ON THE ALMOST EVERYWHERE DIVERGENCE OF LAGRANGE INTERPOLATION*

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1. INTRODUCTION

In the previous paper P. Erdős stated without proof that if $Z = \{x_{in}\}$, $n = 1, 2, \dots,$ $1 \leq i \leq n$,

$$(1.1) \quad -1 \leq x_{nn} < x_{n-1,n} < \dots < x_{1n} \leq 1 \qquad (n=1,2,\dots)$$

is a triangular matrix, then there exists a continuous function $F(x)$, $-1 \leq x \leq 1$, such that the sequence of Lagrange interpolation polynomials

$$L_n(F, Z, x) = L_n(F, x) = \sum_{k=1}^{n} F(x_{kn}) l_{kn}(x)$$

diverges almost everywhere in $[-1, 1]$, and in fact

$$\overline{\lim_{n\to\infty}} |L_n(F, Z, x)| = \infty$$

for almost all x (see [1]). (Here, as usual

$$(1.2) \qquad l_{kn}(x) = \frac{\omega_n(x)}{\omega_n'(x_{kn})(x - x_{kn})}$$

$$(k = 1, 2, \dots, n; \ \omega_n(x) = \prod_{k=1}^{n} (x - x_{kn}))$$

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* The detailed version will appear in Acta Math. Acad. Sci. Hunga

the corresponding fundamental polynomials,

(1.3) $\lambda_n(x) = \sum_{k=1}^{n} |l_{kn}(x)|, \quad \lambda_n = \max_{-1\leq x\leq 1} \lambda_n(x) \quad (n = 1, 2, \ldots),$

the Lebesgue functions and the Lebesgue constants of the interpolation.)

Here is the sketch of the proof. The detailed proof (about 30 pages) turned out to be quite complicated and several unsuspected difficulties had to be overcome

2. PRELIMINARY RESULTS

In his classical paper [2] G. Faber proved that for any matrix Z

$$\overline{\lim_{n\to\infty}} \lambda_n = \infty$$

from where it follows directly that for every point group there exists a continuous function $f_1(x)$, $-1\leq x\leq 1$ (shortly $f_1 \in C$), such that

$$\overline{\lim_{n\to\infty}} \| L_n(f_1, x) \| = \infty .$$

(Hence $\| g(x) \| = \| g \| = \max_{-1\leq x\leq 1} |g(x)|$ for $g \in C$.) Almost twenty years later, in 1931, S. Bernstein showed that for every Z for which (1.1) holds there exists an $f_2 \in C$ and x_0, $-1\leq x_0 \leq 1$, such that

(2.1)
$$\overline{\lim_{n\to\infty}} | L_n(f_2, x_0)| = \infty$$

(see [3]).

$E = \{-1+2(k-1)/(n-1)\}$ and the function $|x|$

$$\overline{\lim_{n\to\infty}} |L_n(|t|, E, x)| = \infty \quad \text{if} \quad x \in (-1,1), \ x \neq 0.$$

Then, using the "good" Chebyshev matrix

$$T = \left\{x_{kn} = \cos \frac{2k-1}{2n}\pi \ ; \ k=1,2,\ldots,n; \ n=1,2,\ldots\right\},$$

G. Grünwald [4] obtained that there exists a function $f_3 \in C$ for which

$$(2.2) \qquad \overline{\lim} \, | \, L_n(f_3, T, x) = \infty$$

for almost all x in $[-1,1]$. Later he and (independently) J. Marcinkiewicz proved that for a suitable $f_4 \in C$, (2.2) is true for every x from $[-1,1]$ (see [5] and [6]).

Quite recently A. A. Privalov [7] considered the Jacobi matrices

$$Z^{(\alpha,\beta)} = \left\{x_{kn}^{(\alpha,\beta)}, \quad k=1,2,\ldots,n; \ n=1,2,\ldots\right\}, \qquad \alpha, \beta > -1$$

(see e.g. [8], Part 2), and showed that for a certain $f_5 \in C$

$$(2.3) \qquad \overline{\lim_{n\to\infty}} \, | \, L_n(f_5, Z^{(\alpha,\beta)}, X) | = \infty \qquad \text{a.e. on } [-1,1],$$

where "a. e." stands for "almost everywhere". (He considered some further point groups, too.) His proof strongly depends on the properties of the Jacobi roots $x_{kn}^{(\alpha,\beta)}$.

Finally, he proved (2.3) for the whole $(-1,1)$ (see [13]).

3. RESULT

As indicated above, we are going to prove (2.2) for any fixed point group Z, i.e. we state

THEOREM. For any matrix Z for which (1.1) holds one can find a function $F \in C$ such that

$$(3.1)\quad \overline{\lim_{n\to\infty}} |L_n(F, Z, x)| = \infty \quad \text{for almost all } x \text{ in } [-1, 1].$$

On the other hand, considering the special matrix

$$\begin{array}{l} x_1 \\ x_1, x_2 \\ x_1, x_2, x_3 \\ \cdots\cdots\cdots \end{array}$$

can say that (3.1) generally is not true for all $x\in[-1,1]$ (see P. rán [9], Problem III; [1], p. 384).

Finally, let us note that the "$\overline{\lim}$" cannot be replaced by "lim" "$\underline{\lim}$". Indeed, as P. Erdős showed, one can construct a point oup so that for every $f\in C$ and every $x_0\in[-1,1]$ there would exist sequence n_k (depending on f and x_0) such that

$$\lim_{k\to\infty} L_{n_k}(f, x_0) = f(x_0)$$

ee [1], p. 384).

ON THE PROOF

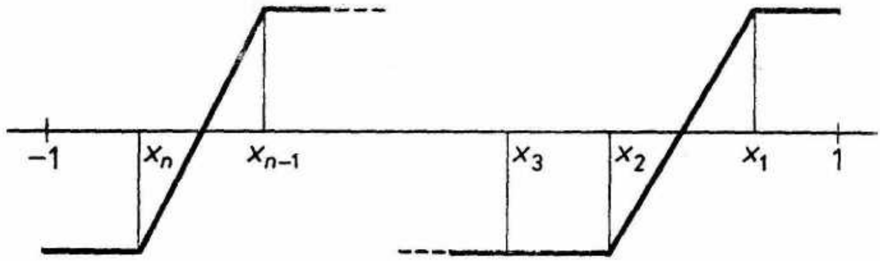
As we mentioned above, the proof is rather long and quite complited although it uses only elementary techniques. Our aim here is sketch it, stressing some characteristic considerations and lemmas.

4.1 The quoted result (2.1) of S. Bernstein can be obtained from e fact that for any matrix Z one can choose the point $x_0\in[-1,1]$ r which

$$\lambda_n(x_0) = \sum_{k=1}^{n} |l_{kn}(x_0)| > (\frac{2}{\pi} + o(1)) \ln n,$$

r infinitely many n (see the same paper, [3]).

Indeed if



then obviously

$$L_n(g_n, x_0) = \sum_{k=1}^{n} g_n(x_k)\, l_k(x_0) = \sum_{k=1}^{n} |l_k(x_0)| = \lambda_n(x_0).$$

I. e., if

$$f(x) \overset{\text{def}}{=} \sum_{k=n_1, n_2, \dots} \frac{1}{\varphi_k} g_k(x),$$

then $f(x) \in C$, moreover

$$L_{n_i}(f, x_0) = \sum_{k<i} \dots + \frac{L_{n_i}(q_{n_i}, x_0)}{\varphi_{n_i}} + \sum_{k>i} \dots,$$

from where we obtain (2. 1) with suitably chosen $\{\varphi_n\}$, $\varphi_n \to \infty$.

4. 2. In 1958 P. Erdős proved, that for any given $A > 0$ and $\varepsilon > 0$ the measure of the set in x $(-\infty < x < \infty)$ for which $\lambda_n(x) \leqslant A$, $n \geqslant n_0(A, \varepsilon)$, holds, is less than ε, whatever is the matrix Z. From this we immediately obtain, that

$$\text{(4. 1)} \qquad \overline{\lim_{n\to\infty}} \sum_{k=1}^{n} |l_k(y)| = \infty \qquad \text{for almost all } y \text{ in } [-1, 1].$$

So, as above, we can obtain an uncountable family of functions $f_y(x) \in C$ such that

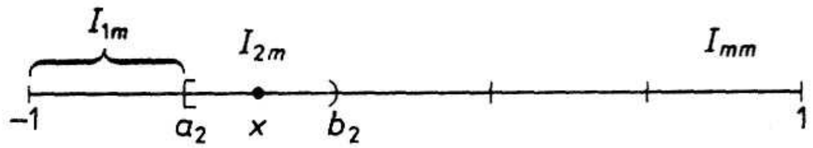
$$\overline{\lim_{n\to\infty}} |L_n(f_y, y)| = \infty \qquad \text{for almost all } y \text{ in } [-1, 1].$$

To prove the original statement (3. 1), we shall have to construct the continuous $F(x)$ using the uncountable family of functions f_y. But is approach does not seem passable. So we choose another method, where we shall have to unite at most countable family of functions.

4.3. At first let us suppose that (with $x_{0n} = 1$, and $x_{n+1,n} = -1$)

$$\Delta x_{kn} \overset{\text{def}}{=} x_{kn} - x_{k+1,n} \leq \delta_n \overset{\text{def}}{=} 1/\ln n \quad (k=0,1,\dots,n;\ n=1,2,\dots).$$

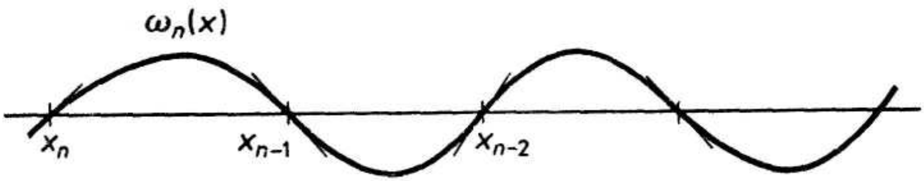
Divide the interval [-1, 1] by equidistant points as follows.



Now, if we condiser, e.g. the expression

$$(4.2)\qquad \Big| \sum_{x_k < a_2} (-1)^k l_k(x) + \sum_{x_k > b_2} (-1)^{k+1} l_k(x) \Big| \quad \text{if} \quad x \in I_2$$

it is a simple, but a very important remark that all the terms in (4.2) have the same sign for any fixed $x \in I_2$. (Indeed if, e.g. $x_k < a_2$, then



sign $[(-1)^k \omega'(x_k)] = S$ where $S = \pm 1$ for any k, $1 \leq k \leq n$, moreover $x - x_k > 0$, which means that we can expect that we have to unite only "few" functions to obtain $F(x)$ where the intervals I_j are of positive measure.

This phenomenon in expressed by the following statement.

LEMMA 4.1. Let $A>0$ be an arbitrary fixed number. Then considering the arbitrary integer $m \geq m_0(A)$, for any $n \geq n_0(m)$ there exists the set $H_n \subset [-1,1]$ for which $\mu(H_n) \leq 1/\ln \ln m$, moreover, whenever $x \in [-1,1] \setminus H_n$

$$(4.3) \quad \sum_{\substack{1 \leq k \leq n \\ x_k \notin I_{j(x),m}}} |l_{kn}(x)| \geq (\ln m)^{1/3} \geq 2A \qquad \text{if} \qquad n \geq n_0(m).$$

Here $I_{j(x),m}$ is the interval containing x; $\mu(\ldots)$ stands for the Lebesgue measure.

This lemma proved to be a very important part of the proof. It is a rather deep generalization of the statement by P. Erdős (quoted in 4.1) because in the sum $\sum_{k=1}^{n} |l_k(x)|$ generally even the terms for which $|x-x_{kn}|$ is "small" are "large". In the proof of (4.3) we use only some basic notions of interpolation theory and combinatorial considerations.

4.4. Using Lemma 4.1, we obtain the finite number of continuous functions $f_i(x)$ whose Lagrange interpolatory polynomials are big on the sets B_i. More exactly we get that

$$\overline{\lim_{n\to\infty}} |L_n(f_i, x)| = \infty \quad \text{on} \quad B_i \quad (1 \leq i \leq s)$$

where $\sum_{i=1}^{s} \mu(B_i) = 2-\rho$, $\rho > 0$ is arbitrary. To combine these f_i we use

LEMMA 4.2. If $r_1(x), r_2(x) \in C$, moreover

$$\overline{\lim_{n\to\infty}} |L_n(r_1, x)| = \infty \quad \text{if} \quad x \in B_1, \quad \mu(B_1) < \infty,$$

$$\overline{\lim_{n\to\infty}} |L_n(r_2, x)| = \infty \quad \text{if} \quad x \in B_2, \quad \mu(B_2) < \infty,$$

then any fixed interval (β_1, β_2) $(\beta_1 < \beta_2)$ contains an α such that

$$\overline{\lim_{n\to\infty}} |L_n(\alpha r_1 + r_2, x)| = \infty \quad \text{a. e. on } B_1 \cup B_2$$

(a. e. = almost everywhere).

Applying these lemmas and some other considerations we obtain the theorem.

4.5. For the intervals $\Delta x_k > \delta_n$, instead of Lemma 4. 1, we can use the following

LEMMA 4. 3. Let $\Delta x_{kn} > \delta_n$ (k is fixed, $0 \leq k \leq n$). Then for any fixed $0 < q < 1/2$ we can define the index $t = t(k, n)$ and the set $h_{kn} \subset [x_{k+1,n}, x_{kn}]$ such that $\mu(h_{kn}) \leq 4q\Delta x_{kn}$, moreover

$$|l_t(x)| \geq 3^{n\delta_n^5} \quad \text{if} \quad x \in [x_{k+1,n}, x_{kn}] \setminus h_{kn}$$

and $n \geq n_1(q)$.

Finally, by a statement analogous to Lemma 4. 2 we can complete the proof for the case of the long intervals as well.

And at last one more problem on Lagrange interpolation which seems to be quite a difficult one: There is a pointgroup $\{x_{kn}\}$ such that for every continuous $f(x)$, $L_n(f, x_0) \to f(x_0)$ holds for at least one x_0 for which $\overline{\lim_{n\to\infty}} \lambda_n(x_0) = \infty$ (see [1]). This is probably true, but at this moment we cannot prove it (the original "proof" was incomplete).

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