Problems and results in number theory and graph theory

Paul Erdös

First I discuss some problems on the iteration of number theoretic functions.

I. There are many attractive and amusing problems in this subject but almost no definitive results. One of the best known conjectures in this subject is an old conjecture of Catalan. Put

$$\sigma_1(n) = \sigma(n) - n, \qquad \sigma(n) = \sum_{d \mid n} d, \qquad \sigma_k(n) = \sigma_1(\sigma_{k-1}(n)).$$

Catalan conjectured that the sequence $\{\sigma_k(n)\}$, $k = 1,2,\ldots$ is bounded for every n, in other words it either leads to 1 or to a cycle. The Lehmers, Guy, Selfridge and Wunderlich have a great deal of numerical evidence about this conjecture; Guy and Selfridge have good heuristic evidence that the conjecture is probably false, in fact probably false for almost all even numbers n. The computations of the Lehmers seem to indicate that the conjecture is probably false for $n = 276$, but so far no proof is in sight and I do not expect any breakthrough in the near (or distant) future. I proved that for fixed k and almost all n

$$\sigma_k(n) = (1 + o(1))n\left(\frac{\sigma_1(n)}{n}\right)^k. \tag{1}$$

It is clear that (1) gives no help at all in deciding Catalan's conjecture. I do not see at all how to estimate $\sigma_n(n)$ or even $\sigma_{[\log n]}(n)$.

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NUMERICAL MATH. AND COMPUTING, 1979, pp. 3-21.

Guy and Wunderlich in a paper presented at this meeting investigated the iterations of the summatory function of the unitary divisors of n. A divisor of n is unitary if $(d,\frac{n}{d}) = 1$ and $s^*(n)$ is the sum of the proper unitary divisors. They do not come to any conclusion whether the sequence of the iterates of $s^*(n)$ is likely to be bounded. When I heard of their investigation I wanted to find a simpler function of slower growth about whose iterates we really can prove boundedness. I considered

$$w_1(n) = n \sum_{p \| n} \frac{1}{p^\alpha}, \qquad w_k(n) = w_1(w_{k-1}(n))$$

and conjectured that $\{w_k(n)\}$ $k = 1,2,\ldots$ is bounded. Unfortunately I could prove nothing. Perhaps $w_k(n) = 1$ for sufficiently large k, i.e., perhaps there are no cycles. I think the smallest solution of $w_2(n) = n$ if it exists must be fairly large - observe that $(w_1(n),n) = 1$ which causes $w_k(n)$ to grow rather slowly. I could prove no analogy to (1) for $w_k(n)$, for the iterations of $s^*(n)$ it is not hard to prove an analogue of (1). One more question about $w_1(n)$ which perhaps is not hopeless. Is it true that the sequence $w_1(n)$, $1 \le n \le x$, contains only $o(x)$ distinct numbers?

Further problems on iterations of number theoretic functions are discussed in my paper "Some recent problems and results in graph theory, combinatorial analysis, and number theory", Proc. Seventh Southeastern Conference, Congressus Num XVII, 11-14. Here I only want to mention a problem which occasionally occupied me for nearly 50 years. Denote by $\varphi(n) = \varphi_1(n)$ Euler's φ function and let $\varphi_k(n) = \varphi(\varphi_{k-1}(n))$. Pillai first investigated nearly 50 years ago the smallest integer $g(n)$ for which $\varphi_{g(n)}(n) = 1$. Pillai proved that

$$\frac{\log n}{\log 3} \leq g(n) \leq \frac{\log n}{\log 2}. \tag{2}$$

H. N. Shapiro a few years later took up the question independently and besides rediscovering (2) also proved that $g(n)$ essentially is an additive function. More precisely he proved that if a and b are not both even then $g(a \cdot b) = g(a) + g(b) - 1$, $g(a \cdot b) = g(a) + g(b) + 1$ otherwise. For a long time I tried hard to prove that $g(n)/\log n$ has a, perhaps degenerate, distribution function but I had absolutely no success. I proved that for almost all n and fixed $k \geq 2$

$$\varphi_k(n) = (1 + o(1))\varphi(n)(\log \log \log n)^{-k+1}. \tag{3}$$

But just as in (1) I can prove nothing if k can tend to infinity together with n, e.g., I have no idea how to estimate the smallest $k = k(n)$ for which, for almost all n,

$$\varphi_k(n) < n^{1/2} \quad \text{or say} \quad \varphi_k(n) < n/\log n.$$

I could not even prove that

$$\lim_{n \to \infty} \frac{1}{n \log n} \sum_{m=1}^{n} g(m)$$

exists and I have no guess about its value. Denote by $g_\alpha(n)$ the smallest integer r for which $\varphi_r(n) < n^\alpha$. I would not be surprised if for every α, $0 < \alpha < 1$,

$$\lim_{n \to \infty} \frac{1}{n \log n} \sum_{m=1}^{n} g_\alpha(m) = 0. \tag{4}$$

In view of (3), (4) has a certain plausibility.

Shapiro denotes by A_k the set of integers m with $g(m) = k$. He observes that the smallest member of A_k seems to be a prime, but states that he cannot even prove that for every k there is a prime in A_k. He proves several interesting results on A_k. Perhaps $|A_{k+1}| > |A_k|$ holds for every k. It would be interesting to obtain an asymptotic formula for $\log|A_k|$. Let Q_n be the smallest prime which does not divide any of the $\varphi_k(n)$. I am sure that for all n, $Q_n/\log n \to 0$, but have not proved it. I proved that for almost all n, $Q_n/(\log \log n)^k \to \infty$ for every k and I really do not know anything more about Q_n.

I can prove that to every ϵ and η there is a $k = k_0(\epsilon,n)$ so that the upper density of the integers n for which $\varphi_k(n)$ has a prime factor, $p > n^{\epsilon}$ is less than η. But again I have no idea how many iterations are needed until all prime factors $p > \exp(\log n)^{1/2})$, or say, $p > \log n$ disappear. It is not impossible that after about $[c \log \log n]$ iterations, only small prime factors remain. More precisely let $k(p,n)$ be the smallest integer for which all prime factors of $\varphi_k(n)$ are $\leq p$. Perhaps for every $p > 3$ (≥ 3?), $k(p,n) = o(\log n)$, or what is perhaps more likely: to every $\epsilon > 0$ there is a p so that the density of the integers n for which ($P(m)$ is the greatest prime factor of m)

$$P(\varphi_{[\epsilon \log n]}(n)) > p$$

is less than ϵ.

Put $\sigma_k(n) = \sigma(\sigma_{k-1}(n))$. It seems certain that for every n

$$\lim_{k \to \infty} \sigma_k(n)^{1/k} = \infty,$$

but again I can prove nothing.

Richard K. Guy and Marion C. Wunderlich, Computing unitary aliquot sequences, this volume. This paper has extensive references.

H. N. Shapiro, An arithmetic function arising from the φ function, Amer. Math. Monthly 50(1943), 18-30.

P. Erdős, On asymptotic properties of aliquot sequences, Math. Comput. 30(1976), 641-645.

II. Now I discuss some older results of Selfridge and myself. In a previous paper Selfridge and I investigated some number theoretic functions. Our paper is not easily accessible, thus I repeat some of our results, but also prove some new ones and state new problems.

We were led to our investigation by the following conjecture of Grimm: Let $n+1,\ldots,n+k$ be a sequence of consecutive composite numbers. Then there always is a sequence of distinct primes p_i satisfying $p_i \mid n+i$. Denote by $V(n,k)$ the number of distinct prime factors of $\Pi_{i=1}^{k}(n+i)$. $f_0(n)$ is the largest integer k for which $V(n,k) \geq k$, $f_1(n)$ is the smallest integer k so that for every $1 \leq \ell \leq k$, $V(n,\ell) \geq \ell$ but $V(n,k+1) = k$. Grimm's

function $f_2(n)$ is the largest k so that for each i, $1 \leq i \leq k$ there is a $p_i \mid n+i$, $p_{i_1} \neq p_{i_2}$ if $i_1 \neq i_2$. $P(m)$ is the greatest prime factor of m. $f_3(n)$ is the largest integer k so that all the primes $P(n+i)$, $1 \leq i \leq k$, are distinct. $f_4(n)$ is the largest integer k so that $P(n+i) \geq i$, $1 \geq i \geq k$, and $f_5(n)$ is the largest integer k so that $P(n+i) \geq k$ for every $1 \leqq i \leqq k$. We observed that trivially

$$(1)\qquad f_0(n) \geq f_1(n) \geq \cdots \geq f_5(n)$$

and we conjectured that for infinitely many n all the inequalities in (1) are strict. This seems quite difficult and we made no progress. We could only prove that for every i and infinitely many n, $f_i(n) > f_{i+1}(n)$. As far as I know it is not known that

$$(2)\qquad \limsup_{n \to \infty}(f_0(n) - f_1(n)) = \infty.$$

(2) is almost certainly true. On the other hand, I conjecture

$$(3)\qquad \lim_{n \to \infty} f_0(n)/f_1(n) = 1.$$

It seems certain that

$$(4)\qquad \limsup_{n \to \infty} f_0(n)/f_2(n) = \limsup_{n \to \infty} f_1(n)/f_2(n) = \infty,$$

and (4) perhaps holds with lim instead of lim sup. I have no guess about $f_2(n)/f_3(n)$. We proved that

$$f_0(n) < cn^{1/2}/\log n$$

and observed that results of Ramachandra give $f_0(n) < n^{\frac{1}{2} - \epsilon}$ for some fixed $\epsilon > 0$ and $n > n_0(\epsilon)$. We proved that for infinitely many n

$$(5)\qquad f_0(n) > cn^{1/e}, \qquad f_1(n) > cn^{1/e}.$$

In fact there is a misprint in the paper; we prove (5) and state that we proved the opposite inequality. To clear the matter up we will prove

Theorem 1. The inequalities

$$f_0(n) > c_1 n^{1/e}, \quad f_0(n) < c_2 n^{1/e}, \quad f_1(n) > c_3 n^{1/e}, \quad f_1(n) < c_4 n^{1/e}$$

all have infinitely many solutions.

It is not impossible that for all n

$$cn^{1/e} < f_1(n) \leq f_0(n) < c' n^{1/e}. \tag{6}$$

Perhaps this conjecture is too optimistic. I am fairly sure though that $f_0(n) < n^{(1/e)+\epsilon}$ holds for all sufficiently large n.

We further proved that for infinitely many n, $f_2(n) > \exp(c_1(\log n \log \log n)^{1/2})$, and we proved an upper bound for $f_2(n)$, hardly better than $f_2(n) = o(n^\epsilon)$. Pomerance and Turk proved that in fact $f_2(n) < \exp(c_2(\log n \log \log n)^{1/2})$ holds for infinitely many n. Perhaps

$$\exp(c_1(\log n \log \log n)^{1/2}) < f_2(n) < \exp(c_2(\log n \log \log n)^{1/2})$$

holds for all n. Perhaps the same holds for $f_3(n)$. We have no results whether $f_i(n) = f_{i+1}(n)$ holds for infinitely many n.

Now I state some new problems. Let $f_6(n)$ be the largest integer k for which

$$P(n+1) < \cdots < P(n+k).$$

Here I cannot even prove that $f_6(n)$ takes on arbitrarily large values, although there is no doubt that this is true. Probably $f_6(n) < (\log n)^c$ holds for some not too large value of c.

Let us modify these functions by replacing p by p^2, denote by $f_i^*(n)$ the modified $f_i(n)$. Thus $f_0^*(n)$ is the largest integer k for which $\Pi_{i=1}^k(n+i)$ has k distinct divisors p_i^2. None of our functions now tend to infinity and it is easy to see that the density α_t of the integers n for which $f_0^*(n) = t$ exists and is positive for every t; further $\Sigma_{t=0}^{\infty} \alpha_t = 1$. The same result holds for $1 \leq i \leq 6$. It is easy to see that $f_i^*(n)$ is unbounded, $0 \leq i \leq 6$, in fact it easily follows from the prime number theorem that for infinitely many n

$$f_0^*(n) \geq (\tfrac{1}{2} + o(1))\log n/\log \log n. \tag{7}$$

Probably (7) holds also for $f_i(n)$, $1 \leq i \leq 6$, but at present I can only prove that there is a $c > 0$ so that for infinitely many n

$$f_i^*(n) > c \log n/\log \log n.$$

Perhaps (7) is not far from being best possible, but I cannot even prove that $f_0^*(n) < n^{\epsilon}$ holds for every n if $n > n_0(\epsilon)$. I am in fact sure that $f_0^*(n) < (\log n)^c$ for some c.

Now we prove Theorem 1. First we prove that for infinitely many n, $f_0(n) < c_2 n^{1/e}$. If this would not hold, then for every c and every sufficiently large x we would have for every $x < n < 2x$

$$(8) \qquad f_0(n) > cx^{1/e}.$$

Put now $0 \leq i \leq t$

$$f_0(x) = y_0, \qquad f_0(x+y_1+\cdots+y_i) = y_{i+1}$$

where

$$(9) \qquad x < \sum_{i=0}^{t} y_i < \sum_{i=0}^{t+1} y_i = x + z.$$

By (8)

$$(10) \qquad y_i > cx^{1/e}, \qquad t < \frac{1}{c}\, x^{1-\frac{1}{e}} \; .$$

By the definition of $f_0(n)$ we have

$$(11) \qquad V(x + \sum_{j=0}^{i} y_j, \, y_{i+1}) = y_{i+1}.$$

Thus by (9) and (11)

$$(12) \qquad \sum_{i=0}^{t} V(x + \sum_{j=0}^{i} y_j, \, y_{i+1}) = x + z.$$

Now we count the contribution of the primes to the sum (12). By (10) each prime $p \leq cx^{1/e}$ contributes to each summand of (12) exactly one. Thus the total contribution of these primes to the sum (12) is by (10) and the prime number theorem at most

$$(13) \qquad t\pi(cx^{1/e}) < \frac{10x}{\log x} .$$

The primes $p > cx^{1/e}$ each contribute at most

$$(14) \qquad [\frac{2x+z}{p}] - [\frac{x}{p}] \leq \frac{x + z}{p} + 1$$

since the contribution of these primes p is at most the number of multiples of p in $(x+1, 2x+z)$. Thus from (14) the contribution of all these primes is less than ($z < x$ since $f_0(n) = o(n)$),

$$(15) \qquad \pi(2x+z) + (x+z) \Sigma' 1/p$$

where in Σ' $cx^{1/e} < p < 3x$ (we use $z < x$). Now by the theorem of Mertens we have for sufficiently large c

$$(16) \qquad \Sigma' 1/p < 1 - \frac{100}{\log x}.$$

Thus from (13), (15) and (16) and $\pi(2x+z) < \frac{4x}{\log x}$ we have

$$(17) \qquad \sum_{i=0}^{t} V(x + \sum_{j=0}^{i} y_j, y_{i+1}) < \frac{14x}{\log x} + (x+z)(1 - \frac{100}{\log x}) < x + z.$$

(17) contradicts (12). This contradiction proves that $f_0(n) < c_2 n^{1/e}$ holds for some sufficiently large c_2 for infinitely many n.

To complete the proof of Theorem 1 we now have to show that for infinitely many n, $f_1(n) > c_3 n^{1/e}$. The proof will be very similar to the previous one. If $f_1(n) > c_3 n^{1/e}$ does not hold for some fixed $c_3 > 0$ and infinitely many n (in other words if $f_1(n)/n^{1/e} \to 0$), then for every ε and every sufficiently large x we would have for every $x < n < 2x$

$$(18) \qquad 0 \le i \le t', \quad f_1(n) < \epsilon x^{1/e}.$$

Put now

$$f_1(x) = y_0', \quad f_1(x+y'+\cdots+y_i') = y_{i+1}'$$

where

$$(19) \qquad x < \sum_{i=0}^{t'} y_i < \sum_{i=0}^{t'+1} y_i = x + z'$$

From (18) and (19) we have

$$(20) \qquad y_i' < \epsilon x^{1/e}, \quad t' > \frac{1}{\epsilon} x^{1 - \frac{1}{e}}.$$

By the definition of $f_1(n)$ we have

$$(21) \qquad V(x + \sum_{j=0}^{i} y_j', y_{j+1}') = y_{i+1}'.$$

Thus by (19) and (21)

$$(22) \qquad \sum_{i=0}^{t'} V(x + \sum_{j=0}^{i} y_j', y_{j+1}') = x + z'.$$

Now we count the contribution of the primes p to the sum (22). We ignore the primes $p < \epsilon x^{1 - 1/e}$. By (19) the contribution of the primes $p > \epsilon x^{1 - 1/e}$ is exactly the number of times a multiple of p occurs in $(x, 2x+z')$. Thus the contribution of these primes is at least (in Σ' the summation is extended over the p satisfying $\epsilon x^{1-1/e} < p < 2x+z'$)

$$(23)\quad \Sigma'\left(\left[\frac{2x+z'}{p}\right] - \left[\frac{x}{p}\right]\right) > \Sigma'\left(\frac{x+z'}{p} - 1\right) > (x+z')\Sigma' \frac{1}{p} - \pi(3x) > x + z'$$

for sufficiently small ϵ. In the last inequality of (23) we used the prime number theorem and the theorem of Mertens. (23) contradicts (22) and hence the proof of our Theorem is complete.

This method should give that the average order of $f_0(n)$ and $f_1(n)$ is $cx^{1-1/e}$, but we have not yet succeeded in proving this.

In another paper with Selfridge we investigate the following questions: Let

$$(24)\quad 0 \leq n < a_1 < \cdots < a_t \leq n + k, \quad (a_1, a_j) = 1, \quad 1 \leq i < j \leq t.$$

We call $\{a_1, \ldots, a_t\}$ complete if for every s in $n < s \leq n + t$, $(s, a_i) > 1$ for some i. Put

$$F(n,k) = \max t, \qquad f(n,k) = \min t$$

where the maximum and minimum are taken with respect to all complete sequences satisfying (24). We investigated the following four functions:

$$\max_n F(n,k) = \overline{F(k)}, \qquad \min_n F(n,k) = \underline{F(k)},$$

$$\max_n f(n,k) = \overline{f(k)}, \qquad \min_n f(n,k) = \underline{f(k)}$$

We observed that $\overline{F(k)}$, $\underline{F(k)}$ are trivially non-decreasing but $\overline{f(6)} = 3$,

$\overline{f(5)} = 4$. We do not know if there are other values of n for which $\overline{f(n+1)} < \overline{f(n)}$. This seems to us a curious and interesting question, but unfortunately we cannot decide it. We stated that perhaps

$$f(k) - \pi(k) \to -\infty \text{ as } k \to \infty. \tag{25}$$

(25) is almost certainly incorrect; this follows from the results of Hensley and Richards and the prime k-tuple conjecture, but perhaps (25) can be disproved without the prime k-tuple conjecture. Observe that every complete sequence must contain all the integers $n < m \leq n + k$ with $p(m) \geq k$, ($p(m)$ denotes the least prime factor of m). Put $n + i = a_i b_i$, $i=1,\ldots,k$, where $P(a_i) < k$ and $P(b_i) \geq k$. It follows from a slight modification of the prime k-tuple conjecture that for every k there are infinitely many values of n for which the b_i are all primes and in fact there are infinitely many values of n for which $a_i = i$ and the b_i are all primes. It would be of interest to find the smallest such n (say n_k) for small values of k, e.g., $k = 10$.

We stated on page 8 of our paper that we cannot improve $\overline{f(k)} \geq (1 + o(1)) \frac{e^{-\lambda} k}{\log k}$ where γ is Euler's constant. After our paper was published we noticed that this is nonsense; $\overline{f(k)} \geq (1 + o(1)) \frac{k}{\log k}$ immediately follows from the periodicity of $f(n,k)$ and from $f(0,k) = (1 + o(1)) \frac{k}{\log k}$.

Put

$$\min_n(F(n,k) - f(n,k)) = g(k).$$

We conjecture that $g(k) \to \infty$ as $k \to \infty$. We hope to investigate this attractive conjecture (if we live).

P. Erdős and J. L. Selfridge, Some problems on the prime factors of consecutive integers II, Proc. Washington State Univ. Conference Number Theory, 1971, 13-21.

C. A. Grimm, A conjecture on consecutive composite numbers, Amer. Math. Monthly 76(1969), 1126-1128.

P. Erdős and J. L. Selfridge, Complete prime subsets of consecutive integers, Conf. Numerical Math., Winnipeg 1971, 1-14.

D. Hensley and I. Richards, Primes in intervals, Acta Arithmetica 25(1979), 376-391.

P. Erdős and I. Richards, Density functions for prime and relatively prime numbers, Monatshefte für Math. 83(1977), 99-112.

III. Finally I report on some results V. Neumann-Lara and I found over the last two years; detailed proofs will be published elsewhere. First some notations. $G(n;e)$ will denote a graph of n vertices and e edges. $G(n)$ a graph of n vertices and G_e a graph of e edges. Let G be a directed graph. Neumann-Lara defines $d_k(G)$, the dichromatic number of G, as the smallest integer so that the vertex set of G can be decomposed into $d_k(G)$ disjoint subsets none of which span a directed circuit. He is preparing a paper on his function $d_k(G)$. When I first visited Mexico City two years ago we started to investigate a modification of this function. Let G be (an undirected) graph. The dichromatic number $d_k(G)$ is the smallest integer so that for any orientation of the edges of G one can always divide the vertex set into $d_k(G)$ or fewer disjoint sets, none of which span a directed circuit of G (in the given orientation).

It is surprisingly difficult to determine $d_k(G)$ even for the simplest graphs. We proved ($k(n)$ is the complete graph of n vertices)

$$c_1 n/\log n < d_k(k(n)) < c_2 n/\log n. \tag{1}$$

Probably there is no simple explicit formula $d_k(k(n))$. We could not even prove that

$$d_k(k(n)) \log n \cdot n^{-1} \to c. \tag{2}$$

Let $f(n)$ be the smallest integer for which there is a $G_{f(n)}$ of dichromatic number n. Perhaps then $G_{f(n)}$ must be a complete graph. This is easy to prove for the ordinary chromatic number but we could not prove it for the dichromatic number, and in fact we could not even prove that $f(n)/n^2$ tends to infinity - we have no doubt that this is true.

Let me state two of our most striking problems. Is it true that if every vertex of G has degree at most n then $d_k(G) = o(n)$? Perhaps in fact $d_k(G) < cn/\log n$. Is it true that $k(G)$ (the chromatic number of G) is large then $d_k(G)$ is also large? More precisely put

$$h(\ell) = \min d_k(G)$$

where the minimum is extended over all G with $k(G) = \ell$. Determine or estimate $h(\ell)$ as accurately as possible. We could not even prove that $h(\ell) > 2$ for $\ell > \ell_0$. Is it true that for every ℓ there is a $g(\ell)$ so that if $d_k(G) \geq g(\ell)$ and we direct the edges of G in an arbitrary way then there always is an induced subgraph G_1 of G, which contains no directed circuit satisfying $k(G_1) \geq \ell$? Perhaps

$$g(\ell) < \exp c\ell. \tag{3}$$

(3) holds for complete graphs - this easily follows from (1), but we know nothing about the general case.

It is easy to see that if G has fewer than n^2 vertices and $k(G) = n$ then $d_k(G) < n$. On the other hand, we proved $d_k(k_n(n)) = n$ where $k_n(n)$ is the complete n-partite graph with n vertices of each color. In fact, we proved a somewhat stronger result. Denote by $V(G)$ the largest integer so that for every orientation of the edges of G there is a set of at least $V(G)$ vertices which does not span a directed circuit. We proved $V(k_n(n)) = n + 1$, which immediately implies $d_k(k_n(n)) = n$.

Denote by $I(G)$ the largest independent set of G. Clearly $V(G) \geq I(G) + 1$. We showed that there is a sufficiently large absolute constant c so that for $n > n_0(c)$ there is a $G(n)$ with $I(G(n)) < c \log n$ and

$$V(G(n)) = I(G(n)) + 1.$$

Let G and H be two graphs. Following Harary, we denote by $G[H]$ the composition of G and H. Assume that $k(G) = \ell$, then for sufficiently large m, $d_k(G(\overline{k(m)}) = \ell$, $\overline{k(m)}$ is the complement of $k(m)$, i.e. consists of m isolated vertices.

Following G. Dirac we call a graph G vertex critical if the omission of every one of its vertices decreases the dichromatic number; it is edge critical if the omission of every one of its edges decreases the dichromatic number. $k_n(n)$ is vertex critical but not edge critical. The following extremal problem might be of some interest. Denote by $f_d(e,n)$ the largest integer for which there is a G_e satisfying $d_k(G_e) = n$. G_e is vertex critical and has a subgraph $G_{e'}$ for which $e - e' = f_d(e,n)$. $k_n(n)$ has $e = n^2\binom{n}{2}$, and we proved that e' can be chosen to be less than $n^{4-\eta}$ (we do not know the best possible value of η). Thus $f_d(e,n) = e(1 + o(1))$ is possible, but we cannot determine the exact value of $f_d(e,n)$. We do not know what happens if for fixed n, e tends to infinity. Does $f_d(e,n)$ remain bounded? $f(e,n)$ can be defined for ordinary chromatic numbers too. We have no example where $f(e,n) = (1 + o(1))e$ $(n \to \infty)$.

It is well known that for almost all $G(n)$

$$c_1 n/\log n < k(G(n)) < c_2 n/\log n. \tag{4}$$

In other words (4) holds for all but $o(2^{\binom{n}{2}})$ graphs $G(n)$. We proved that (4) also holds for $d_k(G(n))$ but with different values of the constants c_1 and c_2. We could not decide whether

$$d_k(G(n)) = (1 + o(1))k(G(n))$$

holds for almost all $G(n)$. It seems more likely that there is an absolute constant $c < 1$ that for almost all $G(n)$

$$d_k(G(n))/k(G(n)) \to c.$$

The proof of (4) for $d_k(G(n))$ follows from the following lemma of independent interest. Let $\alpha > 0$, $\beta > 0$ be two suitably chosen constants. Assume that $G(n)$ is such that every induced subgraph of $[\beta \log n]$ vertices contains at least $\alpha(\log n)^2$ edges (it is not hard to see that for sufficiently large n almost all $G(n)$ have this property). Then there is a constant $c_1(\alpha,\beta)$ for which

$$c_1(\alpha,\beta)n/\log n < d_k(G(n)).$$

We showed that for every r and ℓ there is a G whose girth is greater than r and for which $d_k(G) \geq \ell$ (the girth of G is the length of the smallest circuit). We do not know the smallest $n = n_0(r,\ell)$ for which there is such a $G(n)$, but the analogous problem for ordinary chromatic numbers is also not solved.

Finally we proved that if $f_d(n,k)$ is the smallest integer for which $d_k(G(n;\ f_d(n,k)) \geq k$ holds for every $G(n;\ f_d(n,k))$. Then

$$f_d(n,k) = \frac{n^2}{2(k-1)} + O(n^{2-1/k}).$$

The exponent $2 - 1/k$ is surely not best possible, but we do not know the best possible value for the exponent.

To complete this paper, I just say a few words about a different generalization of the notion of chromatic number. The acyclic chromatic number of G is defined by B. Grünbaum as follows: A coloring of the vertices of G is said to be acyclic if no two vertices of the same color are joined and if the subgraph induced by the vertices of any two colors is acyclic. The acyclic chromatic number a(G) is the minimum value of k for which G has an acyclic k-coloring. The acyclic chromatic number has also been investigated by M. O. Albertson and D. M. Bergman.

Let m be the maximum degree of G. Denote by $f(m)$ the largest value of $a(G)$, where the maximum is extended over all G whose maximum degree does not exceed m. Grünbaum observed that $f(m) < m^2$ and wanted a better estimation from above and below. I proved that $f(m) > m^{4/3-\epsilon}$ and conjectured that $f(m) = o(m^2)$.

More generally define the acyclic chromatic number $a_r(G)$ of order r as the minimum value of k for which G has a coloring with k colors so that the union of any r color classes is acyclic. $f_r(m)$ is the largest value of $a_r(G)$ where the maximum is extended over all G whose maximum degree does not exceed m. Is it true that for every α there is an r for which for every $m > m_0(r,\alpha)$, $f_r(m) > m^{\alpha}$?

B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math. 14(1973), 390-408.

M. O. Albertson and D. M. Berman, The acyclic chromatic number, Proc. Seventh Conf. on Combinatorics, etc., L.S.V. Baton Rouge 1976. Cong. Num. XVII, 51-60.