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ON THE MAXIMAL VALUE OF ADDITIVE FUNCTIONS IN SHORT INTERVALS AND ON SOME RELATED QUESTIONS

By

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1. Let (a, b) and $[a, b]$ be the greatest common divisor and the least common multiple of a and b, respectively. p_n denotes the n'th prime; $p, q, q_1, q_2, \dots$ are prime numbers. A sum $\sum\limits_p$ and a product $\prod\limits_p$ denote a summation and a multiplication, respectively, over primes indicated. The symbol $\#\{\dots\}$ denotes the number of elements indicated in the bracket $\{\ \}$. P_μ is the product of the first μ primes.

The aim of this paper is to continue our investigation on the distribution of the maximal value of additive functions in small intervals.

In the sequel let $g(n)$ be a non-negative strongly additive function,

(1.1) $$f_k(n) = \max_{j=1,\dots,k} g(n+j).$$

Let

(1.2) $$\varrho(k, \varepsilon) = \sup_{x\geqq 1} \frac{1}{x}\ \#\{n \leqq x | f_k(n) > (1+\varepsilon) f_k(0)\},$$

(1.3) $$\delta(k_0, \varepsilon) = \sup_{x\geqq 1} \frac{1}{x}\ \#\{n \leqq x | \exists k,\ k > k_0, f_k(n) > (1+\varepsilon) f_k(0)\},$$

$$\theta(k, \varepsilon) = \limsup_{x=\infty} \frac{1}{x}\ \#\{n \leqq x | f_k(n) > f_k(0)(1+\varepsilon)\}.$$

It is obvious that

(1.4) $$\theta(k, \varepsilon) \leqq \varrho(k, \varepsilon),$$

and that

(1.5) $$\delta(k_0, \varepsilon) \geqq \sup_{k\geqq k_0} \varrho(k, \varepsilon).$$

In [1] we tried to determine those additive $g(n)$ for which the relation

(1.6) $$\delta(k_0, \varepsilon) \to 0 \quad (k_0 \to \infty), \quad \forall\, \varepsilon > 0$$

holds. There we noticed that (1.6) implies

(1.7) $$\sum_p \frac{\min(1, g(p))}{p} < \infty,$$

but we could not decide if the condition

(1.8) $$\sum_p \frac{g(p)}{p} < \infty$$

were necessary. Now we shall prove this. More exactly, we shall prove the following assertion.

THEOREM 1. *If*

$$\theta(k, \varepsilon) \to 0 \quad (k \to \infty) \tag{1.9}$$

for all $\varepsilon > 0$, *then*

$$\sum_p \frac{g(p)^r}{p} < \infty, \tag{1.10}$$

for every $r \geqq 1$.

Let $F(x)$ be the limit distribution function of $g(n)$, the existence of which is guaranteed by (1.7).

THEOREM 1′. *Assume that*

$$k\big(1 - F(f_k(0)(1+\varepsilon))\big) \to 0 \tag{1.11}$$

holds for every $\varepsilon > 0$. *Then* (1.10) *holds for every* $r \geqq 1$.

Theorem 1 is an immediate consequence of Theorem 1′. Indeed, (1.11) implies that the density of integers n, satisfying $g(n) > (1+\varepsilon) f_k(0)$ is $o(1/k)$, consequently (1.9) holds.

Perhaps (1.11) implies that

$$\sum_p \frac{e^{ug(p)} - 1}{p} < \infty \tag{1.12}$$

for every $u > 0$. We could not give a counter example.

THEOREM 2. *If for some constant* $A > 0$

$$k\big(1 - F(f_k(0) + A)\big) \to 0 \quad (k \to \infty), \tag{1.13}$$

thne (1.12) *holds for every* $u > 0$.

On the other hand, we shall prove that (1.6) does not imply $g(p) = O(1)$. This will follow easily from the following

THEOREM 3. *Let* $L(k)$ *be a function on* $[1, \infty)$ *tending to infinity arbitrary slowly. Then there exists a strongly additive non-negative* $g(n)$ *with* $\overline{\lim}\, g(p) = \infty$, *so that*

$$\sup_{x \geqq 1} \frac{1}{x} \# \{n \leqq x \mid \exists k \geqq k_0, f_k(n) > L(k)\} \to 0 \quad (k_0 \to \infty). \tag{1.14}$$

We are interested in the conditions that imply

$$\sup_{x \geqq 1} \frac{1}{x} \# \{n \leqq x \mid \exists k > k_0, f_k(n) > f_k(0) + A\} \to 0 \quad (k_0 \to \infty), \tag{1.15}$$

with some suitable constant A.

THEOREM 4. *If* $g(p) = \dfrac{1}{p}$, *then*

$$\sup_{x \geqq 1} \frac{1}{x} \# \{n \leqq x \mid \exists k > k_0, f_k(n) > f_k(0) + \lambda_k\} \to 0 \quad (k_0 \to \infty), \tag{1.16}$$

where $\lambda_k = 3/(\log \log k)$.

THEOREM 5. *If* $g(p)=1/p^\delta$, $0<\delta<1$, $\varrho>0$ *being an arbitrary constant, then*

$$\lim_{k\to\infty}\underline{\lim}_{x=\infty}\frac{1}{x}\#\{n\leqq x|f_k(n)>f_k(0)+(\log k)^{1-\delta-\varrho}\}=1. \tag{1.17}$$

By somewhat more trouble we could prove that

$$\sup_{x\geqq 1}\frac{1}{x}\#\{n\leqq x|\exists k>k_0, f_k(n)<f_k(0)+(\log k)^{1-\delta-\varrho}\}\to 0, \tag{1.18}$$

as $k_0\to\infty$.

Let $F_\delta(x)$, $F_\gamma(x)$ denote the limit distribution functions corresponding to $g(p)=1/p^\delta$, $g(p)=(\log p)^{-\gamma}$, respectively; $G_\delta(x)=1-F_\delta(x)$, $G_\gamma(x)=1-F_\gamma(x)$.

We shall consider $G(x)$ for large $x(>0)$.

THEOREM 6. *We have for* $\delta=1$:

$$\log\log\frac{1}{G_1(\tau)}\geqq e^{\tau-a}-c\tau^2e^{-\tau}, \tag{1.19}$$

where $a=\gamma-\sum_{k\geqq 2}\sum_p\frac{1}{kp^k}$; γ *being Euler's constant, c denotes a suitable constant.* Furthermore, if $0<\delta<1$,

$$\log\frac{1}{G_\delta(\tau)}\geqq(\tau\log\tau)^{1/(1-\delta)}\big(1+O(\log\tau)^{-1}\big)\quad(\tau>1), \tag{1.20}$$

and

$$\log\frac{1}{G_\gamma(\tau)}\geqq\tau(\log\tau)^{\gamma+1}-c_1\tau(\log\tau)^\gamma, \tag{1.21}$$

c_1 being a positive constant depending on γ.

REMARK. It is easy to see that the previous inequalities are quite sharp. Indeed, if g is monotonically decreasing on the set of primes p, then for $P_\mu\leqq k<P_{\mu+1}$ we have

$$1-F\big(g(P_\mu)\big)\geqq\frac{1}{P_\mu}\geqq\frac{1}{k}.$$

Hence, after some simple computation, we have the following inequalities for $\tau>1$:

(i) $\log\log\frac{1}{G_{\delta=1}(\tau)}\leqq e^{\tau-a}+O(e^{-B\tau})$, B being an arbitrary but fixed number;

(ii) $\log\frac{1}{G_\delta(\tau)}\leqq(\tau\log\tau)^{1/(1-\delta)}\big(1+O((\log\tau)^{-1})\big)$, if $0<\delta<1$;

(iii) $\log\frac{1}{G_\gamma(\tau)}\leqq\tau(\log\tau)^{\gamma+1}\big(1+O((\log\tau)^{-1})\big)$.

Let now

$$\sum_p \frac{g(p)}{p} = \infty; \quad \sum_p \frac{g^2(p)}{p} < \infty, \tag{1.22}$$

$$A_x = \sum_{p \leqq x} \frac{g(p)}{p}; \tag{1.23}$$

$$\psi(y) = \sum_{p \leqq y} g(p), \tag{1.24}$$

$$F_k(n) = \max_{1 \leqq j \leqq k} \{g(n+j) - A_{n+j}\}. \tag{1.25}$$

THEOREM 7. *Let* $0 < t(x)$ *monotonically tend to zero in* $[1, \infty)$, *let* $g(n)$ *be strongly additive defined for primes p by* $g(p) = t(p)$. *If* (1.22) *holds, then for every fixed* k, $P_\mu \leqq k < P_{\mu+1}$, *we have*

$$F_k(n) \geqq \psi(P_\mu) + A_{\log k} - \varepsilon_k \tag{1.26}$$

for every but $O(\delta_k x)$ *of* $n \leqq x$; $\varepsilon_k \to 0$, $\delta_k \to 0$ *as* $k \to \infty$.

Suppose, in addition, that

$$\lim_{y \to \infty} \frac{\psi(y)}{y t(e^{y^\delta})} = \infty \tag{1.27}$$

for every $\delta > 0$, *and that*

$$\sum_{p > y} \frac{t^2(p)}{p} \ll t^2(y) (\log \log y)^\gamma \quad (y \to \infty) \tag{1.28}$$

for a suitable $\gamma > 0$. *Then*

$$\limsup_{k_0 \to \infty} \sup_{x \geqq 1} \frac{1}{x} \# \left\{ n \leqq x \,|\, \exists k > k_0, \left| \frac{F_k(n)}{\psi(\log k)} - 1 \right| \geqq \varepsilon \right\} = 0, \tag{1.29}$$

for every $\varepsilon > 0$.

2. Asymptotic of distribution functions for large values. Let $g(n) \geqq 0$ be strongly additive. Then for every $u \geqq 0$

$$\sum_{n \leqq x} e^{ug(n)} \leqq x \prod_{p \leqq x} \left(1 + \frac{e^{ug(p)} - 1}{p}\right). \tag{2.1}$$

As it is well known

$$\frac{1}{x} \sum_{n \leqq x} e^{ug(n)} \to K(u) = \prod_p \left(1 + \frac{e^{ug(p)} - 1}{p}\right), \tag{2.2}$$

if the infinite product on the right hand side converges. Let $F(\tau)$ be the distribution function of $g(n)$. Then

$$1 - F(\tau) \leqq K(u) e^{-u\tau} \quad (0 < u < \infty). \tag{2.3}$$

By choosing u appropriately, we shall use (2.3) to give an upper estimate for $G(\tau) = 1 - F(\tau)$ for some special additive functions.

Let $t(x)$, $x\in[1,\infty)$, tend to zero monotonically, $g(p)=t(p)$ for primes p, $\psi(y)=\sum_{p\le y} t(p)$. Suppose that $t(x)$ is differentiable.

Let the values t_0, t_1 be defined by the relations

$$ut(t_0)=\log t_0+H;\quad ut(t_1)=\log t_1-H, \tag{2.4}$$

where $H>1$. Let

$$K(u)=K_1(u)K_2(u)K_3(u),$$

where in $K_i(u)$ ($i=1,2,3$) the product is extended over the primes in the intervals $(1,t_0]$, $(t_0,t_1]$, (t_1,∞), respectively.

For $p\in(1,t_0)$ we use the inequality

$$\log\left(1+\frac{e^{ug(p)}-1}{p}\right)<\log\frac{e^{ug(p)}}{p}+e^{-ug(p)}p\le ug(p)-\log p+e^{-H},$$

and deduce

$$\log K_1(u)<u\psi(t_0)-\sum_{p\le t_0}\log p+\sum_{p\le t_0}pe^{-ug(p)}. \tag{2.5}$$

Since

$$1+\frac{e^{ug(p)}-1}{p}\le 1-\frac{1}{p}+e^{H}<e^{H+1}$$

in $p\in(t_0,t_1]$, therefore

$$\log K_2(u)<(H+1)\big(\pi(t_1)-\pi(t_0)\big). \tag{2.6}$$

Furthermore

$$\log K_3(u)<\sum_{p>t_1}\frac{e^{ug(p)}-1}{p}. \tag{2.7}$$

We shall give an upper estimate for the right hand side of the last inequality when $t(x)=x^{-\delta}$ $(0<\delta\le 1)$; $t(x)=(\log x)^{-\gamma}$. For this we use the prime number theorem in the form

$$\pi(x)=\operatorname{li}x+R(x),\quad |R(x)|\le c_2x(\log x)^{-c_3},$$

where c_3 is a large constant. Let

$$f(x)=\frac{e^{ut(x)}-1}{x}. \tag{2.8}$$

Then

$$\sum_{p>t_1}\frac{e^{ug(p)}-1}{p}=I_1+I_2,\quad I_1=\int_{t_1}^{\infty}\frac{f(x)}{\log x}\,dx,\quad I_2=\int_{t_1}^{\infty}f(x)\,dR(x).$$

For the estimation of I_2 we integrate by parts:

$$I_2=R(x)f(x)\Big|_{t_1}^{\infty}-\int_{t_1}^{\infty}R(x)f'(x)\,dx. \tag{2.9}$$

Suppose that

$$f'(x)=\frac{e^{ut(x)}\big(ut'(x)x-1\big)+1}{x^2}$$

changes its sign in $[t_1, \infty)$ at most once, for example at z_0. Then, by integrating by parts, we have

$$\int_{t_1}^{\infty} |R(x)||f'(x)|\,dx \leqq c_2\left|\int_{t_1}^{z_0} \frac{x}{(\log x)^{c_3}} f'(x)\,dx\right| + c_2\left|\int_{z_0}^{\infty} \frac{x}{(\log x)^{c_3}} f'(x)\,dx\right| \ll$$

$$\ll f(t_1)\frac{t_1}{(\log t_1)^{c_3}} + \int_{t_1}^{\infty} \frac{f(x)}{(\log x)^{c_3}}\,dx.$$

So, observing that

$$f(t_1) = \frac{e^{-H}t_1 - 1}{t_1} \leqq e^{-H},$$

we get

$$I_2 \ll e^{-H}\frac{t_1}{(\log t_1)^{c_3}} + \frac{1}{(\log t_1)^{c_3-1}} \cdot I_1. \tag{2.10}$$

To estimate I_1, we write

$$I_1 = \int_{\log t_1}^{\infty} \frac{e^{ut(e^\lambda)} - 1}{\lambda}\,d\lambda = \sum_{k=1}^{\infty} \frac{u^k}{k!} \int_{\log t_1}^{\infty} \frac{t(e^\lambda)^k}{\lambda}\,d\lambda \doteq \mathscr{H}(g; \log t_1). \tag{2.11}$$

For the integral

$$J(y, h) = \int_{y}^{\infty} \lambda^h e^{-\lambda}\,d\lambda$$

we have

$$J(y, h) = y^h e^{-y} + hJ(y, h-1).$$

Let now $t(p) = p^{-\delta}$ $(0 < \delta \leqq 1)$. Then

$$\int_{\log t_1}^{\infty} \frac{t(e^\lambda)^k}{\lambda}\,d\lambda = \int_{\log t_1}^{\infty} \frac{e^{-\lambda\delta k}}{\lambda}\,d\lambda = J(\delta k \log t_1, -1) < \frac{e^{-\delta k \log t_1}}{\delta k \log t_1},$$

and so

$$\mathscr{H}\left(\frac{1}{p^\delta}; \log t_1\right) \leqq \sum_{k=1}^{\infty} \frac{(ut_1^{-\delta})^k}{k!\,k\delta \log t_1}.$$

Since $ut_1^{-\delta} = \log t_1 - H$, we have

$$I_1 \leqq \frac{4e^{-H}t_1}{\delta(\log t_1)^2}, \tag{2.12}$$

if $H < \frac{1}{2}\log t_1$.

Let now $t(p) = (\log p)^{-\gamma}$, $(\gamma > 0)$. Then, from (2.11),

$$\mathscr{H}\left((\log p)^{-\gamma}; \log t_1\right) = \sum_{k=1}^{\infty} \frac{u^k}{k!} \int_{\log t_1}^{\infty} \lambda^{-k\gamma-1}\,d\lambda =$$

$$= \sum_{k \geqq 1} \frac{\left(u(\log t_1)^{-\gamma}\right)^k}{k!\,(k\gamma+1)} = \sum_{k \geqq 1} \frac{(\log t_1 - H)^k}{k!(k\gamma+1)} \leqq \frac{4e^{-H}t_1}{\gamma \log t_1},$$

if $H < \frac{1}{2}\log t_1$.

So for $t(p)=p^{-\delta}$ $(0<\delta\leqq 1)$

$$\log K_3(u) \leqq Be^{-H}\frac{t_1}{(\log t_1)^2}, \tag{2.13}$$

while for $t(p)=(\log p)^{-\gamma}$ $(\gamma>0)$

$$\log K_3(u) \leqq Be^{-H}\frac{t_1}{\log t_1},$$

B being a constant.

For the sake of brevity we shall write $u_1=\log u$, $u_2=\log u_1$, $u_3=\log u_2$.

Let us first consider the case $t(p)=p^{-1}$. By choosing $H=1$, and collecting our inequalities we have

$$\log K(u) < u\sum_{p\leqq t_0}\frac{1}{p}-t_0+O\left(\frac{t_0}{\log t_0}\right),$$

where

$$t_0=\frac{u}{\log t_0+1},\quad t_1=\frac{u}{\log t_1-1}.$$

Since, from the prime number theorem

$$\sum_{p\leqq t_0}\frac{1}{p}=\log\log t_0+a+O(u_1^{-2}),$$

where

$$a=\gamma-\sum_{k\geqq 2}\sum_{p}\frac{1}{kp^k},$$

(γ being Euler's constant), and observing that

$$\log\log t_0=u_2-\frac{u_2}{u_1}+O(u_2u_1^{-2}),\quad t_0=\frac{u}{u_1}+O(uu_2u_1^{-2}),$$

we get

$$\log K(u) < u\left[u_2+a-\frac{u_2+1}{u_1}\right]+O(uu_2^2u_1^{-2}).$$

So, from (2.3),

$$\log\bigl(1-F(\tau)\bigr)\leqq u\left[u_2+a-\tau-\frac{u_2+1}{u_1}\right]+O(uu_2^2u_1^{-2}).$$

Let u be chosen according to the equation

$$\tau=u_2+a-u_2u_1^{-1}.$$

Then, by an easy calculation, we get

$$\log\bigl(1-F(\tau)\bigr)\leqq -\frac{u}{u_1}+O(uu_2^2u_1^{-2}),$$

$$\mathscr{L}\overset{\text{def}}{=}\log\log\frac{1}{1-F(\tau)}\geqq u_1-u_2+O(u_2^2u_1^{-1}).$$

Since

$$u_1 = e^{\tau-a}+\frac{u_2}{u_1} = e^{\tau-a}\left(1+\frac{u_2}{u_1}+O\left(\frac{u_2^2}{u_1^2}\right)\right) = e^{\tau-a}+u_2+O\left(\frac{u_2^2}{u_1}\right),$$

we have $\mathscr{L} \geqq e^{\tau-a}-c\tau^2 e^{-\tau}$, that is (1.19) holds.

Now we consider the case $t(p)=p^{-\delta}$, $0<\delta<1$. By choosing $H=1$, we have

$$t_0^\delta = \frac{u}{\log t_0+1} < \frac{u}{\log t_1-1} = t_1^\delta,$$

and so $t_1/t_0 \leqq e^2$. Consequently, by (2.3)

$$\log\frac{1}{1-F(\tau)} \geqq \tau u - u\psi(t_0)+t_0+O(t_0/(\log t_0)).$$

Since

$$\psi(t_0) = \sum_{p\leqq t_0} 1/p^\delta = \frac{t_0^{1-\delta}}{(1-\delta)\log t_0}\left(1+O\left(\frac{1}{\log t_0}\right)\right),$$

and $u=t_0^\delta(\log t_0+1)$, we have

$$u\psi(t_0) = \frac{t_0}{1-\delta}\left(1+O\left(\frac{1}{\log t_0}\right)\right),$$

and so

$$\log\frac{1}{1-F(\tau)} \geqq \tau u - \frac{\delta}{1-\delta}t_0+O(t_0/(\log t_0)).$$

By choosing t_0 to satisfy

$$\tau = \frac{t_0^{1-\delta}}{(1-\delta)\log t_0},$$

we have

$$\log\frac{1}{1-F(\tau)} \geqq t_0+O\left(\frac{t_0}{\log t_0}\right) = (\tau\log\tau)^{1/(1-\delta)}\left(1+O\left(\frac{1}{\log\tau}\right)\right),$$

and so (1.20) holds.

To prove (1.21), we observe that

$$\log\frac{1}{1-F(\tau)} \geqq \tau u - \log K(u) \geqq u\tau + t_0 - \frac{ut_0}{(\log t_0)^{\gamma+1}} - \frac{c_4 t_0}{\log t_0}.$$

By choosing $u=(\log\tau)^{\gamma+1}$, we have

$$\log\frac{1}{1-F(\tau)} \geqq \tau(\log\tau)^{\gamma+1} - c_1\tau(\log\tau)^\gamma$$

and this proves (1.21).

Now we shall prove Theorem 4. Let $g(p)=1/p$,

$$g_y(n) = \sum_{\substack{p|n\\ p<y}} g(p); \quad g(y;n) = g(n)-g_y(n).$$

Then

$$\mathcal{S}_\Delta \overset{\text{def}}{=} \frac{1}{x} \# \{n \leqq x | g_{t_0}(n) \geqq \psi(t_0)+\Delta\} \leqq e^{-u(\psi(t_0)+\Delta)} \prod_{p \leqq t_0} \left(1+\frac{e^{ug(p)}-1}{p}\right),$$

where $u=u_{t_0}$ is defined according to (2.4), i.e. $u_{t_0}=t_0(\log t_0+H)$. By using (2.5), we get

$$\log \mathcal{S}_\Delta < -\Delta u - t_0 + O\left(\frac{t_0}{(\log t_0)^c}\right) + \sum_{p \leqq t_0} p e^{-u/p},$$

where c is an arbitrary large constant. Since

$$\sum_{\frac{y}{2}<p<y} p e^{-u/p} < y\pi(y) e^{-u/y} \ll \frac{y^2}{\log y} e^{-u/y},$$

by choosing $y=y_k=\frac{t_0}{2^k}$ $(k=0, 1, 2, \ldots)$, we have

$$\sum_{p \leqq t_0} p e^{-u/p} \ll \frac{t_0^2 e^{-u/t_0}}{\log t_0} = \frac{e^{-H} t_0}{\log t_0}.$$

By choosing $H=c \log\log t_0$, with a fixed c,

$$\log \mathcal{S}_\Delta < -\Delta u_{t_0} - t_0 + B \frac{t_0}{(\log t_0)^c}, \tag{2.14}$$

B being a constant.

Let $u_{t_1}=t_1(\log t_1 - H)$. Then, by choosing $H=c \log\log t_1$,

$$\frac{1}{x} \# \{n \leqq x | g(t_1, n) \geqq R\} \leqq \exp\left(-Ru_{t_1} + B \frac{t_1}{(\log t_1)^{c+2}}\right). \tag{2.15}$$

Let

$$t_0 = t_1 = (\log k)^{1+\varepsilon_k}, \quad \varepsilon_k = \frac{\log\log\log k}{\log\log k};$$

$$f_k^{(1)}(n) = \max_{j=1,\ldots,k} g_{t_0}(n+j); \quad f_k^{(2)}(n) = \max_{j=1,\ldots,k} g(t_0;\, n+j).$$

Let

$$H_k \overset{\text{def}}{=} \psi(t_0) - \log k = \log(1+\varepsilon_k) + O\left(\frac{1}{\log\log k}\right) = \frac{\log\log\log k}{\log\log k} + O\left(\frac{1}{\log\log k}\right).$$

Let k be so large that $H_k < 2\varepsilon_k$. Then, by (2.14),

$$a(x, k, 2\varepsilon_k) \overset{\text{def}}{=} \frac{1}{x} \# \{n \leqq x | f_k^{(1)}(n) \geqq \psi(\log k) + 2\varepsilon_k\} \leqq \tag{2.16}$$

$$\leqq \left(1+\frac{k}{x}\right) \frac{k}{x+k} \# \{n \leqq x+k | g_{t_0}(n) \geqq \psi(t_0)\} \leqq$$

$$\leqq \left(1+\frac{k}{x}\right) k \exp\left(-t_0 + B \frac{t_0}{(\log t_0)^c}\right) \leqq \left(1+\frac{k}{x}\right) k^{-\log\log k + c},$$

c being a constant. Similarly, from (2.15),

$$(2.17)\qquad b(x, k, \varepsilon_k) = \frac{1}{x} \# \{n \leqq x | f_k^{(2)}(n) \geqq \varepsilon_k\} \leqq$$

$$\leqq \left(1+\frac{k}{x}\right) k \exp\left(-\varepsilon_k u_{t_1} + O\left(\frac{t_1}{(\log t_1)^c}\right)\right) \leqq \left(1+\frac{k}{x}\right) k^{-\log\log k}.$$

So for $k \leqq x$ we have

$$(2.18)\qquad \frac{1}{x} \# \{n \leqq x | f_k(n) > \psi(\log k) + 3\varepsilon_k\} < 1/k^3,$$

if k is large. For $k > x$, $n \leqq x$ we have

$$f_k(0) \leqq f_k(n) \leqq f_{k+x}(0) = \psi(\log k) + O\left(\frac{1}{\log k}\right).$$

Hence it follows immediately that

$$\frac{1}{x} \# \{n \leqq x | \exists k > k_0, f_k(n) \geqq \psi(\log k) + 3\varepsilon_k\} < \frac{1}{k_0^2}.$$

By this, Theorem 4 has been proved.

3. Proof of Theorem 7. Suppose that the conditions of Theorem 7 are satisfied. Let $\tilde{g}(n)$ be strongly additive defined for primes by

$$\tilde{g}(p) = \begin{cases} g(p) & \text{if} \quad p > p_\mu \\ 0 & \text{if} \quad p \leqq p_\mu. \end{cases}$$

It is obvious that $g(P_\mu m) = g(P_\mu) + \tilde{g}(m)$. From the Turán—Kubilius inequality

$$\sum_{m \leqq x/P_\mu} \{\tilde{g}(m) - A'\}^2 \ll \frac{x}{P_\mu} \sum_{p > p_\mu} \frac{g^2(p)}{p},$$

if $P_\mu < x$; $A' = A_{x/P_\mu} - A_{p_\mu}$. Hence we get immediately

$$(3.1)\qquad M_B \overset{\text{def}}{=} \# \left\{m \leqq \frac{x}{P_\mu} \,\middle|\, |\tilde{g}(m) - A'| \geqq B\right\} \ll \frac{x}{P_\mu B^2} \sum_{p > p_\mu} \frac{g^2(p)}{p}.$$

If $\tilde{g}(m) - A' \geqq -B$, then

$$g(P_\mu m) = \psi(p_\mu) + \tilde{g}(m) \geqq \psi(p_\mu) + A' - B.$$

So for $P_\mu(m-1) < n < P_\mu m$ we get

$$(3.2)\qquad F_{P_\mu}(n) \geqq g(P_\mu m) - A_{(m+1)P_\mu} \geqq \psi(p_\mu) + A_{x/P_\mu} - A_{(m+1)P_\mu} - A_{p_\mu} - B.$$

Let now $x \to \infty$. For $m \geqq \sqrt{x}$ we have

$$A_{x/P_\mu} - A_{(m+1)p_\mu} \ll \left(\sum \frac{1}{p}\right)^{1/2} \left(\sum \frac{g^2(p)}{p}\right)^{1/2} \to 0 \quad (x \to \infty),$$

where the summation is over the primes in $\left[(m+1)p_\mu, \frac{x}{P_\mu}\right]$. By choosing

$$B_\mu = B = \left(\sum_{p>p_\mu} \frac{g^2(p)}{p}\right)^{1/4}$$

we obtain (1.26) immediately for $k=P_\mu$.

Let now $P_\mu<k<P_{\mu+1}$. To prove (1.26) it is enough to observe that $F_k(n)\geqq F_{P_\mu}(n)$, and that $A_{\log k}-A_{p_\mu}\to 0$ $(k\to\infty)$.

Now we assume that (1.27), (1.28) hold. If $P_\mu\leqq k<P_{\mu+1}$ then, $\psi(\log k)=\psi(p\mu)(1+o(1))=\psi(p_{\mu+1})(1+o(1))$ and $F_{P_{\mu+1}}(n)\geqq F_k(n)\geqq F_{P_\mu}(n)$, and so it is enough to prove (1.29) for $k=P_\mu$. From (1.28) we have

$$M_B \ll \frac{x}{P_\mu B^2}\, t^2(p_\mu)(\log\log p_\mu)^\gamma.$$

From the monotonicity of t we have

$$\frac{t^2(p_\mu)}{\psi^2(p_\mu)} \leqq 1/\mu^2,$$

so by choosing $B=\lambda_\mu\psi(p_\mu)$, $0<\lambda_\mu<1$, we have

$$M_B \ll \frac{x}{P_\mu\lambda_\mu^2}\,\frac{(\log\log\mu)^\gamma}{\mu^2}.$$

Let $x>P_\mu^3$. In the interval $n\in[1, x]$ we drop the n's for which $n\leqq x^{1/2}$. Observing that $A_{p_\mu}=o(\psi(p_\mu))$, and that $A_y-A_{y^\alpha}=O(1)$ $(0<\alpha<1)$, from (3.2) we get that

$$F_{P_\mu}(n) \geqq (1-2\lambda_\mu)\psi(p_\mu)$$

for all but $\frac{x\,(\log\log\mu)^\gamma}{\mu^2\lambda_\mu^2}$ of $n\leqq x$, if λ_μ tends to zero sufficiently slowly. Let $x<P_\mu^3$. Then, for every $n\leqq x$,

$$F_{P_\mu}(n) = \max_{j=1,\dots,P_\mu} \left(g(n+j)-A_{n+j}\right) \geqq \psi(p_\mu)-A_{x+P_\mu}.$$

Since

$$A_{x+P_\mu}-A_{P_\mu} \ll \left(\sum_{p_\mu<p<P_\mu+x}\frac{1}{p}\right)^{1/2}\left(\sum_{p>p_\mu}\frac{t^2(p)}{p}\right)^{1/2} \ll$$

$$\ll t(p_\mu)(\log\log p_\mu)^\gamma(\log p_\mu)^{1/2} \ll \frac{\psi(p_\mu)}{\mu}(\log\log p_\mu)^\gamma(\log p_\mu)^{1/2} = o(\psi(p_\mu)),$$

therefore

$$F_{P_\mu}(n) \geqq (1-2\lambda_\mu)\psi(p_\mu)$$

holds for every n if μ is large. Applying this argument for the sequence $x=2^\nu$, we get the relation:

$$\forall\varepsilon>0\colon \limsup_{k_0\to\infty}\ \sup_{x\geqq 1}\frac{1}{x}\ \#\{n\leqq x \mid \exists k>k_0,\ F_k(n)<(1-\varepsilon)\psi(\log k)\} = 0.$$

To prove the second half of (1.29) we choose $\log\log t_0 = p_\mu^\delta$, where $0<\delta<\gamma$ (see (1.27), (1.28)), and define $g(t_0, n)$, $g_{t_0}(n)$ to be strongly additive satisfying

$$g(t_0;\, p) = \begin{cases} 0 & \text{if} \quad p \leqq t_0, \\ g(p), & \text{if} \quad p > t_0, \end{cases}$$

$$g_{t_0}(n) = g(n) - g(t_0;\, n).$$

Let $A_x^{t_0} = A_x - A_{t_0}$. For every $u \geqq 0$ we have

$$D(x, u) \overset{\text{def}}{=} \sum_{n \leqq x} e^{u(g(t,n) - A_x^{t_0})} \leqq x \prod_{t_0 < p \leqq x} \left(1 + \frac{e^{ug(p)} - 1}{p}\right) e^{-ug(p)/p},$$

whence it follows that

$$\frac{1}{x} \# \{n \leqq x | g(t_0, n) \geqq \Delta\} \leqq \exp\left(-\Delta u + u^2 \sum_{p > t_0} \frac{g^2(p)}{p}\right),$$

if $u = \dfrac{1}{2t(t_0)}$. Let $\Delta = \eta_\mu \psi(p_\mu)$, $\eta_\mu \to 0$ slowly. Then, from (1.27)

$$\Delta u = u \frac{\psi(p_\mu)}{2t(t_0)} > 4p_\mu,$$

if μ is large. Furthermore, from (1.28)

$$\frac{1}{4t^2(t_0)} \sum_{p > t_0} \frac{g^2(p)}{p} \ll (\log\log t_0)^\gamma = p_\mu^{\delta\gamma} = o(p_\mu),$$

since $\delta\gamma < 1$. Consequently

$$\# \{n \leqq x | g(t;\, n) \geqq \eta_\mu \psi(p_\mu)\} \ll x/P_\mu^3. \tag{3.3}$$

Let $C_r(x)$ be the number of those $n \leqq x$, that have at least r prime factors in $[1, t_0]$. We have by Stirling's formula,

$$C_r(x) \leqq x \cdot \frac{1}{r!} \left(\sum_{p < t_0} \frac{1}{p}\right)^r \leqq x \exp\left(-r \log \frac{r}{e(p_\mu^\delta + O(1))} + O(\log r)\right).$$

Let $r = [(1+4\varrho)\mu]$, ϱ being a small positive constant. Then,

$$r \log \frac{r}{e(p_\mu^\delta + O(1))} \geqq (1+4\varrho)(1-2\delta)\, p_\mu \geqq (1+2\varrho)\, p_\mu,$$

if δ is small enough. Consequently

$$C_r(x) \ll \frac{x}{P_\mu^{1+\varrho}}.$$

Let n be a such number that has $s(>\mu)$ prime factors in $[1, t_0]$. From the monotonicity of $t(y)$ we get

$$g_{t_0}(n) \leqq g(p_1 \dots p_s) \leqq \psi(p_\mu) + (s-\mu)\, t(p_\mu) \leqq \left(\frac{s}{\mu} - 1\right) \psi(p_\mu).$$

So, if $g_{t_0}(n)\geqq(1+4\varrho)\psi(p_\mu)$, then $s\geqq r$. Consequently

$$\# \{n \leqq x | g_{t_0}(n) > (1+4\varrho)\psi(p_\mu)\} \ll \frac{x}{P_\mu^{1+\varrho}}. \tag{3.4}$$

From (3.3) and (3.4) we get immediately that

$$\# \{n \leqq x | \max_{j=1,\dots,k} g(n+j) > (1+5\varrho)\psi(p_\mu)\} \ll \frac{x}{P_\mu^{\varrho}},$$

if $P_\mu < x$.

For $P_\mu > x$ we have

$$F_{P_\mu}(n) \leqq \max_{n \leqq x+P_\mu} g(n) \leqq \psi(p_{\mu+1}) = \psi(p_\mu)+o(1).$$

Applying this estimation for $x=2^\nu$ $(\nu=1, 2, \dots)$ and summing up for $\mu\geqq\mu_0$, we have

$$\sup_{x\geqq 1} \frac{1}{x} \{n \leqq x | \exists \mu > \mu_0, F_{P_\mu}(n) > (1+5\varrho)\psi(p_\mu)\} \ll \frac{1}{P_{\mu_0}^{\varrho}}.$$

By this we proved (1.29).

4. Proof of Theorem 1′ and Theorem 2. To prove Theorem 1′ we suppose that (1.11) holds. From the existence of the distribution function $F(x)$,

$$\sum_p \frac{\min\left(1, g(p)\right)}{p} < \infty.$$

Let $\delta>0$ be fixed, $\mathscr{P}_k$ be the set of those primes p, for which

$$(1+\delta)f_k(0) \leqq g(p) < (1+\delta)f_{2k}(0)$$

holds. Then

$$\sum_{p\in\mathscr{P}_k} 1/p < \infty,$$

if $f_k(0)\neq 0$. Let $b(n)=(n+1)\dots(n+k)$; $R_k=\prod_{p\in\mathscr{P}_k} p$.

From (1.11),

$$\sum_{\substack{n\leqq x \\ (b(n), R_k)=1}} 1 \geqq (1-\varepsilon)x,$$

if $k>k_0(\delta, \varepsilon)$. Since $1-F\left(f_k(0)\right)\geqq 1/k$ for every k, from (1.11) it follows that

$$f_{\nu k}(0) \leqq (1+\varepsilon)f_k(0)$$

for every fixed ν, if k is large. So $f_k(0)=O(k^\varepsilon)$ and for $p\in\mathscr{P}_k$ we have $p/k\to\infty$ $(k\to\infty)$. Consequently

$$\prod_{p\in\mathscr{P}_k}\left(1-\frac{k}{p}\right) > 1-\varepsilon,$$

and

$$\sum_{p\in\mathscr{P}_k} \frac{k}{p} < 2\varepsilon,$$

if k is sufficiently large.

So we have

$$\sum_{g(p)>(1+\delta)f_k(0)} \frac{g(p)^r}{p} < \sum_{2^\nu \geqq k_0} \frac{\varepsilon(1+\delta)^r f_{2^\nu}^r(0)}{2^\nu} \ll \sum \frac{2^{\varepsilon\nu}}{2^\nu} < \infty,$$

and Theorem 1′ has been proved.

The proof of Theorem 2 is almost the same. We need to observe only that from (1.13)

$$f_k(0) = o(\log k) \tag{4.1}$$

follows. Since for fixed ν

$$\nu k\big(1-F(f_{\nu k}(0))\big) \geqq 1,$$

and

$$\nu k\big(1-F(f_k(0)+A)\big) \to 0 \quad (k \to \infty),$$

therefore $f_{\nu k}(0)<f_k(0)+A$ if k is large, that implies (4.1).

5. Proof of Theorem 3. Let $L(k)\nearrow\infty$ be given. We can give $L_1(k)\nearrow\infty$, so that $L_1(k)\leqq L(k)$, $L_1(k+k^2)\leqq 2L_1(k)$, $L_1(k)$ has integer values with jump 1. It is enough to prove our theorem for $L_1(k)$ instead of $L(k)$.

Let $\mathcal{P}=\{q_1<q_2<\dots\}$ be a rare sequence of primes. We shall define $g(n)$ so that $g(q_i)\nearrow\infty$, and $g(p)=0$ for $p\notin\mathcal{P}$.

Let B_k be a sequence tending to infinity monotonically, $\mathcal{P}$ be so rare and the increase of $g(q_i)$ so slow that

$$\sum_{q_i>k} \frac{g(q_i)}{q_i} < \frac{B_k}{k}, \tag{i}$$

$$g\Big(\prod_{q_i\leqq k} q_i\Big) \leqq \frac{1}{4} L_1(k) \tag{ii}$$

hold for every $k\geqq 1$.

So $f_k(0)\leqq \frac{1}{4}L_1(k)$ for every $k\geqq 1$. Let $g_1(n), g_2(n)$ be strongly additive defined for primes as

$$g_1(p) = \begin{cases} 0, & p>k, \\ g(p), & p\leqq k, \end{cases}$$

$$g_2(p) = g(p)-g_1(p), \quad f_k^{(i)}(n) = \max_{j=1,\dots,k} g_i(n+j).$$

It is obvious that

$$f_k^{(1)}(n) \leqq g\Big(\prod_{q_i\leqq k} q_i\Big) \leqq \frac{1}{4}L_1(k).$$

Furthermore

$$\sum_{n\leqq x} f_k^{(2)}(n) \leqq k \sum_{n\leqq x+k} g_2(n) \leqq k \sum_{q_i>k} g(q_i)\frac{x+k}{q_i},$$

and so for $x>k$,

$$\frac{1}{x} \sum_{\substack{n\leqq x \\ f_k^{(2)}(n)>C_k}} 1 \leqq \frac{1}{C_k}\sum_{n\leqq x} f_k^{(2)}(n) \leqq 2\frac{k}{C_k}\sum_{q_i>k}\frac{g(q_i)}{q_i} < \frac{2B_k}{C_k} (=\varrho_k).$$

Let $C_k=\frac{1}{4}L_1(k)$, $B_k=\frac{1}{8}\cdot\sqrt{L_1(k)}$. Then $\varrho_k=(\sqrt{L_1(k)})^{-1}$.

Since, for $k\geqq x$, $n\leqq x$,

$$f_k(n)\leqq f_{k+x}(0)\leqq\frac{1}{4}L_1(k+x)\leqq\frac{1}{4}L_1(2k)\leqq\frac{1}{2}L_1(k).$$

Since $f_k(n)\leqq f_k^{(1)}(n)+f_k^{(2)}(n)$, therefore

$$\sup_{x\geqq 1}\frac{1}{x}\#\left\{n\leqq x|f_k(n)>\frac{1}{2}L_1(k)\right\}\leqq\varrho_k.$$

Let now k_0 be fixed, the sequence $k_1<k_2<\dots$ be defined by

$$k_\nu=\min_{L_1(k)=2L_1(k_{\nu-1})}k.$$

It is clear that

$$\lambda(k_0)=\sum_{\nu=0}^{\infty}\varrho_{k_\nu}<\frac{c}{\sqrt{L_1(k_0)}},$$

$\lambda(k_0)\to 0$ $(k_0\to\infty)$.

Applying this argument for $x=2^\mu$ $(\mu=0,1,2,\dots)$ we deduce that

$$\sup_{x\geqq 1}\frac{1}{x}\#\left\{n\leqq x|\exists\nu: f_{k_\nu}(n)>\frac{1}{2}L_1(k)\right\}\leqq\lambda(k_0).$$

Let now n be such a number for which $f_{k_\nu}(n)<\frac{1}{2}L_1(k_\nu)$ $(\nu=0,1,2,\dots)$ holds. Then for every $k\in(k_{\nu-1},k_\nu)$,

$$f_k(n)\leqq f_{k_\nu}(n)\leqq\frac{1}{2}L_1(k_\nu)=L_1(k_{\nu-1})\leqq L_1(k).$$

This finishes the proof of Theorem 3.

6. Proof of Theorem 5. Let $\varepsilon>0$ and t be given, $\mathscr{P}_1,\mathscr{P}_2,\mathscr{P}_3$ be the set of primes in the intervals $[1,(1-\varepsilon)t]$, $((1-\varepsilon)t, t,]$ $(t,(1+\varepsilon)t,]$ P_i be the product of the elements $\mathscr{P}_i$, i.e.

$$P_i=\prod_{p\in\mathscr{P}_i}p.$$

Let r,s be natural numbers. In this section $b_r, b_r^{(j)}$, $j=1,2,\dots,$ denote a number that is a product of r distinct elements of $\mathscr{P}_2$. Similarily $c_s, c_s^{(1)}, c_s^{(2)},\dots$ denote such numbers that are the product of s distinct primes from $\mathscr{P}_3$. Let H and K be the number of elements in $\mathscr{P}_2$, and in $\mathscr{P}_3$, respectively.

Then the number of b_r's is $\binom{H}{r}$, and the number of c_s's is $\binom{K}{s}$.

From the prime number theorem

$$H=\frac{\varepsilon t}{\log t}+O\left(\frac{t}{(\log t)^2}\right),\quad K=\frac{\varepsilon t}{\log t}+O\left(\frac{t}{(\log t)^2}\right). \tag{6.1}$$

Let $\mathscr{A}$ be the set of those integers that have the form $n=\frac{P_2}{b_r}m$, where $(m, P_2)=1$, and that contains at least s prime factors from $\mathscr{P}_3$. Let

$$F(n) = \sum_{c_s|m} 1,$$

if $n\in\mathscr{A}$, and $F(n)=0$ otherwise.

Let $0<\delta<1$, $r=[t^\delta]$, $s=[cr]$, $c>1$ being a constant.

To prove our theorem we shall deduce a Turán—Kubilius' type inequality for the sum

$$\mathscr{E}(y) \overset{\text{def}}{=} \sum_{n\leqq y}\left[\sum_{i=1}^{P_2} F(n+i)-A\right]^2, \tag{6.1}$$

where

$$A = \left(\sum b_r\right)\left(\sum 1/c_s\right). \tag{6.2}$$

For the sake of simplicity we shall assume that r, s, t are large but temporarily fixed numbers, $y\to\infty$.

Let

$$S(y, i) = \sum_{n\leqq y} F(n)F(n+i). \tag{6.3}$$

Squaring out (6.1) we get easily that

$$\mathscr{E}(y) = \sum_{i=1}^{P_2} 2(P_2-i)S(y, i)+P_2\sum_{n\leqq y}F^2(n)-2AP_2\sum_{n\leqq y}F(n)+ \tag{6.4}$$

$$+A^2y+O(P_2^3y^{1/10}) =$$

$$=\sum{}^{(1)}+P_2\sum{}^{(2)}-2AP_2\sum{}^{(3)}+A^2y+O(P_2^3y^{1/10}).$$

We shall use Eratosthenian sieve for some primes in $\mathscr{P}_2$. We note that

$$\prod_{p\in\mathscr{P}_2}\left(1-\frac{\gamma(p)}{p}\right) = 1+O\left(\frac{\varepsilon}{\log t}\right)\quad (t\to\infty)$$

if $\gamma(p)$ is bounded by an absolute constant.

Then

$$H(z) = \sum_{\substack{n\leqq z\\(n, P_2)=1}} 1 = z\prod_{p\in\mathscr{P}_2}(1-1/p)+O(2^H).$$

Consequently

$$\sum{}^{(3)} = \sum_{b_r}\sum_{\substack{m\leqq \frac{b_r y}{P_2}\\(m, P_2)=1}}\sum_{c_s|m} 1 = \sum_{b_r, c_s} H\left(\frac{b_r y}{P_2 c_s}\right) = \frac{1}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)Ay+O_t(1), \tag{6.5}$$

where t in the order term denotes that the constant involved may depend on t.

We shall give an upper estimate for $\sum^{(2)}$. We have

$$\sum{}^{(2)} = \sum_{b_r}\sum_{c_s^{(1)}, c_s^{(2)}}\sum_{n\leqq\frac{b_r y}{P_2[c_s^{(1)}, c_s^{(2)}]}} 1 \leqq B\frac{y}{P_2}\left(\sum b_r\right), \tag{6.6}$$

where

$$B=\sum\frac{1}{[c_s^{(1)},c_s^{(2)}]}. \tag{6.7}$$

Let ε_μ be a fixed product of μ prime factors from $\mathcal{P}_3$. The equation $\varepsilon_\mu=(c_s^{(1)},c_s^{(2)})$ has

$$\binom{K-\mu}{2(s-\mu)}\binom{2(s-\mu)}{s-\mu}$$

solutions. For all of them $[c_s^{(1)},c_s^{(2)}]\geqq t^{2s-\mu}$ holds. ε_μ can be chosen $\binom{K}{\mu}$ times Consequently

$$B\leqq\sum_{\mu=0}^{s}t^{\mu-2s}\binom{K}{\mu}\binom{K-\mu}{2(s-\mu)}\binom{2(s-\mu)}{s-\mu}. \tag{6.8}$$

Furthermore it is obvious that

$$\sum b_r\leqq t^r\binom{H}{r}.$$

So by the Stirling formula

$$\sum b_r<\frac{(tH)^r}{r!}<\exp\left(2r\log t-r\delta\log t+O(r)\right)=\exp\left((2-\delta)r\log t+O(r)\right).$$

Similarly, from (6.8),

$$B<\sum_{\mu=0}^{s}\frac{K^{2s-\mu}}{t^{2s-\mu}\mu!\,(s-\mu)!^2}<\sum_{\mu=0}^{s}\frac{1}{\mu!\,(s-\mu)!^2}<\exp\left(-s\delta\log t+O(r)\right).$$

Consequently

$$\sum\nolimits^{(2)}\leqq\frac{y}{P_2}\exp\left([(2-\delta)r-\delta s]\log t+O(r)\right). \tag{6.9}$$

Now we estimate A. Counting the b_r's and c_s's we have

$$t^{r-s}\binom{H}{r}\binom{K}{s}\geqq A\geqq\frac{(1-\varepsilon)^r}{(1+\varepsilon)^s}\cdot t^{r-s}\binom{H}{r}\binom{K}{s}.$$

Since

$$\frac{(H-r)^r}{r!}<\binom{H}{r}<\frac{H^r}{r!},$$

from the Stirling formula we deduce easily that

$$\log A=(r-s)\log t+r\log H+O\left(\frac{r^2}{H}\right)+s\log K+O\left(\frac{s^2}{K}\right)-r\log r-s\log s+O(r),$$

and so by (6.1) that

$$\log A=[2r-(r+s)\delta]\log t+O(r\log\log t). \tag{6.10}$$

We choose c $(s=[cr])$ so that

$$\alpha = 2-(1+c)c > 0. \tag{6.11}$$

This guarantees that $A \gg 1$.

Let now consider the sum

$$\sum_B = \sum_{\Delta > P_2} \frac{b_r^{(1)} b_r^{(2)}}{c_s^{(1)} c_s^{(2)}}, \tag{6.12}$$

where

$$\Delta = \frac{P_2(c_s^{(1)}, c_s^{(2)})}{[b_r^{(1)}, b_r^{(2)}]}.$$

The condition $\Delta > P_2$ implies that $(c_s^{(1)}, c_s^{(2)}) \geqq [b_r^{(1)}, b_r^{(2)}]$.

Let $\delta_l, \varepsilon_\mu$ be fixed, where the index denotes the number of its prime divisors, and consider those $b_r^{(1)}, b_r^{(2)}, c_s^{(1)}, c_s^{(2)}$ for which $\delta_l = (b_r^{(1)}, b_r^{(2)})$, $\varepsilon_\mu = (c_s^{(1)}, c_s^{(2)})$. If $\Delta > P_2$, then

$$\{(1+\varepsilon)t\}^\mu \geqq \{(1-\varepsilon)t\}^{2r-l},$$

i.e.

$$\frac{1}{(1-\varepsilon)^{2r-(l+\mu)}} \geqq \frac{(1+\varepsilon)^\mu}{(1-\varepsilon)^{2r-l}} \geqq t^{2r-(l+\mu)},$$

whence

$$1 \geqq [(1-\varepsilon)t]^{2r-(l+\mu)},$$

i.e. $l+\mu \geqq 2r$.

For fixed l and μ the number of $b_r^{(1)}, b_r^{(2)}, c_s^{(1)}, c_s^{(2)}$ that satisfy $\omega\big((b_r^{(1)}, b_r^{(2)})\big) = l$, $\omega([c_s^{(1)}, c_s^{(2)}]) = \mu$ is

$$\binom{H}{l}\binom{H-l}{2(r-l)}\binom{2(r-l)}{r-l}\binom{K}{\mu}\binom{K-\mu}{2(s-\mu)}\binom{2(s-\mu)}{s-\mu} \leqq \frac{H^{r-l}}{l!(r-l)!^2}\cdot\frac{K^{s-\mu}}{\mu!(s-\mu)!^2}.$$

Since $\dfrac{b_r^{(1)} b_r^{(2)}}{c_s^{(1)} c_s^{(2)}} \leqq t^{2(r-s)}$ and $H<t, K<t$, therefore

$$\sum_B \ll t^{2(r-s)} \sum_{l+\mu \geqq 2r} \frac{t^{r+s-l-\mu}}{l!(r-l)!^2 \mu!(s-\mu)!^2} \ll t^{r-s+1}. \tag{6.13}$$

Consider now

$$\sum_C = \left(\sum (b_r^{(1)}, b_r^{(2)})\right)\left(\sum \frac{1}{[c_s^{(1)}, c_s^{(2)}]}\right). \tag{6.14}$$

Arguing as before, we have

$$\sum_C \leqq \left\{H^r \sum_{l=0}^{r} \frac{(t/H)^l}{l!(r-l)!^2}\right\}\left\{\sum_{\mu=0}^{s} \frac{(K/t)^{2s-\mu}}{\mu!(s-\mu)!^2}\right\} = \sum^{(b)} \cdot \sum^{(c)}.$$

By Stirling's formula

$$\frac{1}{l!(r-l)!^2} < \exp\big(-g(l) + O(\log r)\big),$$

where

$$g(l) = l\log l + 2(r-l)\log(r-l) - 2r + l.$$

By differentiating, we see that the smallest value is achieved at $l=l_0$, where l_0 is the solution of $l_0=(r-l_0)^2$. We have easily that

$$g(l_0)=r\log l_0-r+O(\sqrt{r})=r\delta\log t-r+O(\sqrt{r}).$$

Since $H^r(t/H)^l\leqq t^r$,

$$\sum\nolimits^{(b)}<\exp\left(r(1-\delta)\log t-r+O(\sqrt{r})\right).$$

We have similarly that

$$\sum\nolimits^{(c)}<\exp\left(-s\delta\log t+O(s\log\log t)\right).$$

Consequently

$$\sum\nolimits_C<\exp\left([r-\delta(r+s)]\log t+O(s\log\log t)\right). \tag{6.15}$$

Let now consider the sum $S(y,i)$. This is equal to the number of solutions of the equation

$$\frac{P_2}{b_r^{(2)}}c_s^{(2)}v-\frac{P_2}{b_r^{(1)}}c_s^{(1)}u=i,\quad \frac{P_2}{b_r^{(1)}}c_s^{(1)}u\leqq y, \tag{6.16}$$

$(uv,P_2)=1$; in variable $b_r^{(1)},b_r^{(2)},c_s^{(1)},c_s^{(2)},u,v$. Let $b_r^{(j)},c_s^{(j)}$ $(j=1,2)$ be fixed; $\delta=(b_r^{(1)},b_r^{(2)})$; $\varepsilon=(c_s^{(1)},c_s^{(2)})$; $\xi^{(j)},f^{(j)},\Delta$ $(j=1,2)$ be defined by

$$c_s^{(j)}=\xi^{(j)}\varepsilon,\quad \delta f^{(j)}=b_r^{(j)};\quad \Delta=\frac{P_2}{[b_r^{(1)},b_r^{(2)}]}(c_s^{(1)},c_s^{(2)}).$$

If (6.16) has a solution, then $\Delta|i$. Let $i=\Delta i_1$. Dividing by Δ we reduce (6.16) to

$$\xi^{(2)}f^{(1)}v-\xi^{(1)}f^{(2)}u=i_1,\quad (uv,P_2)=1. \tag{6.17}$$

It has a solution if and only if $(i_1,\xi^{(2)}\xi^{(1)})=1$. The solutions u,v are of the forms

$$u=u_0+l\xi^{(2)}f^{(1)},\quad v=v_0+l\xi^{(1)}f^{(2)}\quad (l=0,1,2,\dots).$$

To enumerate the l's for which $(uv,P_2)=1$, we sieve for primes $p\in\mathscr{P}_2$. Since the number $\gamma(p)$ of the solution of $uv\equiv 0\pmod p$ is 1 or 2, we get

$$\prod_{p|P_2}\left(1-\frac{\gamma(p)}{p}\right)=1+O\left(\frac{\varepsilon}{\log t}\right).$$

On the previous assumptions (6.16) has

$$\frac{y(b_r^{(1)},b_r^{(2)})}{P_2[c_s^{(1)},c_s^{(2)}]}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)+O_t(1)$$

solutions. O_t denotes that the constant involved by the order term may depend on t.

Hence we have

$$\sum\nolimits^*\overset{\text{def}}{=}\sum_{i=1}^{P_2}S(y,i)=\frac{y}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)\sum\frac{(b_r^{(1)},b_r^{(2)})}{[c_s^{(1)},c_s^{(2)}]}\cdot\sum_{\substack{i_1\leqq P_2/\Delta\\(i_1,\xi^{(1)}\xi^{(2)})=1}}1+O_t(1). \tag{6.18}$$

Since

$$\sum_{\substack{i_1 \leqq P_2/\Delta \\ (i_1, \xi^{(1)} \xi^{(2)})=1}} 1 = \begin{cases} \frac{P_2}{\Delta}\left(1+O\left(\frac{r}{t}\right)\right)+O(1), & \text{if } \Delta \leqq P_2, \\ 0, & \text{if } \Delta > P_2, \end{cases}$$

and $\frac{r}{t} \ll \frac{\varepsilon}{\log t}$ as $t \to \infty$, we have

$$\sum{}^* = \frac{y}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)(A^2 - \sum{}_B) + O\left(\frac{y}{P_2}\sum{}_C\right) + O_t(1),$$

i.e.

$$\sum{}^* = \frac{y}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)A^2 + O\left(\frac{y}{P_2}(\sum{}_B + \sum{}_C)\right) + O_t(1). \tag{6.19}$$

Similarly, for the sum

$$\sum{}^{**} \stackrel{\text{def}}{=} \sum_{i=1}^{P_2} iS(y, i) \tag{6.20}$$

we have

$$\sum{}^{**} = \frac{y}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)\sum \frac{(b_r^{(1)}, b_r^{(2)})}{[c_s^{(1)}, c_s^{(2)}]} \cdot \Delta\{\sum_{\substack{i_1 \leqq P_2/\Delta \\ (i_1, \xi^{(1)} \xi^{(2)})=1}}\}.$$

Since

$$\sum_{\substack{i_1 \leqq P_2/\Delta \\ (i_1, \xi^{(1)} \xi^{(2)})=1}} i_1 = \frac{P_2^2}{2\Delta^2}\left(1+O\left(\frac{r}{t}\right)\right) + O\left(\frac{P_2}{\Delta}\right)$$

for $\Delta \leqq P_2$, we have, as earlier

$$\sum{}^{**} = \frac{y}{2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)A^2 + O(y(\sum{}_B + \sum{}_C)) + O_t(1).$$

Consequently for $\sum^{(1)}$ defined in (6.4) we have

$$\sum{}^{(1)} = 2(P_2 \sum{}^* - \sum{}^{**}) = y\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)A^2 + O(y(\sum{}_B + \sum{}_C)) + O_t(1). \tag{6.21}$$

So, by (6.21) and (6.5) we have

$$\mathscr{E}(y) \leqq B_1 \frac{\varepsilon}{\log t} A^2 y + B_2 y(\sum{}_B + \sum{}_C) + O(P_2 \sum{}_2) + O_t(1),$$

where B_1, B_2 are absolute constants. Now by (6.10), (6.13), (6.15) we get

$$\sum{}_C < t^{-r/2} A, \quad \sum{}_B < 1.$$

From (6.9) $P_2 \sum_2 \ll Ae^{O(r)}$, and so from (6.10), (6.11),

$$Ae^{O(r)} \ll \frac{\varepsilon}{\log t} A^2.$$

Consequently

$$\mathscr{E}(y) \leqq B \frac{\varepsilon}{\log t} A^2 y + O_t(1). \tag{6.22}$$

Let $M(y)$ be the number of $n \leqq y$, for which no one of $n+1, \dots, n+P_2$ is belonging to $\mathscr{A}$. Then, from (6.22)

$$M(y) \leqq B \frac{\varepsilon}{\log t} y + O_t(1). \tag{6.23}$$

Since

$$\{P_1(n+1), \dots, P_1(n+P_2)\} \subseteqq \{P_1 n+1, \dots, P_1 n + P_1 P_2\},$$

we have immediately the following assertion.

THEOREM 8. *Let* $\varepsilon > 0$, $0 < \delta < 1$, c *be fixed so that*

$$\alpha \overset{\text{def}}{=} 2 - (1+c)\delta > 0,$$

t *a large constant;* $r = [t^\delta]$, $s = [ct^\delta]$. *Let* $\mathscr{B}$ *be the set of those integers* n *for which there exist* b_r *and* c_s *so that*

$$n \equiv 0 \left(\bmod \frac{P_1 P_2}{b_r} c_s \right).$$

Let

$$N(x) = \# \{ n \leqq x \mid \{n+1, \dots, n+P_1 P_2\} \cap \mathscr{B} = \varnothing \}.$$

Then

$$\overline{\lim_x} \frac{N(x)}{x} \leqq B \frac{\varepsilon}{\log t},$$

where B *is an absolute constant.*

Hence we deduce easily Theorem 5. Indeed, if $n \equiv 0 \left(\frac{P_1 P_2}{b_r} c_s \right)$, then

$$g(n) \geqq g(P_1 P_2) + g(c_s) - g(b_r).$$

Let $g(p) = p^{-\delta}$. By choosing $r = [t^\gamma]$, $s = [ct^\gamma]$, $\gamma < 1$,

$$g(c_s) - g(b_r) \geqq \frac{s}{[(1+\varepsilon)t]^\delta} - \frac{r}{[(1-\varepsilon)t)^\delta} \geqq t^{\gamma-\delta} \left\{ \frac{c}{1+\varepsilon} - \frac{1}{1-\varepsilon} \right\} > c_1 t^{\gamma-\delta}$$

$(c_1 > 0$ constant)

if ε is sufficiently small.

Let $P_1 P_2 = p_1 \dots p_\mu \leqq k < P_1 P_2 p_{\mu+1}$. Then $f_k(0) = g(P_1 P_2)$. If we put $t = p_\mu$, we get immediately Theorem 5.

Reference

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