Values of the Divisor Function on Short Intervals

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In this paper we prove (in a rather more precise form) two conjectures of P. Erdös about the maximum and minimum values of the divisor function on intervals of length k.

INTRODUCTION

In this paper we prove two conjectures of P. Erdös concerning the divisor function $\tau(n)$. These are

CONJECTURE A. *For each fixed integer* k, *we have*

$$\sum_{n<x} \max\{\tau(n), \tau(n+1), \ldots, \tau(n+k-1)\} \sim kx \log x.$$

CONJECTURE B. *For each fixed integer* k, *there exists a* $\beta_k < 1$, *such that* $\lim(\beta_k : k \to \infty) = \log 2$, *and such that for every* $\epsilon > 0$ *and* $x > x_0(\epsilon, k)$, *we have*

$$x(\log x)^{\beta_k - \epsilon} < \sum_{n<x} \min\{\tau(n), \tau(n+1), \ldots, \tau(n+k-1)\} < x(\log x)^{\beta_k + \epsilon}.$$

In each case we prove slightly more—it turns out that B is much more difficult than A.

THEOREM 1. *Conjecture* A *is true. Moreover, the formula holds for* $k \to \infty$ *as* $x \to \infty$, *provided*

$$k = o((\log x)^{3-2(2)^{1/2}}).$$

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THEOREM 2. *Conjecture* B *is true. More precisely, let k be fixed,*

$$\alpha_k = k(2^{1/k} - 1).$$

Then for sufficiently large x,

$$\frac{C_7(k)\, x(\log x)^{\alpha_k}}{(\log\log x)^{11k^2}} \leqslant \sum_{n<x} \min\{\tau(n),\ldots, \tau(n+k-1)\} \leqslant C_8(k)\, x(\log x)^{\alpha_k}.$$

Remarks. It would be of interest to know how large k may be, as a function of x, for the formula in Theorem 1 to be valid.

The $11k^2$ appearing in Theorem 2 is not the best that could be obtained from the present technique, but the exponent of loglog x certainly tends to infinity with k. It seems possible that no power of loglog x is needed, so that the sum is determined to within constants: this would need a new idea, and of course an asymptotic formula would be much better.

Before embarking on the proofs we establish several lemmas. Lemma 9, which is rather too technical to be comprehensible standing alone, appears in the middle of the proof of Theorem 2.

0-Constants, and those implied by $\ll$, are independent of all variables. The constants A_i and B in Lemma 9 depend on k. Constants $C_i(k)$ also depend, at most, on k. The usual symbols for arithmetical functions are used: thus $\nu(n)$ and $\omega(n)$ stand for the number of distinct, and the total number of prime, factors of n. The least common multiple of $d_0,\ldots, d_{k-1}$ will be denoted by $[d_0,\ldots, d_{k-1}]$.

LEMMA 1. *For all positive integers α and k, we have*

$$1 + 2^{1/k} + 3^{1/k} + \cdots + \alpha^{1/k} \geqslant \frac{k}{k+1}\, \alpha(\alpha+1)^{1/k}.$$

Proof. For positive integers k and β, we have

$$\left(1 + \frac{1}{k\beta}\right)^k \geqslant 1 + \frac{1}{\beta}.$$

Hence

$$\{k + 1 + k(\beta - 1)\}\, \beta^{1/k} \geqslant k\beta(\beta + 1)^{1/k}$$

and

$$\beta^{1/k} \geqslant \frac{k}{k+1}\{\beta(\beta+1)^{1/k} - (\beta - 1)\, \beta^{1/k}\}.$$

We sum this for $\beta = 1, 2, 3,\ldots, \alpha$.

LEMMA 2. *Let $f_k(n)$ be the multiplicative function generated by*

$$f_k(p^\alpha) = (\alpha + 1)^{1/k} - \alpha^{1/k}, \qquad f_k(1) = 1.$$

Then for all positive integers n, we have

$$\sum_{d|n} f_k(d) \log d \leqslant \frac{\log n}{k+1} \sum_{d|n} f_k(d).$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, and set

$$g(s) = \prod_{i=1}^{r} \left(1 + \frac{f_k(p_i)}{p_i^{s}} + \cdots + \frac{f_k(p_i^{\alpha})}{p_i^{\alpha s}}\right) = \frac{f_k(d)}{d^s}.$$

We have to show that

$$-\frac{g'(0)}{g(0)} \leqslant \frac{\log n}{k+1}.$$

But the left-hand side is

$$\sum_{p^\alpha \| n} \frac{(2^{1/k} - 1) + 2(3^{1/k} - 2^{1/k}) + \cdots + \alpha((\alpha+1)^{1/k} - \alpha^{1/k})}{(\alpha+1)^{1/k}} \log p$$
$$= \sum_{p^\alpha \| n} \left(1 - \frac{1 + 2^{1/k} + 3^{1/k} + \cdots + \alpha^{1/k}}{\alpha(\alpha+1)^{1/k}}\right) \alpha \log p,$$
$$\leqslant \frac{1}{k+1} \sum_{p^\alpha \| n} \alpha \log p = \frac{\log n}{k+1},$$

using the inequality proved in Lemma 1.

LEMMA 3. *For all positive integers k and n, we have*

$$\{\tau(n)\}^{1/k} \leqslant (k+1) \sum_{d|n,\, d<n^{1/k}} f_k(d).$$

Proof. We have

$$\{\tau(n)\}^{1/k} = \sum_{d|n} f_k(d).$$

But

$$\sum_{d|n} \{f_k(d): d \geqslant n^{1/k}\} \leqslant k \sum_{d|n} f_k(d) \frac{\log d}{\log n}$$
$$\leqslant \frac{k}{k+1} \sum_{d|n} f_k(d)$$

by Lemma 2. The result follows.

LEMMA 4. *For each k, there exist a $C_0(k)$ such that for all x,*

$$\sum_{n<x} \{\tau(n)\,\tau(n+1)\cdots\tau(n+k-1)\}^{1/k} \leqslant C_0(k)\, x(\log x)^{\alpha_k}.$$

Proof. Put $y^k = x + k$. By Lemma 3, we have

$$\{\tau(n+j)\}^{1/k} \leqslant (k+1) \sum_{d\mid(n+j),\, d<y} f_k(d).$$

Hence the sum above does not exceed

$$(k+1)^k \sum_{d_0<y} \cdots \sum_{d_{k-1}<y} f_k(d_0) \cdots f_k(d_{k-1})\, \mathrm{card}\{n < x\colon d_j \mid (n+j)\ \forall j\}$$
$$\leqslant (k+1)^k \sum_{d_0<y} \cdots \sum_{d_{k-1}<y} \frac{x f_k(d_0) \cdots f_k(d_{k-1})}{[d_0, d_1, d_2, \ldots, d_{k-1}]}.$$

We have

$$d_0 d_1 d_2 \cdots d_{k-1} \leqslant [d_0, d_1, \ldots, d_{k-1}] \prod_{i<j} (d_i, d_j)$$

and we note that if the congruences $n + j \equiv 0 \pmod{d_j}$ have a solution, then $(d_i, d_j) \mid (j - i)$ for every $i < j$. If we write

$$C_1(k) = \prod_{0\leqslant i<j<k} (j-i),$$

then the sum above does not exceed

$$C_1(k)(k+1)^k\, x \left(\sum_{d<y} \frac{f_k(d)}{d}\right)^k$$
$$\leqslant C_1(k)(k+1)^k\, x \prod_{p<y} \left(1 + \frac{f_k(p)}{p} + \frac{f_k(p^2)}{p^2} + \cdots\right)^k$$
$$\leqslant C_1(k)(k+1)^k\, x \exp\left(k(2^{1/k}-1) \sum_{p<y} \frac{1}{p-1}\right)$$
$$\leqslant C_2(k)\, x(\log y)^{\alpha_k}.$$

We may assume that $x > k$, as otherwise our result is trivial. Thus $y^k < 2x$, and the result follows.

LEMMA 5. *For each integer k and all x, we have*

$$\sum_{n<x} \{\tau(n)\,\tau(n+k)\}^{1/2} \leqslant \frac{\sigma(k)}{k}\, x(\log x)^{\alpha_2}.$$

This is proved in a similar manner to Lemma 4.

LEMMA 6. *For any real numbers $x_j \geqslant 0$ $(0 \leqslant j < k)$ we have*

$$\max_j x_j \geqslant \sum_j x_j - \sum_{i<j} (x_i x_j)^{1/2}.$$

Proof. Let x_0 be the maximum. Plainly

$$\sum_{j=1}^{k-1} x_j \leqslant \sum_{0<j} (x_0 x_j)^{1/2}.$$

LEMMA 7. *For positive integers k, t, and for all positive x,*

$$\sum_{n<x} \max_{0\leqslant j<k} \{\omega^t(n+j)\} \ll k(t!)(x+k)(\operatorname{loglog}(x+k))^t.$$

Proof. For each fixed $y_0 < 2$, we have

$$\sum_{n<x} y^{\omega(n)} \leqslant C(y_0)\, x(\log x)^{y-1},$$

for $0 \leqslant y \leqslant y_0$. Put $y_0 = 3/2$, and for sufficiently large x, $\log y = 1/\operatorname{loglog} x$. Then

$$\sum_{n<x} \frac{(\omega(n))^t}{t!\,(\operatorname{loglog} x)^t} \leqslant \sum_{n<x} y^{\omega(n)} \ll x.$$

Hence

$$\sum_{0\leqslant j<k} \sum_{n<x} (\omega(n+j))^t \ll k(t!)(x+k)(\operatorname{loglog}(x+k))^t$$

and the result follows.

LEMMA 8. *Let $\tau_k(n)$ denote the number of divisors of n which have no prime factor exceeding k. Then*

$$\sum_{n<x} \prod_{j=0}^{k-1} \{\tau_k(n+j)\}^t \ll (x+k)(tk)^{tk^2}.$$

Proof. Write $n = qm$, where the prime factors of q and m are, respectively, $\leqslant k$, and $>k$. Then

$$\sum_{n<x} (\tau_k(n))^t \leqslant \sum_{q<x} (\tau_k(q))^t \sum_{m<x/q} 1 \leqslant x \sum_q \frac{(\tau_k(q))^t}{q}$$

$$\leqslant x \prod_{p\leqslant k} \left(1 + \frac{2^t}{p} + \frac{3^t}{p^2} + \cdots\right).$$

But

$$\sum_{\alpha=1}^{\infty} \frac{(\alpha+1)^t}{p^\alpha} \leqslant \int_0^\infty \frac{(u+2)^t}{p^u}\,du = \sum_{r=0}^{t} \binom{t}{r} 2^{t-r} \frac{r!}{(\log p)^{r+1}}$$

$$\leqslant 2^{t+1} \sum_{r=0}^{t} \frac{t!}{(t-r)!} \ll 2^t t! \ll t^t,$$

using the fact that $\log p > 1/2$. So we have

$$\sum_{n<x} (\tau_k(n))^t \ll x t^{tk}.$$

The result now follows from Hölder's inequality.

Proof of Theorem 1. We have

$$\sum_{n<x} \max\{\tau(n), \tau(n+1), \dots, \tau(n+k-1)\}$$

$$\leqslant \sum_{j=0}^{k-1} \sum_{n<x} \tau(n+j)$$

$$\leqslant k\{x \log x + (2\gamma-1)x\} + O(k^2 \log x + kx^{1/2}).$$

Next, we apply Lemma 6, with $x_j = \tau(n+j)$. We have to estimate, from above,

$$\sum_{i<j} \sum_{n<x} \{\tau(n+i)\,\tau(n+j)\}^{1/2}$$

and, by Lemma 5, this is

$$\ll \sum_{i<j} \frac{\sigma(j-i)}{j-i} x(\log x)^{\alpha_2} \ll k^2 x(\log x)^{\alpha_2}.$$

We therefore have

$$\sum_{n<x} \max\{\tau(n), \tau(n+1), \dots, \tau(n+k-1)\}$$

$$= kx \log x + O(k^2 x(\log x)^{\alpha_2}) \sim kx \log x$$

provided

$$k = o((\log x)^{3-2(2)^{1/2}}).$$

This is the result stated.

Proof of Theorem 2. The upper bound is an immediate deduction from Lemma 4, since

$$\min\{\tau(n), \tau(n+1), \dots, \tau(n+k-1)\} \leqslant \prod_{0 \leqslant j < k} \{\tau(n+j)\}^{1/k}.$$

It remains to prove the lower bound. Let us define

$$T_k(x, v) = \operatorname{card}\{n < x \colon \min_{0 \leqslant j < k} \tau(n+j) \geqslant 2^v\}.$$

Then for each v,

$$\sum_{n<x} \min_j \{\tau(n+j)\} \geqslant 2^v T_k(x, v).$$

Let $M < x^{1/3}$ be squarefree, $\nu(M) = kv$, and suppose $m_0 m_1 \cdots m_{k-1} = M$, $\nu(m_j) = v$ for all j. There exists N, $0 < N \leqslant M$, such that $N \equiv -j \pmod{m_j}$ for each j, and we put $N + j = m_j a_j$. For $l + 1 < x/M$ put

$$q_j = q_j(l) = (M/m_j)l + a_j$$

so that $q_j m_j = Ml + N + j$, for each j. Plainly $n = Ml + N$ is counted by $T_k(x, v)$.

Let $\omega_k(n)$ denote the total number of prime factors of n which exceed k. We restrict l so that

$$\omega_k \left\{ \prod_{0 \leqslant j < k} q_j(l) \right\} \leqslant r;$$

indeed, we denote by $S_r(x; m_0, m_1, \dots, m_{k-1})$ the number of l, $1 \leqslant l < (x/M) - 1$ for which this inequality is satisfied. We have

$$\sum_M \sum_{(m_j)} S_r(x; m_0, m_1, \dots, m_{k-1}) \leqslant {\sum_{n<x}}' R(n)$$

where $\sum'$ is restricted to numbers n contributing to $T_k(x, v)$ and $R(n)$ denotes the number of times n is repeated in our construction. Let us write

$$n + j = q_j m_j = q_j^- q_j^+ m_j,$$

where the prime factors of q_j^-, q_j^+ are, respectively, $\leqslant$, $>k$; moreover, $\omega(q_j^+) = s$. The number of ways of writing $n + j$ in this way is

$$\leqslant \tau_k(n+j) \binom{\omega_k(n+j)}{s}$$

and so

$$R(n) \leqslant \prod_{j=0}^{k-1} \tau_k(n+j) \sum_{s_0+s_1+\cdots+s_{k-1}\leqslant r} \prod_{j=0}^{k-1} \binom{\omega_k(n+j)}{s_j}$$

$$\leqslant \left(\prod_{j=0}^{k-1} \tau_k(n+j)\right) \max_{0\leqslant j<k} \{\omega(n+j)\}^r.$$

Moreover, for any $t > 1$ we have

$$\sum_M \sum_{(m_j)} S_r(x; m_0, m_1, ..., m_{k-1}) \leqslant (T_k(x, v))^{1-1/t} \left(\sum_{n<x} R^t(n)\right)^{1/t}.$$

By Lemmas 7 and 8, and the Schwarz inequality, we have

$$\left(\sum_{n<x} R^t(n)\right)^{1/t} \leqslant \left(\sum_{n<x} \prod_{j=0}^{k-1} \{\tau_k(n+j)\}^{2t}\right)^{1/2t} \left(\sum_{n<x} \max_j \{\omega(n+j)\}^{2rt}\right)^{1/2t}$$

$$\ll x^{1/t}(2tk)^{k^2} k^{1/2t}(2rt \operatorname{loglog} x)^r.$$

We set $t = [\operatorname{loglog} x]$. For this t, we have

$$\sum_M \sum_{(m_j)} S_r(x; m_0, m_1, ..., m_{k-1})$$

$$\ll x^{1/t}(T_k(x, v))^{1-1/t}(2k)^{k^2}(2r)^r(\operatorname{loglog} x)^{k^2+2r}.$$

We require a lower bound for $S_r(x)$, and we employ the Selberg sieve, in the lower bound form given by Ankeny and Onishi [1], and set out in Halberstam and Richert [2], Chapter 7. We do not attempt to give the best result which could be obtained from a weighted sieve procedure, since this would not affect our final result.

LEMMA 9. *In the above notation, we have*

$$S_r(x; m_0, m_1, m_2, ..., m_{k-1}) \geqslant C_4(k)(x/M)(\log x)^{-k},$$

where $C_4(k) > 0$ depends only on k, provided only

$$|\mu(M)| = 1,\ M = x^{1/3}, \qquad \nu(M) = kv,\ v = 0(\operatorname{loglog} x), \qquad r = 5k^2.$$

Proof. Set

$$f(l) = \prod_{j=0}^{k-1} (M_j l + a_j),$$

$$\mathbf{A} = \{f(l): 1 \leqslant l \leqslant X\},$$

$$\mathbf{B} = \{p: k < p\},$$

$$P = P(k, z) = \prod (p: k < p < z).$$

We seek a lower bound for

$$S(\mathbf{A}, \mathbf{B}, z) = \text{card}\{l: 1 = l = X, (f(l), P) = 1\}.$$

We follow the notation of Halberstam and Richert [2]. Let $\rho(p)$ denote the number of solutions of the congruence $f(l) \equiv 0 \pmod{p}$. Now by definition, $a_j m_j - a_i m_i = j - i$, and so we have

$$(M_j l + a_j, M_i l + a_i) \mid (j - i)$$

and

$$(m_j, a_i) | (j - i).$$

Thus

$$(M_i, a_i, P) = 1.$$

It follows that the solutions of the congruences $M_j l + a_j \equiv 0 \pmod{p}$ are distinct, for $p > k$, and that each congruence has precisely 0 or 1 solutions according as $p \mid M_j$ or not. Thus

$$\rho(p) \leqslant k < p, \qquad \frac{\rho(p)}{p} \leqslant 1 - \frac{1}{k+1},$$

and Halberstam and Richert's condition Ω_1 is satisfied, with $A_1 = k + 1$. Since M is squarefree, $p \mid M_j$ for at most one j, and so for $p > k$, we have $\rho(p) = k - 1$ or k according as $p \mid M$ or not. When $p = k$, we just have $0 \leqslant \rho(p) \leqslant p$. Thus for $2 \leqslant \omega < y$, we have (condition $\Omega_2(k, L)$):

$$k \log \frac{y}{\omega} - L \leqslant \sum_{\omega \leqslant p < y} \frac{\rho(p) \log p}{p} \leqslant k \log \frac{y}{\omega} + A_2,$$

where

$$A_2 = \sum_{p \leqslant k} \log p + O(1) = O(k),$$

$$L = \sum_{p \mid M, P > k} \frac{\log p}{p} + \sum_{p \leqslant k} \frac{k \log p}{p} = O((v + k) \log k),.$$

as $\nu(M) = vk$. Next, let d be a squarefree number all of whose prime factors exceed k. (We can write this in the form $(d, \bar{B}) = 1$.) Set

$$R_d = \text{card}\{l: 1 \leqslant l \leqslant X, f(l) \equiv 0 \pmod d\} - X \prod_{p|d} \frac{\rho(p)}{p}.$$

Then

$$| R_d | \leqslant \prod_{p|d} \rho(p) \leqslant k^{\nu(d)},$$

and

$$\sum \{| \mu(d)| \, 3^{\nu(d)} \, | R_d |: d \leqslant y, (d, \bar{\mathbf{B}}) = 1\}$$
$$\leqslant \sum_{d \leqslant y} (3k)^{\nu(d)} \ll y(\log y)^{3k-1}.$$

Hence Halberstam and Richert's condition $R(k, \alpha)$ is satisfied (cf. [2, p. 219]), with $\alpha = 1$, $A_4 = 4k$, $A_5 = 0(1)$. We may therefore apply their Theorem 7.4, and we have (note the misprint!):

$$S(\mathbf{A}, \mathbf{B}, z) \geqslant X \prod_{k<p<z} \left(1 - \frac{\rho(p)}{p}\right)\left\{1 - \eta_k\left(\frac{\log X}{\log z}\right) - BL\, \frac{(\text{loglog } X)^{3k+2}}{\log X}\right\},$$

where $B = B(A_1, A_2, A_4, A_5) = B(k)$, provided

$$z^2 = X(\log X)^{-4k}.$$

Here η_k is related to the function G_k of Ankeny and Onishi [1]: it is strictly decreasing, and $1 - \eta_k(u) > 0$ for $u > \nu_k$. It is known that $\nu_k < 3k$ for positive integers k [2, p. 221]. Let assume $v = 0(\text{loglog } X)$, and put $X = z^{3k}$. Then we have

$$S(\mathbf{A}, \mathbf{B}, z) \geqslant C_3(k)X(\log X)^{-k} \qquad (X > X_0(k)),$$

where $C_3(k) > 0$, and depends on k only. Moreover, the prime factors of $f(l)$, for l counted by $S(\mathbf{A}, \mathbf{B}, z)$, are either $\leqslant k$ or $\geqslant z$, and we have

$$Mf(l) = \prod_{j=0}^{k-1} (Ml + N + j) \leqslant (M(X + 1) + k)^k \leqslant M^k(X + 2)^k,$$

provided $M \geqslant k$. In fact this is automatic, as M has kv distinct prime factors. It follows that

$$\omega_k(f(l)) \leqslant \frac{k \log(M(X + 2))}{\log z} \leqslant 3k^2 \frac{\log(M(X + 2))}{\log X}.$$

In the application to $S_r(x; m_0, m_1, \ldots, m_{k-1})$, we set $X = (x/M)-1 > x^{2/3}-1$, and so $M < x^{1/3} < (1+X)^{1/2}$, and

$$3\,\frac{\log(M(X+2))}{\log X} \leqslant 5$$

for $X > X_1$. Provided k is fixed and $x \to \infty$, this condition, and the condition $X > X_0(k)$, are automatically satisfied. We therefore have $\omega_k(f(l)) = r$ as required.

We now return to the proof of our theorem. We have

$$C_5(k)\,\frac{x}{(\log x)^k}\sum_M \frac{1}{M}\sum_{(m_j)} 1 \ll x^{1/t}(T_k(x, v))^{1-1/t}(\operatorname{loglog} x)^K,$$

where $K = k^2 + 2r = 11k^2$, $t = [\operatorname{loglog} x]$. Given M, there are

$$(kv)!(v!)^{-k}(k!)^{-1}$$

different choices of $m_0, m_1, \ldots, m_{k-1}$; moreover we find that

$$\sum_M \frac{1}{M} \gg \frac{(\operatorname{loglog} x + O(1))^{kv}}{(kv)!}.$$

Thus

$$C_6(k)\,\frac{x}{(\log x)^k}\left(\frac{(\operatorname{loglog} x + O(1))^v}{v!}\right)^k \ll x^{1/t}(T_k(x, v))^{1-1/t}(\operatorname{loglog} x)^K.$$

We choose

$$v = [2^{1/k}\operatorname{loglog} x + 1]$$

and we have

$$C_6(k)\,\frac{x^{1-1/t}}{(\log x)^k}\cdot\frac{e^{vk}}{2^v} \ll (T_k(x, v))^{1-1/t}(\operatorname{loglog} x)^K.$$

Since $t = [\operatorname{loglog} x]$, this gives

$$C_7(k)\,\frac{xe^{vk}}{(\log x)^k} \ll 2^v T_k(x, v)(\operatorname{loglog} x)^K$$

and so for this v,

$$2^v T_k(x, v) \gg C_7(k)\, x(\log x)^{\alpha_k}(\operatorname{loglog} x)^{-11k^2}.$$

This gives the result stated.

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2. H. HALBERSTAM AND H.-E. RICHERT, "Sieve Methods," Academic Press, New York, 1974.