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SOME ASYMPTOTIC FORMULAS ON GENERALIZED DIVISOR FUNCTIONS, IV

by

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1. Throughout this paper, we use the following notation: $c_1, c_2, \ldots, X_0, X_1, \ldots$ denote positive absolute constants. We denote the number of elements of the finite set S by $|S|$. We write $e^x = \exp(x)$. We denote the least prime factor of n by $p(n)$. We write $p^\alpha \| n$ if $p^\alpha | n$ but $p^{\alpha+1} \nmid n$. $\nu(n)$ denotes the number of the distinct prime factors of n, while the number of all the prime factors of n is denoted by $\omega(n)$ so that

$$\nu(n) = \sum_{p|n} 1 \quad \text{and} \quad \omega(n) = \sum_{p^\alpha \| n} \alpha.$$

We write

$$\nu(n, x, y) = \sum_{\substack{p|n \\ x < p \leqq y}} 1, \quad \omega(n, x, y) = \sum_{\substack{p^\alpha \| n \\ x < p \leqq y}} \alpha,$$

$$\nu^+(n, x) = \sum_{\substack{p|n \\ p > x}} 1 \quad \text{and} \quad \omega^+(n, x) = \sum_{\substack{p^\alpha \| n \\ p > x}} \alpha$$

(so that $\nu^+(n, 1) = \nu(n, 1, n) = \nu(n)$, $\omega^+(n, 1) = \omega(n, 1, n) = \omega(n)$, $\nu(n, x, y) = \nu^+(n, x) - \nu^+(n, y)$ and $\omega(n, x, y) = \omega^+(n, x) - \omega^+(n, y)$). The divisor function is denoted by $d(n)$:

$$d(n) = \sum_{d|n} 1.$$

Let A be a finite or infinite sequence of positive integers $a_1 < a_2 < \ldots$. Then we write

$$N_A(x) = \sum_{\substack{a \in A \\ a \leqq x}} 1,$$

$$f_A(x) = \sum_{\substack{a \in A \\ a \leqq x}} \frac{1}{a},$$

$$d_A(n) = \sum_{\substack{a \in A \\ a|n}} 1$$

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(in other words, $d_A(n)$ denotes the number of divisors amongst the a_i's) and

$$D_A(x) = \max_{1 \leq n \leq x} d_A(x).$$

The aim of this series is to investigate the function $D_A(x)$. (See [1], [2] and [3]; see also HALL [5].) Clearly,

$$\sum_{1 \leq n \leq x} d_A(n) = x f_A(x) + O(x)$$

so that we have $D_A(x)/f_A(x) \gg 1$.

In Part I of this paper we proved that for an infinite sequence A, we have

$$\lim_{x \to +\infty} \sup D_A(x)/f_A(x) = +\infty$$

and we proved some other related results.

In Part II, we sharpened this theorem. In fact, we proved that

$$\lim_{x \to +\infty} f_A(x) = +\infty$$

implies that

(1) $$\lim_{x \to +\infty} \sup D_A(x)/\exp\left(c_1(\log f_A(x))^2\right) = +\infty.$$

The proof was based on the fact that

$$f_A(x) > \exp\left((\log \log x)^{1/2}\right)$$

implies that writing $y = \exp\left((\log x)^2\right)$, we have

$$D_A(y) > \exp\left(c_2(\log f_A(x))^2\right).$$

In Part III, we estimated $D_A(y)$ in terms of $f_A(x)$ for $y = x$; in fact, we proved

THEOREM 1. *For all* $\Omega > 0$ *and for* $x > X_0(\Omega)$,

$$f_A(x) > (\log \log x)^{20}$$

implies that

$$D_A(x) > \Omega f_A(x).$$

(In both Parts II and III, we proved also some other related results.)

In this paper, *our aim is to seek for a possibly small function* $y = y(x)$ *such that* $f_A(x) \to +\infty$ *implies that* $D_A(y(x))/f_A(x) \to +\infty$. In fact, we prove

THEOREM 2. *For all* $\Omega > 1$, *there exist constants* $c_3 = c_3(\Omega)$, $X_1 = X_1(\Omega)$ *such that* $x > X_1$,

(2) $$f_A(x) > c_3$$

and

(3) $$[x^{1-1/(f_A(x))^{1/3}}, x] \cap A = \emptyset$$

imply that

(4) $$D_A(x) > \Omega f_A(x).$$

COROLLARY 1. *For all* $\Omega > 0$, *there exist constants* $c_4 = c_4(\Omega)$, $X_2 = X_2(\Omega)$ *such that* $x > X_2$ *and*

$$f_A(x) > c_4$$

imply that writing $y=x^{1+1/(f_A(x))^{1/4}}$, *we have*

(5) $$D_A(y)>\Omega f_A(x).$$

COROLLARY 2. *For all* $\Omega>1$, *there exist constants* $c_5=c_5(\Omega)$, $X_3=X_3(\Omega)$ *such that* $x>X_3$,

(6) $$f_A(x)>c_5$$

and

(7) $$D_A(x)\leqq\Omega f_A(x)$$

imply

$$N_A(x)>x^{1-1/(f_A(x))^{1/3}}.$$

Section 2 is devoted to the proof of Theorem 2, while in Section 3, we deduce Corollaries 1 and 2 from Theorem 2.

On the other hand, we show that Theorem 2 is not true if we replace the exponent $1-1/(f_A(x))^{1/3}$ on the left-hand side of (3) by $1-1/c_6^{f_A(x)}$:

THEOREM 3. *There exist absolute constants* $c_6, c_7, c_8, c_9, c_{10}, c_{11}$ *and* X_4 *such that for*

(8) $$x>X_4$$

and

(9) $$c_6<t<c_7\log\log x,$$

there exists a sequence A *satisfying*

(10) $$c_8 t<f_A(x)<c_9 t,$$

(11) $$\left(x^{1-1/c_{10}^{f_A(x)}},x\right]\cap A=\emptyset$$

and

(12) $$D_A(x)<c_{11}f_A(x).$$

We prove this theorem in Section 4.

Finally, in Section 5, we discuss some other related problems.

2. Proof of Theorem 2. If x is sufficiently large and we have

$$f_A(x)>(\log\log x)^{20}$$

then (4) holds by Theorem 1. Thus we may assume that

(13) $$f_A(x)\leqq(\log\log x)^{20}.$$

Let us put

(14) $$y=x^{1/(f_A(x))^{1/3}}$$

and write A in the form

(15) $$A=A_1\cup A_2$$

where A_1 consists of the integers a such that $a\in A$ and there exists an integer u satisfying

(16) $$(\log x)^3<u<y^{1/2\Omega f_A(x)}$$

and $u|a$, while A_2 consists of the integers a such that $a\in A$ and $u\nmid a$ for all u satisfying (16). We have to distinguish two cases.

Case 1. Assume first that

$$f_{A_1}(x) = \sum_{a\in A_1} \frac{1}{a} \geqq \frac{1}{2} f_A(x). \tag{17}$$

For $a\in A_1$, write a in the form

$$a = u(a)b(a)$$

where $u(a)$ denotes the least integer u such that u satisfies (16) and $u|a$. Then by (3), for $a\in A_1$ we have $b(a)\leqq a\leqq x|y$ and $(\log x)^3<d(a)$ so that

$$\begin{aligned} f_{A_1}(x) &= \sum_{a\in A_1} \frac{1}{a} = \sum_{a\in A_1} \frac{1}{u(a)b(a)} = \\ &= \sum_{b\leqq x|y} \frac{1}{b} \sum_{\substack{a\in A_1 \\ b(a)=b}} \frac{1}{u(a)} < \sum_{b\leqq x|y} \frac{1}{b} \sum_{\substack{a\in A_1 \\ b(a)=b}} \frac{1}{(\log x)^3} = \\ &= \frac{1}{(\log x)^3} \sum_{b\leqq x|y} \frac{1}{b} \sum_{\substack{a=A_1 \\ b(a)=b}} 1 \leqq \frac{1}{(\log x)^3} \left(\max_{b\leqq x|y} \sum_{\substack{a\in A_1 \\ b(a)=b}} 1 \right) \sum_{b\leqq x} \frac{1}{b} < \\ &\leqq \frac{1}{(\log x)^3} \left(\max_{b\leqq x|y} \sum_{\substack{a\in A_1 \\ b(a)=b}} 1 \right) 2\log x = \frac{2}{(\log x)^2} \left(\max_{b\leqq x|y} \sum_{\substack{a\in A_1 \\ b(a)=b}} 1 \right). \end{aligned} \tag{18}$$

If x and c_3 (in (2)) are sufficiently large in terms of Ω then (2), (17) and (18) yield that

$$\begin{aligned} &\max_{b\leqq x|y} \sum_{\substack{a\in A_1 \\ b(a)=b}} 1 > \frac{(\log x)^2}{2} f_{A_1}(x) \geqq \\ &\geqq \frac{(\log x)^2}{2} \frac{1}{2} f_A(x) > (\log x)^2 > \Omega \sum_{n\leqq x} \frac{1}{n} + 1 \geqq \Omega f_A(x) + 1 \end{aligned}$$

so that there exists an integer b_0 for which

$$1 \leqq b_0 \leqq x/y \tag{19}$$

and

$$\sum_{\substack{a\in A_1 \\ b(a)=b_0}} 1 > \Omega f_A(x) + 1. \tag{20}$$

Put $s=[\Omega f_A(x)]+1$. Then by (20), there exist integers $a_1, a_2, \ldots, a_s$ such that $a\in A$ and a_i can be written in the form

$$a_i = b_0 u(a_i) = b_0 u_i$$

where (with respect to (16))

$$((\log x)^3 <) u_i < y^{1/2\Omega f_A(x)}. \tag{21}$$

Let

$$m = b_0 u_1 u_2 \ldots u_s.$$

Then by (2), (19) and (21), for sufficiently large c_3 we have

$$m = b_0 u_1 u_2 \ldots u_s \leqq \frac{x}{y} (y^{1/2\Omega f_A(x)})^s < \frac{x}{y} (y^{1/2\Omega f_A(x)})^{2\Omega f_A(x)} = x, \tag{22}$$

and, obviously, $a_i = b_0 u_i/m$ and $a_i = b_0 u_i \in A$ so that

$$d_A(m) \geqq s = [\Omega f_A(x)] + 1 > \Omega f_A(x). \tag{23}$$

(22) and (23) yield (4) and this completes the proof of Theorem 2 in this case.

Case 2. Assume now that

$$f_{A_1}(x) = \sum_{a \in A_1} \frac{1}{a} < \frac{1}{2} f_A(x). \tag{24}$$

Then (2), (15) and (24) yield that

$$f_{A_2}(x) = \sum_{a \in A_2} \frac{1}{a} \geqq \sum_{a \in A} \frac{1}{a} - \sum_{a \in A_1} \frac{1}{a} > f_A(x) - \frac{1}{2} f_A(x) = \frac{1}{2} f_A(x) \left(> \frac{c_3}{2} \right). \tag{25}$$

Let us write all $a \in A_2$ in the form

$$a = e(a)v(a) \quad \text{where} \quad e(a) \leqq (\log x)^3 \quad \text{and} \quad p(v(a)) \geqq y^{1/2\Omega f_A(x)}.$$

(Note that if x is sufficiently large in terms of Ω then by (13) we have

$$y^{1/2\Omega f_A(x)} = x^{1/2\Omega (f(x))^{4/3}} > (\log x)^3.)$$

Again, we have to distinguish two cases.

Case 2.1. Let

$$\max_{v \leqq x|y} \sum_{\substack{a \in A_2 \\ v(a) = v}} \frac{1}{e(a)} > 2 \log f_A(x). \tag{26}$$

Note that if $v(a) = v$ for some $a \in A_2$ then by (3) we have

$$v \leqq a \leqq x|y. \tag{27}$$

We are going to show that (26) implies

$$\max_{v \leqq x|y} \sum_{\substack{a \in A_2 \\ v(a) = v}} 1 > \Omega f_A(x). \tag{28}$$

In fact,

$$\sum_{\substack{a \in A_2 \\ v(a) = v}} 1 \leqq \Omega f_A(x)$$

implies by (2) that if c_3 is sufficiently large in terms of Ω then we have

$$\sum_{\substack{a \in A_2 \\ v(a) = v}} \frac{1}{e(a)} \leqq \sum_{1 \leqq e \leqq \Omega f_A(x)} \frac{1}{e} < \frac{3}{2} \log \Omega f_A(x) < 2 \log f_A(x).$$

By (26), this cannot hold for all v, which proves (28).

But (28) yields that there exists an integer v_0 such that

$$\sum_{\substack{a \in A_2 \\ v(a) = v_0}} 1 > \Omega f_A(x).$$

This implies that writing $s=[\Omega f_A(x)]+1$, there exist integers $e_1<e_2<\ldots<e_s$ such that $v_0e_i\in A_2$ and

(29) $$e_i \leqq (\log x)^3$$

for $i=1, 2, \ldots, s$. Write

$$h = v_0e_1e_2\ldots e_s.$$

Then by (13), (27) and (29),

(30) $$h = v_0e_1e_2\ldots e_s \leqq \frac{x}{y}\left((\log x)^3\right)^s \leqq$$

$$\leqq \frac{x}{x^{1/(f_A(x))^{1/10}}}\exp\left(4\Omega f_A(x)\log\log x\right) < \frac{x}{x^{1/(\log\log x)^2}}\exp\left((\log\log x)^{22}\right) < x$$

and v_0e_i/A and $v_0e_i\in A$ for $i=1, 2, \ldots, s$ so that

(31) $$d_A(h) \geqq s = \left[\Omega f_A(x)\right]+1 > \Omega f_A(x).$$

(30) and (31) yield (4) and this completes the proof of Theorem 2 in this case.

Case 2.2. Let

(32) $$\sum_{\substack{a\in A_2\\ v(a)=v}}\frac{1}{e(a)} \leqq 2\log f_A(x) \quad \text{for all} \quad v \leqq x/y.$$

Let us write A_2 in the form

$$A_2 = A_3\cup A_4$$

where $a\in A_3$ if and only if $a\in A_2$ and

$$\nu^+(a,\, y^{1/2\Omega f_A(x)}) = \nu^+\left(v(a),\, y^{1/2\Omega f_A(x)}\right) > \frac{2}{5}\log f_A(x)$$

and

$$A_4 = A_2 - A_3.$$

Then by (25) and (32) we have

(33) $$\begin{aligned} f_{A_3}(x) &= f_{A_2}(x) - f_{A_4}(x) > \\ &> \frac{1}{2}f_A(x) - \sum_{\substack{a\in A_2\\ \nu^+(v(a),\, y^{1/2\Omega f_A(x)}) \leqq \frac{2}{5}\log f_A(x)}}\frac{1}{a} = \\ &= \frac{1}{2}f_A(x) - \sum_{\substack{v\leqq x/y\\ p(v)>y^{1/2\Omega f_A(x)}\\ \nu^+(v,\, y^{1/2\Omega f_A(x)})\leqq\frac{2}{5}\log f_A(x)}}\ \sum_{\substack{a\in A_2\\ v(a)=v}}\frac{1}{v(a)e(a)} = \\ &= \frac{1}{2}f_A(x) - \sum_{\substack{v\leqq x/y\\ p(v)>y^{1/2\Omega f_A(x)}\\ \nu^+(v,\, y^{1/2\Omega f_A(x)})\leqq\frac{2}{5}\log f_A(x)}}\frac{1}{v}\sum_{\substack{a\in A_2\\ v(a)=v}}\frac{1}{e(a)} \geqq \\ &\geqq \frac{1}{2}f_A(x) - 2\log f_A(x)\sum_{\substack{v\leqq x\\ p(v)>y^{1/2\Omega f_A(x)}\\ \nu^+(v,\, y^{1/2\Omega f_A(x)})\leqq\frac{2}{5}\log f_A(x)}}\frac{1}{v}. \end{aligned}$$

By using the Stirling-formula and the well-known formula

$$\sum_{p\le u}\sum_{\alpha=1}^{+\infty}\frac{1}{p^\alpha}=\log\log u+c_{12}+o(1)$$

and with respect to (2), for sufficiently large c_3 we obtain that

$$(34)\qquad \sum_{\substack{v\le x\\ p(v)>y^{1/2\Omega f_A(x)}\\ \nu^+(v,y^{1/2\Omega f_A(x)})\le\frac{2}{5}\log f_A(x)}}\frac{1}{v}\le$$

$$\le 1+\sum_{j=1}^{\left[\frac{2}{5}\log f_A(x)\right]}\ \sum_{y^{1/2\Omega f_A(x)}<p_{i_1}<\dots<p_{i_j}\le x}\ \sum_{\alpha_{i_1}=1}^{+\infty}\dots\sum_{\alpha_{i_j}=1}^{+\infty}\frac{1}{p_{i_1}^{\alpha_{i_1}}\dots p_{i_j}^{\alpha_{i_j}}}\le$$

$$\le 1+\sum_{j=1}^{\left[\frac{2}{5}\log f_A(x)\right]}\frac{1}{j!}\left(\sum_{y^{1/2\Omega f_A(x)}<p\le x}\frac{1}{p^\alpha}\right)^j\le 1+\sum_{j=1}^{\left[\frac{2}{5}\log f_A(x)\right]}\frac{1}{j!}\left(\log\frac{\log x}{\log y^{1/2\Omega f_A(x)}}\right)^j=$$

$$=1+\sum_{j=1}^{\left[\frac{2}{5}\log f_A(x)\right]}\frac{1}{j!}\left(\log 2\Omega (f_A(x))^{4/3}+c_{13}\right)^j<1+\sum_{j=1}^{\left[\frac{2}{5}\log f_A(x)\right]}\frac{1}{j!}\left(\frac{134}{100}\log f_A(x)\right)^j<$$

$$<\log f_A(x)\frac{1}{\left[\frac{2}{5}\log f_A(x)\right]!}\left(\frac{134}{100}\log f_A(x)\right)^{\left[\frac{2}{5}\log f_A(x)\right]}<$$

$$<c_{14}(\log f_A(x))^{1/2}\left(\frac{\frac{134}{100}e\log f_A(x)}{\left[\frac{2}{5}\log f_A(x)\right]}\right)^{\left[\frac{2}{5}\log f_A(x)\right]}<c_{14}(\log f_A(x))^{1/2}\left(\frac{\frac{14}{10}e}{\frac{2}{5}}\right)^{\frac{2}{5}\log f_A(x)}=$$

$$=c_{14}(\log f_A(x))^{1/2}(f_A(x))^{\frac{2}{5}\log\frac{7}{2}e}<c_{14}(\log f_A(x))^{1/2}(f_A(x))^{91/100}<(f_A(x))^{92/100}.$$

By (2), (33) and (34) yield for sufficiently large c_3 that

$$(35)\quad f_{A_3}(x)>\frac{1}{2}f_A(x)-2(\log f_A(x))(f_A(x))^{92/100}>\frac{1}{2}f_A(x)-(f_A(x))^{93/100}>\frac{1}{4}f_A(x).$$

Let S denote the set of the integers n such that $n\le x$ and n can be written in the form

$$(36)\qquad n=au \text{ where } a\in A_3 \text{ and } \omega(u,y^{1/2\Omega f_A(x)},y)>\frac{99}{100}\log f_A(x).$$

For fixed $n\in S$, let $\varphi(n)$ denote the number of representations of n in the form (36).

Then we have

(37)
$$\sum_{n\leqq x}\varphi(n)=\sum_{a\in A_3}\ \sum_{\substack{au\leqq x\\ \omega(u,y^{1/2\Omega f_A(x)},y)>\frac{99}{100}\log f_A(x)}}1=\sum_{a\in A_3}\left(\sum_{u\leqq x/a}1-\sum_{\substack{u\leqq x/a\\ \omega(u,y^{1/2\Omega f_A(x)},y)\leqq\frac{99}{100}\log f_A(x)}}1\right).$$

In order to estimate the last sum, we need the following lemma:

LEMMA 1. *Let us write*

$$Q(u)=u-(1+u)\log(1+u). \tag{38}$$

Then for $1\leqq t$, $2t<z\leqq v$, $0\leqq\alpha\leqq 1$ *we have*

$$\sum_{\substack{n\leqq v\\ \omega(n,t,z)\leqq(1-\alpha)\sum_{t<p\leqq z}1/p}}1<c_{15}\,v\exp\left(Q(-\alpha)\log\frac{\log z}{\log t}\right).$$

This lemma is identical with Lemma 2 in [3]; in fact, it is a consequence of a result of K. K. NORTON (see [6]; see also HALÁSZ [4]).

By using Lemma 1 with $y^{1/2\Omega f_A(x)}$, y, x/a and $1/200$ in place of t, z, v and α, respectively (note that $1\leqq t$ and $2t<z\leqq v$ hold by (2), (3), (13) and (14)), we obtain for sufficiently large c_3 that

(39)
$$\sum_{\substack{u\leqq x/a\\ \omega(u,y^{1/2\Omega f_A(x)},y)\leqq\frac{199}{200}\sum_{y^{1/2\Omega f_A(x)}<p\leqq y}1/p}}1<c_{15}\frac{x}{a}\exp\left(Q\left(-\frac{1}{200}\right)\log\frac{\log y}{\log y^{1/2\Omega f_A(x)}}\right)<$$
$$<c_{15}\frac{x}{a}\exp\left(-10^{-5}\log 2\Omega f_A(x)\right)<\frac{1}{4}\frac{x}{a}.$$

Furthermore, by (2), and with respect to the well-known formula

$$\sum_{p\leqq u}\frac{1}{p}=\log\log u+c_{16}+o(1), \tag{40}$$

for sufficiently large c_3 (depending on Ω) we have

(41)
$$\frac{199}{200}\sum_{y^{1/2\Omega f_A(x)}<p\leqq y}\frac{1}{p}>\frac{199}{200}\left(\log\frac{\log y}{\log y^{1/2\Omega f_A(x)}}-c_{17}\right)=$$
$$=\frac{199}{200}\left(\log 2\Omega f_A(x)-c_{17}\right)>\frac{99}{100}\log f_A(x).$$

(39) and (41) yield that

(42)
$$\sum_{\substack{u\leqq x/a\\ \omega(u,y^{1/2\Omega f_A(x)},y)\leqq\frac{99}{100}\log f_A(x)}}1\leqq\sum_{\substack{u\leqq x/a\\ \omega(u,y^{1/2\Omega f_A(x)},y)\leqq\frac{199}{200}\sum_{y^{1/2\Omega f_A(x)}<p\leqq y}1/p}}1<\frac{1}{4}\frac{x}{a}.$$

We obtain from (35), (37) and (42) that

$$\sum_{n\leq x}\varphi(n)>\sum_{a\in A_3}\left(\sum_{u\leq x/a}1-\frac{1}{4}\frac{x}{a}\right)=\sum_{a\in A_3}\left(\left[\frac{x}{a}\right]-\frac{1}{4}\frac{x}{a}\right)>\sum_{a\in A_3}\left(\frac{1}{2}\frac{x}{a}-\frac{1}{4}\frac{x}{a}\right)=$$
(43)
$$=\frac{1}{4}xf_{A_3}(x)>\frac{1}{16}xf_A(x).$$

Now we are going to give an upper estimate for $\sum_{n\leq x}\varphi(n)$. Obviously, for $n\leq x$ we have

$$\varphi(n)\leq d_A(n)\leq D_A(x)$$

hence

(44) $$\sum_{n\leq x}\varphi(n)=\sum_{n\in S}\varphi(n)\leq\sum_{n\in S}D_A(x)=|S|D_A(x).$$

Thus in order to obtain an upper bound for $\sum_{n\leq x}\varphi(n)$, we have to estimate $|S|$.

If $n\in S$ then by (36) and the definition of the set A_3, we have

$$\omega(n,y^{1/2\Omega f_A(x)},x)=\omega(au,y^{1/2\Omega f_A(x)},x)=\omega(a,y^{1/2\Omega f_A(x)},x)+\omega(u,y^{1/2\Omega f_A(x)},x)\geq$$

$$\geq v(a,y^{1/2\Omega f_A(x)},x)+\omega(u,y^{1/2\Omega f_A(x)},y)=v^+(a,y^{1/2\Omega f_A(x)})+\omega(u,y^{1/2\Omega f_A(x)},y)>$$

$$>\frac{2}{5}\log f_A(x)+\frac{99}{100}\log f_A(x)=\frac{139}{100}\log f_A(x)$$

hence

(45) $$|S|\leq\sum_{\substack{n\leq x\\ \omega(n,y^{1/2\Omega f_A(x)},x)>\frac{139}{100}\log f_A(x)}}1.$$

In order to estimate this sum, we need the following

LEMMA 2. *For* $1\leq t$, $2t<z\leq v$, $0<\alpha\leq\beta<1$ *we have*

$$\sum_{\substack{n\leq v\\ \omega(n,t,z)\geq(1+\alpha)\sum_{t<p\leq z}1/p}}1<c_{18}(\beta)\alpha^{-1}v\left(\sum_{t<p\leq z}\frac{1}{p}\right)^{-1/2}\exp\left(Q(\alpha)\log\frac{\log z}{\log t}\right)$$

(*where* $Q(u)$ *is defined by* (37)).

This lemma is identical with Lemma 3 in [3]; in fact, it is a consequence of a result of K. K. NORTON (see [6]; see also HALÁSZ [4]).

By using Lemma 2 with $y^{1/2\Omega f_A(x)}$, x, x, $\frac{1}{30}$ and $\frac{1}{2}$ in place of t, z, v, α and β, respectively (note that $1\leq t$ and $2t<z\leq v$ hold by (2), (13) and (14)), and with

respect to (2), (14) and (40), we obtain for sufficiently large c_3 that

$$\sum_{\substack{n\leqq x\\ \omega(n,y^{1/2\Omega f_A(x)},x)\geqq \frac{31}{30}\sum_{y^{1/2\Omega f_A(x)}<p<x} 1/p}} 1 < c_{19}\, x \exp\left(Q\left(\frac{1}{30}\right)\log\frac{\log x}{\log y^{1/2\Omega f_A(x)}}\right) = \tag{46}$$

$$= c_{19}\, x \exp\left(Q\left(\frac{1}{30}\right)\log\frac{\log x}{\log x^{1/2\Omega(f_A(x))^{4/3}}}\right) = c_{19}\, x \exp\left(Q\left(\frac{1}{30}\right)\log 2\Omega(f_A(x))^{4/3}\right) <$$

$$< c_{19}\, x \exp\left(-5\cdot 10^{-4}\cdot\frac{4}{3}\log f_A(x)\right) < x\left(f_A(x)\right)^{-6\cdot 10^{-4}}.$$

Furthermore, by (2), (14) and (40), we obtain that if c_3 is sufficiently large (in terms of Ω) then

$$\frac{31}{30}\sum_{y^{1/2\Omega f_A(x)}<p\leqq x}\frac{1}{p} < \frac{31}{30}\left(\log\frac{\log x}{\log y^{1/2\Omega f_A(x)}}+c_{20}\right) =$$

$$= \frac{31}{30}\left(\log\frac{\log x}{\log x^{1/2\Omega(f_A(x))^{4/3}}}+c_{20}\right) = \frac{31}{30}\left(\log 2\Omega\left(f_A(x)\right)^{4/3}+c_{20}\right) < \tag{47}$$

$$< \frac{31}{30}\cdot\frac{134}{100}\log f_A(x) < \frac{139}{100}\log f_A(x).$$

We obtain from (45), (46) and (47) that

$$|S| \leqq \sum_{\substack{n\leqq x\\ \omega(n,y^{1/2\Omega f_A(x)},x)>\frac{139}{100}\log f_A(x)}} 1 \leqq \sum_{\substack{n\leqq x\\ \omega(n,y^{1/2\Omega f_A(x)},x)\geqq \frac{31}{30}\sum_{y^{1/2\Omega f_A(x)}<p\leqq x} 1/p}} 1 < \tag{48}$$

$$< x\left(f_A(x)\right)^{-6\cdot 10^{-4}}.$$

(43), (44) and (48) yield that

$$\frac{1}{16}xf_A(x) < \sum_{n\leqq x}\varphi(n) \leqq |S|D_A(x) < x\left(f_A(x)\right)^{-6\cdot 10^{-4}}D_A(x)$$

hence

$$D_A(x) > \frac{(f_A(x))^{6\cdot 10^{-4}}}{16}f_A(x).$$

If c_3 in (2) is sufficiently large in terms of Ω then this implies (4). Thus (4) holds also in Case 2.2 and this completes the proof of Theorem 2.

3. Proof of Corollary 1. By using Theorem 2 with $x^{1+1/(f_A(x))^{1/4}}$ in place of x, we obtain (5) (with $y=x^{1+1/(f_A(x))^{1/4}}$).

PROOF of Corollary 2.
Put

$$c_6 = \max(2c_3, 3). \tag{49}$$

Then by using Theorem 1 with $A\cap[0, x^{1-1/(f_A(x))^{1/3}}]$ and 2Ω in place of A and Ω, respectively, we obtain that for sufficiently large x (6) and (7) imply

$$f_A(x^{1-1/(f_A(x))^{1/3}}) \leqq \frac{f_A(x)}{2}.$$

Thus we have

$$\sum_{x^{1-1/(f_A(x))^{1/3}}<a\leqq x} \frac{1}{a} = f_A(x) - f_A\big(x^{1-1/(f_A(x))^{1/3}}\big) \geqq \frac{f_A(x)}{2}. \tag{50}$$

On the other hand,

$$\begin{aligned}\sum_{x^{1-1/(f_A(x))^{1/3}}<a\leqq x} \frac{1}{a} &\leqq \sum_{x^{1-1/(f_A(x))^{1/3}}<a\leqq x} \frac{1}{x^{1-1/(f_A(x))^{1/3}}} = \\ &= \frac{1}{x^{1-1/(f_A(x))^{1/3}}} \sum_{x^{1-1/(f_A(x))^{1/3}}<a\leqq x} 1 \leqq \frac{N_A(x)}{x^{1-1/(f_A(x))^{1/3}}}.\end{aligned} \tag{51}$$

With respect to (6) and (49), (50) and (51) yield that

$$N_A(x) \geqq \frac{f_A(x)}{2} x^{1-1/(f_A(x)^{1/3})} > x^{1-1/(f_A(x))^{1/3}}$$

which completes the proof of Corollary 2.

4. Proof of Theorem 3. Assume that (8) and (9) hold, and define y by

$$y^{2^t} = x^{1/2},$$

i.e.,

$$y = x^{1/2^{t+1}}. \tag{52}$$

For $j=1, 2, \dots, t$, let A_j denote the set consisting of the integers a such that

$$x/y^{2^j} < a \leqq x/y^{2^{j-1}}$$

and

$$p(a) > y^{2^j}.$$

Let

$$A = \bigcup_{j=1}^{t} A_j.$$

We are going to show that this sequence A satisfies (10), (11) and (12) (provided that c_6, c_8 are sufficiently small and c_7, c_9, c_{10}, c_{11}, X_4 are sufficiently large).

In order to estimate $f_A(x)$, we need the following

LEMMA 3. *There exist absolute constants c_{21}, c_{22} such that if $u\geqq 3$, $v\geqq u^2$ then we have*

$$c_{21} < \sum_{\substack{v<n\leqq vu \\ p(n)>u^2}} \frac{1}{n} < c_{22}.$$

This lemma is a consequence of the estimate

$$c_{23}\frac{x}{\log y} < \sum_{\substack{n\le x\\ p(n)>y}} 1 < c_{24}\frac{x}{\log y} \quad (\text{where } 3\le y<x^{5/6})$$

which can be proved easily by using standard methods of the prime number theory (see e.g. [7]).

For $1\le j\le t$, put $v=x/y^{2^j}$, $u=y^{2^{j-1}}$. Then by (8), (9) and (52), for sufficiently small c_7 and sufficiently large X_4 we have

$$u=y^{2^{j-1}}\ge y=x^{1/2^{t+1}}>x^{1/2^{c_7\log\log x+1}}=\exp\left(\frac{\log x}{2(\log x)^{c_7\log 2}}\right)\ge\exp\left((\log x)^{1/2}\right)\ge 3$$

and

$$u^2=\frac{u^2}{v}v=\frac{y^{2^j}}{x/y^{2^j}}v=\frac{y^{2^{j+1}}}{x}v\le\frac{y^{2^{t+1}}}{x}v=v,$$

thus Lemma 3 can be applied. We obtain that

$$c_{21}<f_{A_j}(x)=\sum_{a\in A_j}\frac{1}{a}=\sum_{\substack{x/y^{2^j}<a\le x/y^{2^{j-1}}\\ p(a)>y^{2^j}}}\frac{1}{a}<c_{22}$$

hence

$$c_{21}t<f_A(x)=\sum_{j=1}^{t}f_{A_j}(x)<c_{22}t. \tag{53}$$

Furthermore, by the construction of the sequence A we have

$$A\cap\left(\frac{x}{y},x\right]=\emptyset \tag{54}$$

where, by (53),

$$\frac{x}{y}=x^{1-1/2^{t+1}}<x^{1-1/2^{c_{21}^{-1}f_A(x)+1}}<x^{1-1/c_{23}^{f_A(x)}}. \tag{55}$$

(54) and (55) yield (11).

Finally, by the construction of the sequences A_j, if $n\le x$, $a\in A_j$ and $a'\in A_j$ then $a|n$, $a'|n$ cannot hold simultaneously so that

$$d_{A_j}(n)\le 1 \quad\text{for all}\quad 1\le j\le t \quad\text{and}\quad n\le x,$$

hence with respect to (53),

$$d_A(n)=\sum_{j=1}^{t}d_{A_j}(n)\le\sum_{j=1}^{t}1<t<\frac{1}{c_{21}}f_A(x)\quad\text{for all}\quad n\le x$$

which proves (12) and this completes the proof of Theorem 3.

5. Theorem 2 shows that for relatively small y, $D_A(y)/f_A(x)\to+\infty$, while for $y=\exp((\log x)^2)$, the proof of Theorem 2 in [2] yields a good (near best possible) lower bound for $D_A(y)$ (in terms of $f_A(x)$). One might like to seek for results "midway" these theorems, i.e., one might like to estimate $D_A(y)$ (in terms of $f_A(x)$) e.g. for $y=x^2$. In fact, the following problems of this type can be raised:

PROBLEM 1. Find a possibly small function $\varphi(x)$ such that for all $\Omega>0$, $x>X_5(\Omega)$ and

$$f_A(x)>\varphi(x)$$

imply that

$$D_A(x^2)>(\log x)^\Omega. \tag{56}$$

In fact, we can show that (56) follows from $f_A(x)>(\log x)^\varepsilon$, $x>X_6(\varepsilon,\Omega)$ where ε is arbitrary small but fixed positive number (independent of Ω). However, perhaps, it is sufficient to assume that

$$f_A(x)>\exp\left(c_{24}(\Omega)(\log\log x)^{1/2}\right).$$

Our results in Part II suggest that our assumption for $f_A(x)$ if true cannot be improved very much.

PROBLEM 2. Is it true that for all $\Omega>0$, there exist constants $c_{25}=c_{25}(\Omega)$ and $X_7=X_7(\Omega)$ such that $x>X_7$ and

$$f_A(x)>c_{25}$$

imply that

$$D_A(x^2)>\left(f_A(x)\right)^\Omega?$$

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