) \tag{2}$$

(2) is probably far from being best possible, we have no non-trivial upper bound for $f_w^{(3)}(n)$ and can not even prove

$$\left(f_w^{(3)}(n)\right)^{1/n} \to 1\,. \tag{3}$$

Our paper with Szemerédi poses many new unsolved problems. Here I state only one of them: Let S be a set of n elements and consider subsets $A_k \subset S$ with $A_k = [(\log n)^2]$. Can we have more than $n^{2+\varepsilon}$ sets $\{A_k\}$ no three of which form a weak Δ system.

Hajnal and I conjectured (I, p. 414) that for every l and n there is a function $f(n, l)$ tending to infinity for every fixed l if $n \to \infty$ so that if $\chi(G) \geqslant n$ then G has a subgraph G_1 which contains no C_r, $3 \leqslant r \leqslant l$ and which has chromatic number $\geqslant f(n, l)$. Rödl proved this conjecture for $l = 3$. The infinite form of our conjecture is open even in this case. It seems likely that every G with $\chi(G) = m \geqslant w$ contains a subgraph G_1 which has no triangle and for which $\chi(G_1) = m$. This question is open for every $m \geqslant \chi_1$.

Two further conjectures of ours stated (I, p. 415): Is there a G, $|G| = \aleph_2$, $\chi(G) = \aleph_2$ so that for every subgraph G_1 with $|G_1| \leqslant \aleph_1$ we have $\chi(G) \leqslant \aleph_0$? As far as I know this problem is still open. Our other conjecture stated: Is there a G satisfying $|G| = \aleph_{\omega+1}$, $\chi(G) = \aleph_1$ so that every subgraph G_1 with $|G_1| \leqslant \aleph_\omega$ has chromatic number $\leqslant \aleph_0$. I believe that this has been proved to be consistent.

I. J. Beck. On three-chromatic hypergraphs. *Discrete Math. 29* (1978), 127-137.

L. Lovász. I. Knerer's conjecture, chromatic number and homotopy. *J. Comb. Theory (A) 25* (1978), 319-324. I. Báráry. A short proof of Kneser's conjecture. *ibid.* 325-326.

N. Hindman. Finite sums from sequences within cells of a partition. *J. Comb. Theory 17* (1974), 1-11. J. Baumgartner. A short proof of Hindman's theorem. *ibid.* 384-386.

W. COMFORT. Ultrafilters some old and some new results. *Bull. Amer. Math. Soc. 83* (1977), 417-455.
M. CATES, P. ERDÖS, N. HINDMAN and B. ROTHSCHILD. Partition theorems for subspaces of vector spaces. *ibid. 20 (A)* (1976), 279-291.
J. SPENCER. Interaction theorems for systems of sets. *Canad. Math. Bull. 20* (1977), 249-254.
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J. NESETRIL and V. RÖDL. Partitions of finite relational and set systems. *J. Comb. Theory A (22)* (1977), 289-312. This deep paper has a very extensive list of references.

5. In this final chapter I state a few solved and unsolved problems.

I. One of our old problems with Hajnal was settled two years ago by Mills and Prikry. Let $|A| = |B| = |C| = \aleph_1$ be three disjoint sets. Divide the set of all triples (X, Y, Z), $X \in A$, $Y \in B$, $Z \in C$ into two classes. Is it then true that there are subsets $A_1 \subset A$, $B_1 \subset B$, $C_1 \subset C$, $|A_1| = |B_1| = |C_1| = \aleph_0$ so that all triples (X, Y, Z), $X \in A_1$, $Y \in B_1$, $Z \in C_1$ are in the same class? Mills and Prikry proved that the answer is negative, for further results I have to refer to their paper which will soon appear.

II. Assume $c = \aleph_1$. I conjectured that E_n (the n-dimensional euclidean space) can be decomposed as the union of $\aleph_0$ sets S_n, $n = 1, 2, \ldots$ so that for every n all the distances between two points of S_n are distinct (i.e., every set of four points of S_n determines six distinct distances). For $n = 1$ this (and more) follows from an old result of Kakutani and myself. For $n = 2$ R. V. Davies proved it and very recently K. Kunen settled the general case. Our paper with Kakutani shows that if $c > \aleph_1$ thenthe result fails even for $n = 1$.

P. ERDÖS and S. KAKUTANI. On non-denumerable graphs. *Bull. Amer. Math. Soc. 49* (1943), 457-461.
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III. Galvin and I proved the following theorem: Divide the r-tuples of the integers into k classes. Then there always is an infinite subsequence $\{a_1 < \ldots\} = A$ satisfying $\sum_{a_i < n} 1 > c \log_{(r-1)} n$ (the $r - 1$ times iterated logarithm) for infinitely many n, all r-tuples of which belong to at most 2^{r-1} classes. 2^{r-1} can not be replaced by a smaller number, $\log_{r-1} n$ is probably best possible too but we know this only for $r = 2$.

The following conjecture of ours seems to be of interest. Assume $k = r = 2$. Is it true that to every C there is a set A all pairs of which are in the same class and

$$\sum_{a_i \in A} \frac{1}{a_i} > C . \tag{1}$$

(1) very likely remains true for k classes too. We observe that (1) can not be strengthened to $\sum_{a_i \in A} \frac{1}{a_i} = \infty$. (Added in proof) Rödl proved that (1) is false for $k > 2$.

We further prove the following result: Let $k = r = 2$, then there is a monochromatic path $\{a_1, a_2, ...\}$ (i.e., the edges (a_i, a_{i+1}) are all in the same class, the a's are all distinct but $a_i < a_{i+1}$ is not assumed) satisfying

$$\limsup_{n \to \infty} \frac{n}{\max\limits_{1 \leq k \leq n} a_k} > 0 . \tag{2}$$

The proof of (2) is surprisingly difficult.

It is easy to see that there is a monochromatic path for which

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{a_i \leq n} 1 \geqslant \frac{1}{2} . \tag{3}$$

We expect that in (3) $\frac{1}{2}$ can be replaced by a larger constant but so far we could not do this. It is easy to see that in (3) $\frac{1}{2}$ can not be replaced by $\frac{9}{10}$. For more details I have to refer to our forthcoming paper, which I hope will appear in a finite time.

IV. Several years ago the following problem occured to me: Let $K(n)$ be a complete graph on n vertices ($n < \omega$). Two players alternatingly choose an edge of $K(n)$. The game is finished if all the edges of $K(n)$ have been chosen by one (and only one) of the players. The first player wins if the largest clique of his graph is larger than the largest clique of the graph of the second player; otherwise he loses. Trivially the first player wins for

$n = 2$ and I think if $n > 2$ the second player can force a win. I could not even prove that the second player can force a win if n is sufficiently large.

One could modify the rules as follows: The first player wins if he has the larger clique or if the largest cliques are of the same size; he wins if he has more cliques of maximal size than the second player. Perhaps here the first player wins for all sufficiently large n.

Hajnal and his colleagues did recently a great deal of interesting work on infinite Ramsey games, the relevant papers will soon appear. These results lead me to the following question: Let $S = \{a_\alpha\}$, $\alpha = 1, 2, \ldots$ be a set $|S| = \aleph_n \geqslant \aleph_1$, the a_α are assumed to be linearly independent. Let $S(r)$ be the set of all element $\Sigma\, r_\alpha\, a_\alpha$ where the r_α are rational and the sums are finite. Clearly $|S| = |S(r)| = \aleph_n$. Now two players play the following game in $S(r)$. They move alternatingly, the first player always chooses one element which has not yet been chosen and then the second player chooses $\aleph_0$ elements which have not yet been chosen. The game continues until $S(r)$ has been completely divided between the players, after β moves where β has no predecessor it always is the first players move. The aim of the first player is to get as long an arithmetic progression as possible.

Hajnal and I observed that the second player can always prevent the first player from getting an infinite arithmetic progression. In fact he can do this even if he is also permitted only the choice of one element. We further observed that for $n = 1$ the second player can prevent the first player from getting an arithmetic progression of three terms and for $n = 2$ he can prevent an arithmetic progression of four terms, but not of three terms. Galvin and Nagy proved that the first player alsways can force an arithmetic progression of $n + 1$ terms but the second player can prevent him from getting an arithmetic progression of $n + 2$ terms.

To end the paper I state two problems. The first is a beautiful old problem of Kemperman: Let $f(X)$, $\infty < x < \infty$ be a real function. Assume that

$$2\, f(X) \leqslant f(X+h) + f(X+2h)$$

holds for every x and every positive h. Does it then follow that $f(x)$ is non-decreasing i.e., if $y > x$ then $f(y) \geqslant f(x)$. One would expect that it will be easy to prove this or get a counterexample. If $f(x)$ is assumed to be measurable the proof is indeed easy but the general case seems to present difficulties. (Added in proof) Laczkovics proved that $f(x)$ is monotonic, his proof will soon appear in *Acta Math. Acad. Sci. Hungar.*

The second problem is an old question of mine which perhaps will not be difficult: Let E be an infinite set of real numbers. Is there always a set of real numbers S of positive measure which does not contain a set E_1 similar to E? (similar here means homotetic i.e., E_1 can be obtained from E by translation and dilation or contraction). It clearly would suffice to prove this if E is a sequence of positive numbers tending to 0.

I expect that such a set S of positive measure always exist.

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