ON THE CHROMATIC NUMBER OF GEOMETRIC GRAPHS

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Abstract

Let S be a finite or infinite set in the Euclidean space $\mathbb{E}^h$. We definte the graph $G(S)$ on the vertex-set S by joining $x,y \in S$ iff $\rho(x,y)$ / = their distance / is 1. In this paper we investigate various chromatic properties and the dimension of such graphs. Thus, for example, $\chi_e^{(\mathbb{E}^h)}$ will be defined as the maximum t such that if $G^n = G(S)$, $S \subseteq \mathbb{E}^h$, then one can omit $o(n^2)$ edges so that the remaining graph be $\le t$-chromatic. The dependence of $\chi_e^{(\mathbb{E}^h)}$ on h will be investigated among other related questions.

I. *Introduction*. Let S be a finite or infinite metric space. We define the graph $G(S)$ as follows: the vertex set is S and $x,y \in S$ are jointed iff their distance $\rho(x,y) = 1$. Many interesting questions can be asked and were investigated in connection with the graph theoretical properties of such graphs. The results of this type can be interesting in themselves and on the other hand they give information on the metric of S. In the introduction we list some of the known results and open problems but first we fix a few standard notations.

The graphs considered here will have no loops or multiple edges. G^n, H^n, ... will denote graphs with n vertices, and if G is a graph, $E(G)$, $e(G)$, $V(G)$ and $v(G)$ will denote the set of edges, number of edges,

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set of vertices, number of vertices respectively. The chromatic number of G is $\chi(G)$. K_p is a complete p-graph, $K_p(n_1,\ldots,n_p)$ is the complete p-partite graph with n_i vertices in its ith class.

Problem 1. Let $\mathbb{E}^h$ be the h-dimensional Euclidean space, and $S \subseteq \mathbb{E}^h$ be an n-element set. How large can $e(G(S))$ be (as a function of n)?

Erdős gave sufficiently sharp answer to Problem 1 if $h \geq 4$, but the results for $h = 2,3$ are far from being satisfactory. For example, if $h = 2$, Erdős [4] proved that

$$e(G(S)) = O(|S|^{\frac{3}{2}}) \tag{1}$$

and it took great efforts for Józsa and Szemerédi [12] to push this estimate down to

$$e(G(S)) = o(|S|^{\frac{3}{2}}) \quad (|S| \to \infty) \tag{2}$$

while probably even

$$e(G(S)) = O(|S|^{1+\varepsilon})$$

holds for every $\varepsilon > 0$.

For a metric space S $\chi(G(S))$ will be abbreviated by $\chi(S)$. Hadwiger [11] and Nelson (see [11]), independently asked for the determination of $\chi(\mathbb{E}^h)$.

Problem 2. How large is $\chi(\mathbb{E}^h)$?
(By the de Bruijn-Erdős theorem, [3], $\chi(\mathbb{E}^h) = \max(\chi(S): S \subset \mathbb{E}^h$, S is finite) !).

Klee proved the finiteness of $\chi(\mathbb{E}^h)$ for each h (easy!), Larman and Rogers [16] proved that

$$\chi(\mathbb{E}^h) \leq (3+o(1))^h \quad (h \to \infty). \tag{3}$$

It was conjectured that

(4) $$\chi(\mathbb{E}^h) \geq (1+c)^h$$

for some constant $c > 0$ but the best lower bound (due to P. Frankl [10]) is much weaker. It states that for every γ

(5) $$\chi(E^h)/h^\gamma \to \infty \quad (h\to\infty).$$

It is surprising that even for $h = 2$ $\chi(\mathbb{E}^h)$ is unknown. Hadwiger [11], L. and W. Moser [15] and Woodall [17] proved that

(6) $$4 \leq \chi(\mathbb{E}^2) \leq 7.$$

Another notion connected with geometric graphs is the dimension ($\dim(G)$) of a graph G, introduced by Erdős, Harary and Tutte [9]. The dimension of G is the minimum h such that G can be embedded into $\mathbb{E}^h$ so that for each edge the two end points have distance 1. One can easily see [9] that

(7) $$\dim(G) \leq 2\chi(G).$$

To prove (7) we may choose a $K_d(m,\dots,m) \supseteq G$ for $d = \chi(G)$ and prove (7) for this graph:

(7*) $$\dim(K_d(m,\dots,m)) \leq 2d.$$

Indeed, (7*) immediately implies (7). To prove (7*) put

(8) $$C_i = \{(x_1,\dots,x_{2d}) : x_{2i-1}^2 + x_{2i}^2 = \tfrac{1}{2},\ x_j = 0 \ \text{ if } \ j \neq 2i-1, 2i\}.$$

Clearly, if $\underline{x} \in C_i$, $y \in C_j$, then $\rho(\underline{x},\underline{y}) = 1$, i.e., putting m vertices (of $K_d(m,\dots,m)$) onto C_i we embedded $K_d(m,\dots,m)$ into $\mathbb{E}^{2d}$. We shall refer to this embedding as to Lenz' construction [5].

Before turning to the new results we would like to show that in some sense the problems posed above cover more than what one would think.

First of all, if $\varepsilon > 0$ is a small positive constant and S^{h-1} is the sphere of diameter $1+\varepsilon$ in $\mathbb{E}^h$, then a famous theorem of Borsuk [2] asserts exactly that

$$\chi(S) = h + 1.$$

Another question was the longstanding Kneser conjecture, finally proved by Lovász [14] (whose proof was simplified by Bárány [1]).

Kneser conjecture [13]. Let G^N be a graph, the vertices of which are the $\binom{2n+\ell}{n} = N$ n-tuples of a $(2n+\ell)$-element set U and two vertices (= n-tuples) $A \subseteq U$, $B \subseteq U$ are jointed if $A \cap B = \emptyset$. Prove that $\chi(G^N) = \ell + 2$.

If we consider all the n-tuples of U ($|U| = 2n+\ell$) and introduce the metric $\rho(A,B) = \frac{1}{2n} |\Delta(A,B)|$, where $\Delta(A,B)$ is the symmetric difference, $|\Delta(A,B)|$ is its cardinality, then Lovász Theorem asserts that for the above metric space S of N points $\chi(S) = \ell + 2$.

2. *Main Results.* As we stated in the previous chapter, if G can be embedded into the plane $\mathbb{E}^2$, then $\chi(G) \leq 7$. On the other hand, there are graphs G in the plane with $\chi(G) \geq 4$. The next theorem shows that the high chromatic number is not typical in $\mathbb{E}^2$, not even in $\mathbb{E}^4$!

THEOREM 1. *Let $S \subseteq \mathbb{E}^4$ be a set of n points, $G^n = G(S)$. One can omit $O(n^{\frac{7}{4}})$ edges from G^n so that the obtained graph is bipartite.*

The above theorem motivates the following definition:

Definition 1. If U is a metric space with infinitely many points, $\chi_e(U)$ - the "essential chromatic number" of U is the minimum t such that for any n-element subset $S \subseteq U$ we can omit $o(n^2)$ edges from $G(S)$ so that the obtained graph G^n is $\leq t$ chromatic (as $n \to \infty$).

Remark 1. As we mentioned, $\chi(\mathbb{E}^h) < \infty$, and obviously,

(9) $$\chi_e(U) \le \chi(U),$$

thus $\chi_e(\mathbb{E}^h) < \infty$. On the other hand, we have seen in the introduction that $K_d(\frac{n}{d},\ldots,\frac{n}{d})$ can be embedded into $\mathbb{E}^{2d}$. Obviously, one must omit at least $\approx \frac{n^2}{d^2}$ edges from $K_d([\frac{n}{d}],\ldots,[\frac{n}{d}])$ to decrease its chromatic number, thus $\chi_e(\mathbb{E}^h) \ge [\frac{h}{2}] = d$. This shows that $\chi_e(\mathbb{E}^h) \to \infty$ as $h \to \infty$.

THEOREM 2. *For $h \ge 2$, $\chi_e(\mathbb{E}^h) \ge h - 2$.*

Remark 2. By Theorem 1 $\chi_e(\mathbb{E}^4) = 2$. Theorem 3 (below) implies that $\chi_e(\mathbb{E}^3) = 1$, while $\chi_e(\mathbb{E}^2) = \chi_e(\mathbb{E}^1) = 1$ is trivial.

Conjecture 1. There exists a constant $q > 1$ such that

$$\chi_e(\mathbb{E}^h) \ge q^h.$$

We shall give the motivations of this conjecture later.

To formulate our next theorem we need the following definition, partly motivated by the Lenz construction.

Definition 2. Let $P_1,\ldots,P_m$ be 2-dimensional subspaces of $\mathbb{E}^h$, i.e. planes going through the origin. Let $\hat{G}(P_1,\ldots,P_m)$ be the graph whose vertices are $P_1,\ldots,P_m$ and P_i and P_j are joined iff $P_i \perp P_j$. The orthogonal chromatic number $\chi_\perp(\mathbb{E}^h)$ is defined as $\max \chi(\hat{G}(P_1,\ldots P_m))$ for all possible finite collections $P_1,\ldots,P_m$.

THEOREM 3. *If $h \ge 2$, then $\chi_\perp(\mathbb{E}^h) = \chi_e(\mathbb{E}^h)$. Further, if $G^n = G(S)$ for some $S \subseteq \mathbb{E}^h$, then we can omit $\le 6n^{2-\frac{1}{h}}$ edges from G^n so that the obtained $\tilde{G}^n$ is $\le \chi_\perp(\mathbb{E}^h)$-chromatic.*

Remark 3. Obviously, $\chi_{\perp}(\mathbb{E}^4) = 2$, thus Theorem 3 is a generalization of Theorem 1. It is easy to see, that $\chi_{\perp}(\mathbb{E}^h) \le \chi_e(\mathbb{E}^h)$. Indeed, assume that $\chi_{\perp}(\mathbb{E}^h) = t$. Fix the planes $P_1,\dots,P_m \ni 0$, so that $\chi(\hat{G}(P_1,\dots,P_m)) = t$. Let $C_i = \{\underline{x}\in P_i : |x| = \frac{1}{\sqrt{2}}\}$. Fixing $\simeq \frac{n}{m}$ points on each C_i we obtain a t-chromatic graph G^n, since each $\underline{x} \in C_i$ and $\underline{y} \in C_j$ have distance 1 if $P_i \perp P_j$ (i.e. if $(i,j) \in E(\hat{G}(P_1,\dots,P_m))$. It is easy to see that one must omit $\simeq \frac{n^2}{m^2}$ edges, or more to turn G^n into a $(t-1)$-chromatic graph. Here m is fixed, $n \to \infty$, thus $\chi_e(\mathbb{E}^h) \ge t = \chi_{\perp}(\mathbb{E}^h)$.

As a matter of fact, we shall prove a sharpening of Theorem 3 as well:

THEOREM 4. *Let $G^n = G(S)$ for an n-element set $S \subseteq E^h$. We can subdivide S into $V^*, V_1,\dots,V_{\ell_0}$ so that*

(i) $3n^{\frac{1}{h}} \le V_i \le \left\lceil 3n^{1-\frac{1}{h}} \right\rceil$ *(where $\lceil x \rceil$ denotes the upper integer part of x), $i = 1,2,\dots,\ell_0$.*

(ii) If V_i and V_j are joined by more than $2|V_i||V_j|n^{-\frac{1}{h}}$ edges, then their affine closures are orthogonal, $1 \le i < j \le \ell_0$.

(iii) Each $x \in S$ is joined to at most $\left\lceil 3n^{1-\frac{1}{h}} \right\rceil$ points of V^*.

Theorem 4 implies Theorem 3: if a V_i is one-dimensional, any $x \in S$ is joined to at most 2 of its vertices. Therefore these V_i's can be put into V^*. For the others we choose a plane $P_i \ni \underline{0}$ parallel with some plane $P^*_i \subseteq V_i$ and colour the planes P_i by $t = \chi_{\perp}(\mathbb{E}^h)$ colours so that P_i and P_j have different colours if $P_i \perp P_j$. We colour the points of V_i by the colour of P_i. If we omit all the edges (x,y), $x \in V_i$, $y \in V_j$ for which V_i and V_j are not orthogonal and all the edges (x,y), $x \in V^*$, then the colouring given above is a good colouring of the remaining graph $\tilde{G}^h$ and we omitted at most

$$|V^*| \cdot \left\lceil 3n^{1-\frac{1}{h}} \right\rceil + 2\sum\sum |V_i||V_j| \cdot h^{-\frac{1}{h}} \le n \cdot \left\lceil 3n^{1-\frac{1}{h}} \right\rceil + 2n^2 \cdot n^{-\frac{1}{h}} .$$

This proves that $\chi_\perp(\mathbb{E}^h) \ge \chi_e(\mathbb{E}^h)$. This and Remark prove Theorem 3.

THEOREM 5. *Let S^{h-1} be a sphere of radius $\frac{1}{\sqrt{2}}$ on $\mathbb{E}^h$. Then*

$$\chi(S^{h-1}) \le \chi_e(\mathbb{E}^{2h}) \le \chi_e(\mathbb{E}^{2h+1}) \le \chi(S^{2h}). \tag{10}$$

The meaning of Theorem 5 is that the ordinary chromatic number of the sphere S^{h-1} and the essential chromatic number of $\mathbb{E}^h$ tend to infinity equally fast. We do not think that the ordinary chromatic number of S^{h-1} and $\mathbb{E}^h$ differ very much, this is why we think that Conjecture 1 must hold.

Knowing Theorem 3, Theorem 5 becomes almost trivial and, therefore, the proof is left to the reader.

3. On the Faithful Dimension of a Graph.

Let a graph G^n be given. As we have seen, G^n can be embedded into $\mathbb{E}^{2t}$ if $t = \chi(G^n)$. One can easily see that this dimension is the lowest possible for $K_t(m,\dots,m)$ if m is sufficiently large. Here, embedding G^n into $\mathbb{E}^h$ we ask for finding a set $S \subseteq \mathbb{E}^h$ such that $G^n \subseteq G(S)$. If $G^n = G(S)$, the embedding will be called faithful and the smallest h such that G^n can faithfully be embedded into $\mathbb{E}^h$ is the faithful dimension $\mathrm{Dim}(G^n)$ of G^n. The question is whether there exists a sharp difference between the notions of dimension and faithful dimension.

While (7) and the example of K_t show that the dimension of a graph is strongly related to its chromatic number, we show that $\mathrm{Dim}(G^n)$ has a similar strong connection to the maximum valence $\Delta(G^n)$ of G^n.

<u>THEOREM 6</u>. *$\mathrm{Dim}(G^n) \le 2\Delta(G^n) + 1$.*

<u>Conjecture 2</u>. *Let* $G^n \ne K_2(3,3)$. *Then* $\dim(G^n) \le \Delta(G^n)$.

<u>Proposition 1</u>. If G is the graph obtained from $K_2(m,m)$ $(m \ge 2)$ by omitting a 1-factor, then

$$m - 2 \le \mathrm{Dim}(G) \le m - 1.$$

<u>Remark 4</u>. The important part of Proposition 1 is that in spite of the fact that $\chi(G) = 2$ and (hence) $\dim(G) \le 4$, $\mathrm{Dim}(G)$ is large. For $m = 3$, 4 the $\mathrm{Dim}(G) = 2$! Anyway, this shows that $\mathrm{Dim}(G)$ can be unbounded even if $\chi(G) = 2$, i.e. $\mathrm{Dim}(G)$ is related to $\Delta(G)$ and not $\chi(G)$ in general. Conjecture 2 is sharp, if it holds: $\mathrm{Dim}(K_m) = \Delta(K_m) = m - 1$.

Finally, we shall prove

<u>Proposition 2</u>. $\dim(G^n) \le \Delta(G^n) + 2$.

This assertion is weaker than Conjecture 2 but sometimes stronger than (7).

<u>*4. Proofs of the Results on Chromatic Number.*</u>

<u>Definition 3</u>. Given a set $U \subseteq \mathbb{E}^h$, we denote its affine closure (not necessarily containing $\underline{0}$) by $L(U)$. $M(U)$ is the set of points x such that for every $\underline{y} \in U$ $\rho(x,y) = 1$. If $M(U) \ne \emptyset$, then there exists a unique sphere $Q(U) \subseteq L(U)$ containing U. (Here the "sphere" in $L(U)$ always means one spanning the whole $L(U)$.) To show the existence of $Q(U)$ put $Q(U) = L(U) \cap M(\{x\})$ for some $x \in M(U)$. Obviously, $Q(U) \supseteq U$ and is a sphere in $L(U)$. If $H \ne Q(U)$ is another sphere in $L(U)$ containing U, then $U \subseteq H \cap Q(U)$ but $\dim(H \cap Q(U)) = \dim U - 1$, which is a contradiction. (dim A is the dimension of $L(A)$, further,

for $A, B \subseteq E^h$ $A \perp B$ is an abbreviation of $L(A) \perp L(B)$. $A \not\perp B$, $A \| B$, $A \nparallel B$ are used in similar ways.

LEMMA 1. *If* $M(U) \neq \emptyset$, *then* $U \perp M(U)$. $M(U)$ *is a sphere of* $L(M(U))$ *and* $\dim U + \dim M(U) = h$.

Further, if $x \in Q(U)$, $y \in M(U)$, *then* $\rho(x,y) = 1$.

Proof. We may assume that

$$(11) \qquad Q(U) = \{(y_1,\dots,y_k,0,\dots,0) \in E^h : \sum y_i^2 = r^2\}.$$

As we have seen, if $\underline{x} \in M(U)$, then $Q(U) = M(\{x\}) \cap L(U)$. Thus x has distance 1 from each point of $Q(U)$. Clearly, if e_j is the jth basis vector: $e_j = (0,0,\dots,0,1,0,\dots,0)$ with 1 in its jth position, then $\pm re_j \in Q(U)$, thus x has distance 1 from re_j and $-re_j$, $j = 1,\dots,k$. Thus for $x = (x_1,\dots,x_h)$ $x_j = 0$, $j = 1,\dots,k$. This means that

$$(12) \qquad \underline{x} = (0,\dots,0,x_{k+1},\dots,x_n), \ \sum x_i^2 = 1 - r^2.$$

On the other hand, each x satisfying (12) has distance 1 from each $y \in Q(U)$. Thus $x \in M(U)$ iff (12) holds. This proves the lemma.

Proof of Theorem 4. By Remark 3 we know that

$$t = \chi_\perp(E^h) \le \chi_e(E^h).$$

We show that if $S \subseteq E^h$, $G^n = G(S)$, then one can omit $\le 6n^{2-\frac{1}{h}}$ edges of G^n so that the obtained $\tilde{G}^n$ has chromatic number $\le t$. For each

$U \subseteq S$ satisfying $|U| \ge 3n^{1-\frac{1}{h}}$ and $M(U) \neq \emptyset$ we define a $V = f(U) \subseteq U$ as follows. We consider all the $W \subseteq U$ such that for

$k = \dim M(W)$, $|W| \geq \left\lceil 3n^{1-\frac{k+1}{h}} \right\rceil$.

There exist such W's; e.g. $W = U$ satisfies the condition. Let $V = f(U)$ be a W for which k is maximal. Clearly for $k = h - 1$ $|W| \geq 3$ on the other hand, by Lemma 1 $\dim(W) = 1$ and $Q(W)$ is a "sphere" in $L(W)$, thus $|W| \leq 2$. This contradiction shows that $k \leq h - 2$.

Thus

$$(12) \qquad |W| \geq 3n^{\frac{1}{h}}.$$

Now we select a $U_1 \subseteq S$ such that $M(U_1) \neq \emptyset$ and $|U_1| = \left\lceil 3n^{1-\frac{1}{h}} \right\rceil$. (If such a U_1 does not exist, we put $V^* = S$, $\ell_0 = 0$). $U_{\ell+1}$ and $V_{\ell+1}$ are defined recursively: if $S - \bigcup_{i\leq\ell} V_i$ contains no $U_{\ell+1}$ such that

$$(13) \qquad M(U_{\ell+1}) = \emptyset \quad \text{and} \quad |U_{\ell+1}| = \left\lceil 3n^{1-\frac{1}{h}} \right\rceil,$$

then the recursion stops and we put $V^* = S - \bigcup_{i\leq\ell} V_i$, $\ell_0 = \ell$. Otherwise we select a $U_{\ell+1}$ satisfying (13) and put $V_{\ell+1} = f(U_{\ell+1})$.

The fact that (13) does not hold for any $U_{\ell+1} \subseteq V^*$ is just another form of (iii) of Theorem 4; (i) follows from (12). Thus we have to prove only that if V_i and V_j are not orthogonal, then they are joined by $\leq 2|V_i||V_j|h^{-\frac{1}{h}}$ edges.

Assume that $V_i \not\perp V_j$. If H is a hyperplane in $L(V_i)$, it contains $< |V_i| \cdot n^{-\frac{1}{h}}$ vertices: otherwise for $\tilde{V} = H \cap V_i$ we had $|\tilde{V}| \geq |V_i| n^{-\frac{1}{h}}$ and

(14) $$\dim M(\tilde{V}) = h - \dim \tilde{V} = h - (\dim V_i - 1) = \dim V_i + 1$$

would contradict the maximality of k in the definition of $V_i = f(U_i)$. Thus $|V_i \cap H| < |V_i| n^{-\frac{1}{h}}$. Now, if $x \in V_j - M(V_i)$, then $M(\{x\})$ does not contain $Q(V_i)$, hence $H = Q(V_i) \cap M(\{x\})$ has lower dimension than $Q(V_i)$, therefore $H \cap V_i$ contains at most $|V_i| n^{-\frac{1}{h}}$ points. Thus the number of edges joining V_i and $V_j - M(V_i)$ is at most $|V_i||V_j| n^{-\frac{1}{h}}$. Further, $L(M(V_i))$ cannot contain V_j, since $V_i \not\perp V_j$. Therefore $H = L(M(V_i)) \cap L(V_j)$ has lower dimension that $L(V_j)$ which implies that $|M(V_i) \cap V_j| \le |H \cap V_j| \le |V_i||V_j| h^{-\frac{1}{h}}$. Thus the number of edges between $V_j \cap M(V_i)$ and V_i is $\le |V_i||V_j| n^{-\frac{1}{h}}$. Consequently, the number of edges between V_i and V_j is at most $2|V_i||V_j| n^{-\frac{1}{h}}$. This completes the proof.

As we have seen, Theorem 4 implies Theorem 3.

Proof of Theorem 2. Let

$$P_{k,\ell} = \{(x_1,\ldots,x_h) \in E^h,\ x_i = 0 \text{ unless } i = k \text{ or } i = \ell\}.$$

If we consider these $\binom{h}{2}$ 2-dimensional planes, then the corresponding $\hat{G}(\ldots,P_{k,\ell},\ldots)$ is obviously the Kneser graph corresponding to the pairs of an h-element set: $P_{k,\ell} \perp P_{k',\ell'}$ if $\{k,\ell\} \cap \{k',\ell'\} = \emptyset$. Thus $\chi(\hat{G}(\ldots,P_{k,\ell},\ldots)) = h - 2$, that is, $\chi_i(\mathbb{E}^h) \ge h - 2$. By Theorem 3, more precisely, by the trivial Remark 3, $\chi_e(\mathbb{E}^h) \ge h - 2$. This is just Theorem 2.

Remark 5. So if the Kneser conjecture is used, in fact for pairs only, then Theorem 2 is trivial. Further, we think that it is very far from the right order of magnitude. Since $\chi_e(\mathbb{E}^h) \ge [\frac{h}{2}]$ is trivial, one can ask, what is the point in proving a slightly stronger result, like Theorem 2. There are two points: On the one hand it shows that $[\frac{h}{2}]$ is not sharp, on the other hand, though Conjecture 1 states that Theorem 2 is very far from the truth, still there is some chance that Conjecture 1 is false and Theorem 2 is sharp. We cannot improve Theorem 2 even for $h = 5$.

Remark 6. If $h = 5$, the Kneser graph on the pairs is just the Petersen graph. Thus the Proof above shows that the Petersen graph can be obtained as $\hat{G}(P_1,\dots,P_{10})$ from 10 planes $\mathbb{E}^5$. Let $Q(m)$ be the graph obtained from Q by replacing each $x \in V(Q)$ by m new vertices and joining two vertices of $Q(m)$ if the original vertices of Q were joined in Q. The above proof shows that if Q is the Kneser graph of the pairs of an h-element set, then $Q(m)$ is embeddable into $\mathbb{E}^h$.

5. *Proofs of the Results on the Dimension of a Graph*

Proof of Proposition 1. Assume that the graph G has $2m$ vertices $x_1,\dots,x_m$ and $y_1,\dots,y_m$ and $(x_i,y_j) \in E(G)$ iff $i \ne j$. We have to show that $m - 2 \le \mathrm{Dim}(G) \le m - 1$.

Assume first that G is embedded into $\mathbb{E}^{m-3}$. We choose a minimal subset $\tilde{X} \subseteq \{x_1,\dots,x_m\}$ for which $L(\tilde{X}) = L(\{x_1,\dots,x_m\})$. We may assume without loss of generality that $\tilde{X} = \{x_1,\dots,x_\ell\}$ for some $\ell \le m - 2$. Let $U = \{x_1,\dots,x_{\ell+1}\}$. We prove that $Q(\tilde{X}) \ni x_{\ell+1}$. First of all, $M(U) \ne \emptyset$, $M(\tilde{X}) \ne \emptyset$, because both contain $y_{\ell+2}$. $L(\tilde{X}) = L(U)$ by definition and as we have seen in Definition 3, $Q(U)$ and $Q(\tilde{X})$ are the uniquely determined spheres of $L(\tilde{X}) = L(U)$ containing U and $\tilde{X} \cong U$ respectively. Thus (by the uniqueness of $Q(\tilde{X})$) $Q(U) = Q(\tilde{X})$. In other words, $x_{\ell+1} \in Q(\tilde{X})$.

Clearly, $y_{\ell+1} \in M(\tilde{X})$. By Lemma 1, $\rho(x, y_{\ell+1}) = 1$ for every $x \in Q(\tilde{X})$, thus $\rho(x_{\ell+1}, y_{\ell+1}) = 1$ but $(x_{\ell+1}, y_{\ell+1}) \notin E(G)$. Thus the embedding is not faithful.

$$\mathrm{Dim}(G) \geq m - 2.$$

Now we embed G into $\mathbb{E}^{m-1}$ faithfully. Let

$$\hat{x}_i = (a, a, \ldots, -(m-1)a, \ldots, a) \in \mathbb{E}^m,$$

(the ith coordinate is the exceptional $-(m-1)a$). Clearly, if $\hat{y}_i = -\hat{x}_i$, then

$$\rho^2(\hat{x}_i, \hat{y}_j) = 4(m-2)a^2 + 2(m-2)^2a^2 = 2(m^2-2m)a^2 = 1$$

if $i \neq j$ and $a = (2m^2-4m)^{-\frac{1}{2}}$. Now

$$\rho^2(\hat{x}_i, \hat{x}_j) = \rho^2(\hat{y}_i, \hat{y}_j) = 2m^2a^2 > 1$$

if $i \neq j$ and

$$\rho(\hat{x}_i, \hat{y}_i) = 4(m-1)a^2 + 4(m-1)^2a^2 = 4(m^2-m+2)a^2 > 1$$

if $m > 2$. Thus the embedding is faithful and the vertices $\hat{x}_i$, $\hat{y}_i$ belong to the hyperplane $\{\underline{t} : \sum t_i = 0\}$. This completes the proof.

Remark 7. The geometric background of the above proof is clear: $(\hat{x}_1, \ldots, \hat{x}_m)$ and $(\hat{y}_1, \ldots, \hat{y}_m)$ were regular simplices of $\mathbb{E}^{m-1}$ and the whole picture had a lot of (rotational) symmetries.

In the sequel $S^h \subseteq \mathbb{E}^{h+1}$ denotes $\{a \in \mathbb{E}^{h+1}, |a| = \frac{1}{\sqrt{2}}\}$.

Proof of Proposition 8. We use induction on $\Delta = \Delta(G)$ to prove the following stronger statement:

(*) One can embed G into $S^{\Delta+1}$.

For $\Delta = 1$ (*) is obvious. For a fixed Δ we choose a maximal independent set $A \subseteq V(G)$ and put $G^* = G - A$. Clearly, $\Delta(G-A) \le \Delta - 1$, therefore $G - A$ can be embedded into $S^{\Delta} \subseteq S^{\Delta+1} \subseteq \mathbb{E}^{\Delta+2}$. For each $x \in A$ there is a $\le \Delta$-dimensional linear subspace L_x (containing $\underline{0}$!) containing $\{\hat{y}: (x,y) \in E(G)\}$. (The image of a $u \in V(G)$ at a given embedding will be denoted by $\hat{u}$ unless we compare two different embeddings, when one image will be denoted by $\hat{u}$, the other by $\tilde{u}$). We fix a plane $P_x \ni \underline{0}$ ($\dim p_x = 2$) orthogonal to L_x. Choosing any $\hat{x} \in P_x \cap S^{\Delta+1}$ we ensure that $\hat{x} \perp u$ if $(x,u) \in \mathbb{E}(G)$. Since $\hat{x}$ can be chosen in infinitely many ways, we may choose $\hat{x}$'s one by one so that $\hat{x} \in A$ is different from $\hat{y} \in V(G)$ if $x \ne y$. This completes the proof.

Proof of Theorem 6. Again, we embed G^n into $S^{2\Delta}$ faithfully. We know by Proposition 2 that G^n is embeddable into $S^{2\Delta}$ if faithfulness is not required. We start with an arbitrary embedding and modify it step by step, first achieving that if $x_1, \dots, x_{\Delta+1} \in V(G^n)$ are different, then $\hat{x}_1, \dots, \hat{x}_{\Delta+1}$ are linearly independent. Let $L_0(U)$ denote the linear subspace generated by U. Assume that

$$\hat{x}_{\Delta+1} \in L_0(\hat{x}_1, \dots, \hat{x}_{\Delta}).$$

We fix all the vertices of $V(G)$ but $\hat{x}_{\Delta+1}$. The conditions

$$|\hat{x}_{\Delta+1} - \hat{u}| = 1 \quad \text{if} \quad (x_{\Delta+1}, u) \in E(G^n)$$

keep $x_{\Delta+1}$ on a $\ge \Delta+1$-dimensional sphere S. (S^{λ} is counted $\lambda+1$-dimensional !). Since the dimension of $L_0(\{\hat{x}_1, \dots, \hat{x}_{\Delta}\})$ is $\le \Delta$, it does not contain S, thus we can replace $\hat{x}_{\Delta+1}$ by an $\tilde{x}_{\Delta+1} \notin L_0(x_1, \dots, x_{\Delta})$, moreover, $x_{\Delta+1}$ can be chosen arbitrarily near to $x_{\Delta+1}$.

We iterate the step above until no $\hat{x}_{\Delta+1}$ belongs to the linear closure of Δ others. If in the ith step $\hat{x}$ is replaced by $\tilde{x}$, first we choose ε_j such that each linear subspace $L(\{\hat{y}_1,\ldots,\hat{x}_\Delta\})$ not containing $\hat{x}$ has distance $> \varepsilon_j$ from $\hat{x}$ and then choose an $\tilde{x}$ for which $|\tilde{x}-\hat{x}| < \varepsilon_j$. Thus we shall not ruin the results of earlier steps in later steps. Finally each $\Delta+1$-tuple $\hat{x}_1,\ldots,\hat{x}_{\Delta+1}$ will become linearly independent.

Now, if we have embedding with $|\hat{x}-\hat{y}| = 1$ for some $(x,y) \notin E(G)$, and the $(\Delta + 1)$-tuples are independent, then we change $\hat{x}$ to $\tilde{x}$ as follows. Let $U_x = \{\hat{u} : (x,u) \in E(G)\}$. Above we have achieved that $y \notin L_0(U_x)$. Thus

$$\dim(U_x) < \dim(U_x+\{\hat{y}\}).$$

This implies that there is an $\tilde{x} \perp U_x$ $\tilde{x} \neq \hat{y}$, moreover, this $\tilde{x}\in S^{2\Delta}$ can be chosen arbitrarily close to $\hat{x}$. Clearly, $|\tilde{x}-\hat{u}| = 1$ if $(x,u) \in E(G)$, $|\tilde{x}-\hat{y}| \neq 1$ and $|\tilde{x}-\hat{u}| \neq 1$ if $|\hat{x}-\hat{u}| \neq 1$, (if $|\tilde{x}-\hat{x}|$ is small enough). Iterating this step we obtain the embedding wanted.

6. *Unsolved problems.*

We have already stated some open problems about the embedding of graphs into Euclidean spaces. Below we shall formulate some further ones.

Problem 3. Determine $X_\perp(\mathbb{E}^5)$. Characterize the graphs embeddable into $S^4 \subseteq \mathbb{E}^5$. Can every 3-chromatic G not containing K_3 be embedded into S^4?

Problem 4. Determine $\dim(G^n))$ if G^n is a random graph. More precisely, let G^n be a random graph, where each edge is chosen with probability c, where $c \in (0,1)$ is fixed. As $n \to \infty$, almost all G^n

have chromatic number at most $c_1\, n/\log n$, (see the next remark!), and therefore the dimension of almost all G^n is at most $2c_1 n/\log n$. Since almost all the graphs G^n contain a K_m for $m = [c_2 \log n]$, thus $c_2 \log n$ lower bound. Is $\dim(G^n) = o(n/\log n)$ with probability tending to 1? Find good lower and upper bounds $f_1(n)$ and $f_2(n)$ such that

$$f_1(n) \le \dim(G^n) \le f_2(n)$$

with probability tending to 1.

Remark. In [8] it is implicitely proved that almost all G^n are at most $c_1 n/\log n$ chromatic: it is well-known that for almost all the graphs G^n if $m=\omega(G^n)$ denotes the largest complete graph in G^n, then $m < c_3 \log n$. The second part of theorem of [8] asserts that for every graph G^n, $\chi(G^n) < c_4 \omega(G^n) .\, n/\log^2 n$. This proves the assertion. The other inequality, asserting that almost all graphs G^n have chromatic number at least $c_5 n/\log n$ is trivial from the fact that for almost all G^n the maximal size of an independent set of G^n is also at most $c_3 \log n$.

Problem 5. Let

$$\binom{p}{2} \le e(G^n) < \binom{p+1}{2}.$$

Is it true that

$$\dim(G^n) \le \dim(K_p) = p - 1?$$

Problem 6. Let $S \subseteq \mathbb{E}^2$ and fix k numbers $\alpha_1, \ldots, \alpha_k$. Let $x, y \in S$ be joined by an edge iff $\rho(x,y) = \alpha_\ell$ for some $\ell \le k$. Let $t_k(n)$ be the maximum of the chromatic number of this graph when S, $\{\alpha_1, \ldots, \alpha_k\}$ vary but k and n are fixed. How large is $t_k(n)$?

(For some further results and unsolved problems see [6], [7] and [9].)

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