Studia Scientiarum Mathematicarum Hungarica 13 (1978), 151—155.

ON SOME PARTITION PROPERTIES OF FAMILIES OF SETS

by

G. ELEKES, P. ERDŐS and A. HAJNAL

§ 0.

This write up contains a list of results and problems concerning questions we have stated and investigated in some earlier papers [1], [2], [3]. Though we do not give proofs the experienced reader will be able to reconstruct most of them, by checking the lemmas we are going to state and following the hints we give after stating some of the theorems.

We deal with questions of the following type. Let S be (a relatively large) family of sets. In most of the questions we will ask S will be of the form $P(\varkappa)$, the set of all subsets of an infinite cardinal $\varkappa$. There will be given a mapping $f: P(\varkappa)\to\gamma$. f will be called a partition of $P(\varkappa)$ with γ colors. A subfamily $S'\subset S$ will be called homogeneous for f if $S'\subset f^{-1}(\{\eta\})$ for some $\eta<\gamma$. As usual we will ask for the existence of relatively large homogeneous subfamilies.

DEFINITION 0.1. $\mathscr{A}=\{A_\nu\colon \nu<\varkappa\}$ is a $\varkappa$, *Δ-system* if there is a set D such that $A_\nu\cap A_\mu=D$, $A_\nu\neq A_\mu$ for $\nu\neq\mu$; $\nu,\mu<\varkappa$. D is the *kernel* of the *Δ-system* $\mathscr{A}$.

DEFINITION 0.2. a) Let $\mathscr{A}=\{A_\nu\colon \nu<\varkappa\}$ be a sequence of sets. Put $\mathscr{A}(N)=\cup\{A_\nu\colon \nu\in N\}$ for $N\subset\varkappa$.

b) A family $\mathscr{F}$ of sets is said to be a $(\varkappa,\lambda)$*-system* determined by $\mathscr{A}=\{A_\nu\colon \nu<\varkappa\}$ if $\mathscr{F}=\{\mathscr{A}(N)\colon N\in[\varkappa]^{<\lambda}\setminus\{\emptyset\}\}$ and

$$\mathscr{A}(N_0)\neq\mathscr{A}(N_1)\quad\text{for}\quad N_0\neq N_1,\quad N_0,N_1\in[\varkappa]^{<\lambda}.$$

DEFINITION 0.3. A family $\mathscr{F}$ is said to be a $(\varkappa,\lambda)$, *Δ-system* if there is a $(\varkappa,\lambda)$-system $\mathscr{F}'$ determined by $\mathscr{A}=\{A_\nu\colon \nu<\varkappa\}$ such that $\mathscr{A}$ is a Δ-system with kernel D and $\mathscr{F}=\mathscr{F}'\cup\{D\}$.

To have short notation we introduce relations bearing some resemblance to partition relations investigated earlier.

DEFINITION 0.4. $S\to\Delta(\varkappa)_\gamma$, $S\to([\varkappa]^{<\lambda})_\gamma$, $S\to\Delta([\varkappa]^{<\varkappa})_\gamma$ mean that for all partitions $f\colon S\to\gamma$ of S with γ colors there is a $\varkappa$, Δ-system, a $(\varkappa,\lambda)$-system and a $(\varkappa,\lambda)$, Δ-system homogeneous for f, respectively. $A\nrightarrow$ indicates that the respective statements are false.

§ 1. Positive arrow relations for the first two symbols

DEFINITION 1.1. For a $U\subset P(\varkappa)$, $A, B\subset\varkappa$ we write $A\subset_U B$ if $A\subset B$ and there is a $C\in U$ with $B-A\supset C$.

a) $S\subset P(\varkappa)$ is *dense* in $[A, B]$ for U if $A\subset_U B$ and

$$\forall A'B'(A\subset A'\subset_U B'\subset B\Rightarrow\exists C\in S(A'\subset_U C\subset_U B')),$$

b) $S\subset P(\varkappa)$ is *left (right) dense* in $[A, B]$ for U if $A\subset_U B$ and

$$\forall A'B'(A\subset A'\subset_U B'\subset B\Rightarrow\exists C\in S(A'\subset_U C\subset B')$$

$$(\forall A'B'(A\subset A'\subset_U B'\subset B\Rightarrow\exists C\in S(A'\subset C\subset_U B'))).$$

If $S\subset P(\varkappa)$ is not dense (not left or right dense) for U in any $[A, B]$ then S is *nowhere dense (nowhere left or right dense)* for U in $P(\varkappa)$.

The following sequence of Baire-type lemmas serve as a basis of our proofs. Note that they all imply existence of dense (in a certain sense) sets homogeneous for some partitions.

LEMMA 1.1. *Let* $U=[\omega]^{\omega}$. $P(\omega)$ *is not the union of countably many sets nowhere dense for* U.

This lemma has been proved in [1]. The next lemma is due to J. BAUMGARTNER and is included here with his permission.

LEMMA 1.2. *Let* $\varkappa>\omega$ *be a regular cardinal, and* U *a normal filter in* $P(\varkappa)$. *Then* $P(\varkappa)$ *is not the union of* $\varkappa$*-sets nowhere left dense (right dense) for* U.

LEMMA 1.3. *Let* $\varkappa\geqq\omega$ *be a regular cardinal. Let* $R(\varkappa)=\{g\in{}^{\varkappa}2\colon\exists\xi<\varkappa(g(\xi)=1\wedge\forall\xi<\eta<\varkappa(g(\eta)=0))\}$, *i.e., the well-known Hausdorff set of* 0—1*-sequences of length* $\varkappa$, *with a last* 1*-digit. Let* $<_{\varkappa}$ *denote the usual lexicographic ordering of* $R(\varkappa)$, *and* $U_{\varkappa}$ *the set of non-empty open intervals of* $\langle R(\varkappa), <_{\varkappa}\rangle$. *Then* $P(R(\varkappa))$ *is not the union of* $\varkappa$*-sets nowhere dense for* $U_{\varkappa}$.

COROLLARY 1.1. *Let* $\varkappa\geqq\omega$ *be regular and* $2^{\underset{\sim}{\varkappa}}=\varkappa$. *There is a non-empty* $U\subset P(\varkappa)$ *such that* $P(\varkappa)$ *is not the union of* $\varkappa$ *sets nowhere dense for* U *and* U *satisfies the following conditions* a) b):

a) *If* $\{I_{\eta}\colon\eta<\varphi\}$ *is a decreasing sequence of type* $\varphi<\varkappa$ *of elements of* U *then there is an* $I\in U$ *such that* $I\subset I_{\eta}$ *for* $\eta<\varphi$.

b) *Each member of* U *contains* $\varkappa$*-pairwise disjoint members of* U.

The next lemma transfers the above results for the case of singular $\varkappa$'s.

LEMMA 1.4. *Let* $\lambda=\mathrm{cf}(\varkappa)<\varkappa$ *be a singular cardinal. Assume* $\varkappa_{\alpha}<\varkappa$ *for* $\alpha<\lambda$ *and* $\varkappa=\sup\{\varkappa_{\alpha}\colon\alpha<\lambda\}$. *Let* $\varkappa=\bigcup\{X_{\alpha}\colon\alpha<\lambda\}$ *be a decomposition of* $\varkappa$ *and assume* $U_{\alpha}\subset P(X_{\alpha})$ *and* $P(X_{\alpha})$ *is not the union* $\varkappa_{\alpha}$*-sets nowhere dense [nowhere left (right) dense] for* U_{α} *for* $\alpha<\varkappa$. *Let*

$$V_{\alpha}=\{Y\subset\varkappa\colon\forall\alpha\leqq\beta<\lambda(Y\cap X_{\beta}\in U_{\beta})\}\quad\textit{for}\quad\alpha<\lambda.$$

Then for each decomposition $\bigcup\{S_{\eta}\colon\eta<\varkappa\}=P(\varkappa)$ *of* $P(\varkappa)$ *there are* $\eta<\varkappa$ *and* $\alpha<\lambda$ *such that* S_{η} *is dense (left or right dense) for* V_{α} *in* $P(\varkappa)$.

Note that it is consistent with $2^{\aleph_0}=\aleph_2$ that Lemma 1.1 remains true for $\aleph_1$ sets instead of countably-many. This follows from a result of S. SHELAH [4], and from the fact that forcing a Silver real is proper forcing.

THEOREM 1.1. a) $P(\varkappa)\to\Delta(\lambda)_\varkappa$ *for* $\lambda<\varkappa\geqq\omega$;
b) $P(\varkappa)\to\Delta(\varkappa)_\varkappa$ *for all regular* $\varkappa\geqq\omega$.

For $\varkappa=\omega$ use Lemma 1.1. For $\varkappa>\omega$ use the left dense forms of Lemmas 1.4 and 1.2, respectively. For $\varkappa>\omega$ we originally proved this result under the assumption $2^{\underset{\smile}{\varkappa}}=\varkappa$, using Lemma 1.3. The more general theorem stated above is due to Baumgartner.

PROBLEM 1. Does $P(\aleph_\omega)\to\Delta(\aleph_\omega)_{\aleph_\omega}$ hold? We do not know the answer even assuming G.C.H.

THEOREM 1.2. *Assume* $\varkappa\geqq\omega$ *and* $2^{\underset{\smile}{\varkappa}}=\varkappa$. *Then*

$$P(\varkappa)\to([\omega]^{<\omega})_\varkappa \quad for \quad \varkappa\geqq\omega.$$

Note that this implies $P(\omega)\to([\omega]^{<\omega})_\omega$ without any assumption, as it was announced in [1].

PROBLEM 2. Can one prove $P(\omega_1)\to([\omega]^{<\omega})_{\omega_1}$ without assuming $2^{\aleph_0}=\aleph_1$?

THEOREM 1.3. a) $P(\varkappa)\to([\lambda]^{<n})_\varkappa$ *for* $n<\omega, \varkappa\geqq\omega, \lambda<\varkappa$.
b) $P(\varkappa)\to([\varkappa]^{<n})_\varkappa$ *for* $n<\omega$ *and for all regular* $\varkappa\geqq\omega$.

As to the proofs of Theorems 1.2 and 1.3 consider a partition $f\colon P(\varkappa)\to\varkappa$ of $P(\varkappa)$. By the density lemmas, there is a $\nu<\varkappa$ such that $S=f^{-1}(\{\nu\})$ is appropriately dense in some $[A, B]$. In cases $\varkappa=\omega$ we use Lemma 1.1 and $U=[\omega]^\omega$ density in both proofs. In case $\varkappa>\omega$ regular, for Theorem 1.2 we use the U described in Corollary 1.1 and for Theorem 1.3 we use the normal filter induced by the clubs. In case $\varkappa$ is singular we apply Lemma 1.4 to obtain appropriate U'-s. We finish the proofs by showing that all S dense in $[A, B]$ for U contain (ω, ω)-systems (λ, η)-systems and $(\varkappa, \eta)$-systems, respectively. In both proofs the (τ, σ)-systems are constructed according to the following pattern. We first define a sequence $\{B_N\colon N\in[\tau]^{<\sigma}\setminus\{\emptyset\}\}\subset S$, and prove later that for $A_\nu=B_{\{\nu\}}, \nu<\tau$ $\mathscr{A}(N)=B_N$ holds for $N\in[\tau]^{<\sigma}\setminus\{\emptyset\}$. The B_N are constructed by induction according to a fixed well-ordering of $[\tau]^{<\sigma}\setminus\{\emptyset\}$. In both proofs we need special tricks to make sure that if $B_M\colon M<N$ is defined there is room enough to find B_N. We omit the details of this constructions.

§ 2. Negative arrow relations for the second symbol. Generalizations

As we have already mentioned in [1] it is easy to see that

THEOREM 2.1. $P(\omega)\nrightarrow([\omega]^{<\omega_1})_2$ *holds.*

PROBLEM 3. a) Does $P(\varkappa)\to([\omega]^{<\omega_1})_2$ hold for any $\varkappa$?
b) Does $[\varkappa]^\omega\nrightarrow([\omega]^{<\omega_1})_2$ hold for any $\varkappa$?

The following result shows that in Theorem 1.2 we cannot get a larger homogeneous $[\lambda]^{<\omega}$ system.

THEOREM 2.2. *Assume* $2^{\varkappa}=\varkappa^{+}$. *Then*

$$P(\varkappa)\nrightarrow([\omega_1]^{<\omega})_{\varkappa}.$$

PROBLEM 4. a) Can one prove $P(\omega)\nrightarrow([\omega_1]^{<\omega})_{\omega}$ without assuming C.H.?
b) Can one prove $P(\varkappa)\rightarrow([\omega_1]^{<\omega})_2$ for any $\varkappa$?

PROBLEM 5. a) $P(\omega)\rightarrow([\omega_1]^{<3})_2$?
b) $P(\omega)\rightarrow([\omega_1]^{<3})_{\omega}$?

We can neither prove a) nor disprove b) in any reasonable extension of ZFC. Theorem 2.2 is a consequence of the following lemma due to P. KOMJÁTH.

LEMMA 2.1. *Let* $<$ *be a well-ordering of* $T=[\omega_1]^{<\omega}\setminus\{\emptyset\}$. *Then there are* $L<M<N$; $L, M, N\in T$ *such that* $L\cup M=N$.

LEMMA 2.2. *Assume* $\varkappa>\tau\geqq\omega$. *Let* $\{R_\alpha\colon \alpha<\varkappa\}$ *be an increasing continuous sequence of fields of sets, i.e.,* $R_\beta\subset R_\alpha$ *for* $\beta<\alpha<\varkappa$ *and* $R_\alpha=\bigcup\{R_\beta\colon \beta<\alpha\}$ *for limit* α, $\alpha<\varkappa$. *Assume that* $\bigcup\{R_\alpha\colon \alpha<\varkappa\}\supset\mathscr{F}$ *for some* (τ^{+},ω)*-system* $\mathscr{F}$. *Then there are* $\alpha<\varkappa$ *and a* (τ,ω)*-system* $\mathscr{F}'$ *with* $\mathscr{F}'\subset R_{\alpha+1}\setminus R_\alpha$.

Let now $\varkappa^{+(\mu)}$ denote the μ-th successor of $\varkappa$. We get

THEOREM 2.3. *Assume* S *is a system of sets,* $|S|=\varkappa^{+(\mu)}$ *for some* $\varkappa\geqq\omega$ *and* $\mu<\varkappa^{+}$. *Then* $S\nrightarrow([\aleph_\mu]^{<\omega})_{\varkappa}$.

This result actually yields a stronger theorem then 2.2. We also get

COROLLARY 2.1. *Assume* $\varkappa\geqq\omega$ *is regular and* $2^{\varkappa}<\varkappa^{+(\varkappa^{+})}$. *Then* $P(\varkappa)\nrightarrow([2^{\varkappa}]^{<\omega})_{\varkappa}$.

For example $P(\omega)\nrightarrow([2^{\omega}]^{<\omega})_{\omega}$ provided $2^{\omega}<\aleph_{\omega_1}$.
There is nothing to prevent this $\nrightarrow$ from being true in ZFC but we cannot prove it.
Finally we state one very special result which only shows how one *cannot* solve Problem 5.

THEOREM 2.4. *Assume* $\varkappa$ *is regular and* $2^{<\varkappa}=\varkappa$. *For every coloring* $f\colon P(\varkappa)\rightarrow\varkappa$, *of* $P(\varkappa)$ *with* $\varkappa$ *colors there is a* $\nu<\varkappa$ *such that* $S=f^{-1}(\{\nu\})$ *contains a set system of the following form* $\mathscr{F}=\{A_\mu\colon \mu<\varkappa\}\cup\{B_\nu\colon \nu<\varkappa^{+}\}\cup\{A_\mu\cup B_\nu\colon \mu<\varkappa\wedge\nu<\varkappa^{+}\}$ *where all the sets* $A_\mu, B_\nu, A_\mu\cup B_\nu$ *are different.*

This could be expressed by the symbol

$$P(\varkappa)\rightarrow([\varkappa,\varkappa^{+}]^{<2,<2})_{\varkappa}$$

had we defined this in this generality.

§ 3. The third symbol

Here we only mention a result and one problem.

THEOREM 3.1. *Let* $\lambda \geqq \omega$ *then*

$$2^{\aleph_0} \geqq \lambda^{+(n)} \Leftrightarrow P(\omega) \to \Delta([n]^{<n+1})_\lambda \quad \text{for} \quad 1 \leqq n < \omega.$$

PROBLEM 6. Does $2^{\aleph_0} > \aleph_\omega$ imply

$$P(\omega) \to \Delta([\omega]^{<\omega})_\omega ?$$

REFERENCES

[1] ELEKES, G., On a partition property of infinite subsets of a set, *Period. Math. Hungar.* **5** (1974), 215—218.

[2] ERDŐS, P., Problems and results on finite and infinite combinatorial analysis, *Infinite and finite sets* (Colloq. Math. Soc. J. Bolyai **10**), North-Holland Publishing Co., Amsterdam, 1975, 403—424.

[3] ERDŐS, P.—HAJNAL, A., Solved and unsolved problems in set theory, *Proceedings of the Tarski Symposium* (Berkeley Calif., 1971), Amer. Math. Soc., Providence, R. I., 1974, 269—287.

[4] SHELAH, S., (preprint).

[5] ELEKES, G., Colouring of infinite subsets of ω, *Infinite and finite sets* (Colloq. Math. Soc. J. Bolyai 10), North-Holland Publ. Co., Amsterdam—New York, 1975, 393—396.

Department for Analysis I, Roland Eötvös University
H—1088 Budapest, Múzeum krt. 6—8
and
Mathematical Institute of the Hungarian Academy of Sciences
H—1053 Budapest, Reáltanoda u. 13—15

(Received June 4, 1979)