Multiplicative Functions Whose Values are Uniformly Distributed in $(0,\infty)$

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1. Introduction. Let $n \longmapsto \psi(n)$ be a positive valued arithmetic function which tends to infinity as $n \longrightarrow \infty$. We shall say that the values of ψ are uniformly distributed in $(0,\infty)$ (briefly, ψ u.d. in $(0,\infty)$) if there exists a positive constant δ such that as $y \longrightarrow \infty$

$$N(y) = N(y,\psi) \overset{\text{def}}{=} \# \{n : \psi(n) \le y\} \sim \delta y.$$

The number δ will be called the density of values.

It is known (e.g. [1], [2], [4]) that Euler's function is u.d. in $(0,\infty)$, and the following general condition that a function ψ be u.d. in $(0,\infty)$ was given by Wooldridge [9]:

Theorem 0. *Suppose that $\psi(n)$ is a positive multiplicative function and $\psi(n)/n$ is strongly multiplicative* (i.e. $\psi(p^\alpha)/p^\alpha = \psi(p)/p$ *for each prime p and positive integer α*). *If*

$$\sum_p \frac{|p - \psi(p)|}{p\,\psi(p)} \log (p + \psi(p)) < \infty,$$

then ψ is u.d. in $(0,\infty)$.

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We shall give other conditions for uniform distribution of values of a multiplicative function. For certain classes of functions our conditions are both necessary and sufficient.

Our results are analogous to ones on mean values of multiplicative functions (cf. [3], [5], [8]). There is one direct connection between the two theories, involving the distribution of values of a function ψ and the mean value of the function $n \longmapsto n/\psi(n)$. We shall show when each of these conditions implies the validity of the other.

We shall characterize the cases in which the distribution of values of an arithmetic function has zero or infinite density. In conclusion we shall give several examples showing the limitation of some of our theorems.

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2. Statement of Results

Theorem 1. *Let ψ be a positive valued multiplicative function satisfying*

$$\psi(p) \sim p \quad \text{as} \quad p \longrightarrow \infty \tag{1}$$

and

$$\sum_{p,\alpha\geq 2} \psi(p^{\alpha})^{-1} < \infty. \tag{2}$$

Then ψ is u.d. in $(0,\infty)$ iff

$$F(\sigma) \overset{\text{def}}{=} \sum_{n\geq 1} \psi(n)^{-\sigma} \sim \delta/(\sigma - 1) \qquad (\sigma \longrightarrow 1+) \tag{3}$$

holds for some positive constant δ. The density of values

$$= \delta = \lim_{\sigma\to 1+} \prod_{p} \{(1 - p^{-\sigma}) \sum_{\alpha\geq 0} \psi(p^{\alpha})^{-\sigma}\}.$$

Corollary 1. *Let* ψ *be a positive valued multiplicative fu* *satisfying* (1) *and* (2). *Then* ψ *is u.d. in* $(0,\infty)$ *iff*

(4) $$\sum_p \{\frac{1}{\psi(p)} - \frac{1}{p}\} \text{ converges.}$$

Suppose that the relation $\psi(p) \sim p$ fails to hold on a spar set of primes. In this case we may be able to show that ψ is u.d. i $(0,\infty)$ by a perturbation argument. Indeed we have

Theorem 2. *Suppose that* ψ *is a positive valued multiplica* *function which satisfies* (4) *and for each* $\varepsilon > 0$ *the series*

(5) $$\sum_{|\psi(p) - p| > \varepsilon p} |\frac{1}{\psi(p)} - \frac{1}{p}| \text{ converges.}$$

If (2) *holds, then* ψ *is u.d. in* $(0,\infty)$ *and has density*

$$\delta = \prod_p \{(1 - \frac{1}{p}) \sum_{\alpha \geq 0} \psi(p^\alpha)^{-1}\}.$$

If (2) *does not hold, then* $N(y,\psi)/y \longrightarrow \infty$ *as* $y \longrightarrow \infty$.

Corollary 2. *If* ψ *is a positive valued multiplicativ* *which satisfies* (2) *and if* $\sum |\psi(p)^{-1} - p^{-1}| < \infty$, *then* ψ *is u.d.*

There are simple examples of uniform distribution wher is seldom near 1. If we consider functions ψ for which $\psi(p)$ "too small too often," then uniform distribution of the values of equivalent to a condition on the generating function in the comple

Theorem 3. Let ψ be a positive valued multiplicative function satisfying (2) and

(6) $$\sum_{\psi(p)\leq x} 1 \ll x/\log x.$$

Then ψ is u.d. in $(0,\infty)$ iff for each (fixed) $T > 0$

(7) $$F(s) \overset{\text{def}}{=} \sum_{n\geq 1} \psi(n)^{-s} = \frac{\delta}{s-1} + \underline{o}\left(\frac{1}{\sigma-1}\right)$$

holds uniformly for $-T \leq t \leq T$ as $\sigma \longrightarrow 1+$. (Here and throughout this article we adhere to the curious convention of denoting a complex number s as $\sigma + it$ with σ and t real.)

Condition (7) is the analogue of the hypothesis of Halász's theorem on mean values of multiplicative functions. Can one, by analogy with that theory replace (7) by a condition on the generating function on the vertical line $\sigma = 1$? We have

Theorem 4. Suppose that ψ is a positive valued multiplicative function which satisfies (2) and

(8) $$1 \ll \psi(p)/p \ll 1.$$

Then ψ is u.d. in $(0,\infty)$ iff

$$\sum_{p} p^{-1} - \text{Re}\ \psi(p)^{-1-it}$$

converges to a finite number for $t = 0$ and diverges to $+\infty$ for each real $t \neq 0$.

The generating function for uniform distribution of values of ψ is $\sum_{n\geq 1} \psi(n)^{-s}$; that for the mean value of the function $h: n \longmapsto n/\psi(n)$

is $\sum_{n\geq 1} n^{1-s}/\psi(n)$, which is formally similar at $s = 1$. The following two theorems connect uniform distribution of the values of a multiplicat function ψ with the existence of a mean value of the associated functi These theorems contain different hypotheses upon the values of $\psi(p)/p$, and we shall show by examples that each theorem can fail with its hypoth weakened.

Theorem 5. Suppose that ψ is a positive valued multiplica function which satisfies (2) and (8), and that for each prime p and eac real t

$$\sum_{\alpha\geq 0} p^{-i\alpha t}\, \psi(p^{\alpha})^{-1} \neq 0.$$

If the values of ψ are u.d. in $(0,\infty)$, then $h: n \longmapsto n/\psi(n)$ has a mea value and the density of values of ψ equals the mean value of h.

Theorem 5 has an interesting consequence. Suppose ψ rest to primes is a 1-1 mapping of the primes onto the primes and that $1 \ll \psi(p)/p \ll 1$. In this case we say that ψ is a rearrangement of t primes of bounded ratio.

Corollary 3. Let ψ be a completely multiplicative funct which is a rearrangement of bounded ratio on the primes. Then $n/\psi(n)$ a mean value and it is unity.

There is a converse of Theorem 5, valid with condition (8) replaced by the more stringent condition $\psi(p) \sim p$.

Theorem 6. *Suppose that* ψ *is a positive valued multiplicative function which satisfies* (1) *and* (2). *If* $h: n \longmapsto n/\psi(n)$ *has a mean value, then* ψ *is u.d. in* $(0,\infty)$ *and the density of values of* ψ *equals the mean value of* h.

The two preceding theorems give

Corollary 4. *Suppose that* ψ *is completely multiplicative and satisfies* (1) *and* (2). *Then* ψ *is u.d. in* $(0,\infty)$ *iff* h *has a mean value*.

We say that the values of a positive arithmetic function ψ are distributed with zero (respectively infinite) density in $(0,\infty)$ if $N(y)/y$ tends to zero (respectively infinity) as $y \longrightarrow \infty$. We have

Theorem 7. *Suppose that* ψ *is a positive multiplicative function satisfying* (2) *and* (6). *Then the values of* ψ *have zero density in* $(0,\infty)$ *iff*

$$\lim_{\sigma\to 1+} (\sigma - 1) \sum_{n\geq 1} \psi(n)^{-\sigma} = 0.$$

Theorem 8. *Suppose that* ψ *is a positive completely multiplicative function satisfying* (2) *and*

$$x \ll \sum_{\psi(p)\leq x} \log \psi(p); \qquad \sum_{\psi(p)\leq x} \frac{\log \psi(p)}{\psi(p)} \ll \log x.$$

Then the value of ψ *have infinite density in* $(0,\infty)$ *iff*

$$\lim_{\sigma\to 1+} (\sigma - 1) \sum_{n\geq 1} \psi(n)^{-\sigma} = \infty.$$

3. Remarks on the distribution of values $\psi(p)$. Here we shall compare the various hypotheses made upon the distribution of values of $\psi(p)$. The only data relevant for the count $N(y)$ are the values $\{\psi(p)\colon p \text{ prime}\}$ (with appropriate multiplicities); the order of occurrence does not play a role. The proof of Corollary 3 exploits this fact.

However, we shall see that it is "easier" to establish uniform distribution results for functions ψ for which $\psi(p)/p$ is close to 1. Moreover, some of our theorems involve p and $\psi(p)$ together. To exploit such relations we may need to insure that $|\log \psi(p)/p|$ is not too large.

The following relations hold between the various hypotheses upon $\psi(p)$:

$$\frac{\psi(p)}{p} \gg 1 \Rightarrow \sum_{\psi(p)\le x} 1 \ll \frac{x}{\log x} \Leftrightarrow$$

$$\sum_{\psi(p)\le x} \log \psi(p) \ll x \Rightarrow \sum_{\psi(p)\le x} \frac{\log \psi(p)}{\psi(p)} \ll \log x;$$

$$\frac{\psi(p)}{p} \ll 1 \Rightarrow \sum_{\psi(p)\le x} 1 \gg \frac{x}{\log x};$$

$$\frac{x}{\log x} \ll \sum_{\psi(p)\le x} 1 = o(x) \Rightarrow \sum_{\psi(p)\le x} \log \psi(p) \gg x.$$

The proofs of these relations are immediate. Note that the last relation will not hold if $\psi(p)$ is small too often.

4. *Perturbation of multiplicative functions*. It is convenient to prove Theorems 1 and 3 for functions which are completely multiplicative (c.m.). Given a multiplicative function ψ we shall define a c.m. function by altering ψ on the higher prime powers (and on any primes for which $\psi(p) \leq 1$ to prevent the new function from being bounded on an infinite set). Also, in the course of proving Theorem 2 we shall alter our multiplicative function on a certain set of "bad" primes. In each case we show that the values of the altered function are u.d. in $(0,\infty)$. Then we shall show that the original function is u.d. in $(0,\infty)$ by the following result.

Let U be a subset of $\{p^\alpha: \ p \ \text{prime}, \alpha \geq 1\}$ with the property that if $p^\alpha \in U$, then $p^{\alpha+1} \in U$. We say that a prime p "meets U" if $p^\alpha \in U$ for some positive integer α.

Lemma 1. Let U *be as above. Suppose that* ψ *and* ψ^* *are multiplicative functions which satisfy*

$$\psi(p^\alpha) = \psi^*(p^\alpha), \text{ if } \ p^\alpha \notin U \tag{9}$$

$$\psi^*(p^\alpha) = \psi^*(p)^\alpha, \text{ if } \ p \ \text{ meets U} \tag{10}$$

$$\sum_{p^\alpha \in U} \psi(p^\alpha)^{-1} + \psi^*(p^\alpha)^{-1} < \infty \tag{11}$$

and ψ^* *is u.d. in* $(0,\infty)$ *with density* δ^*. *Then* ψ *is also u.d. in* $(0,\infty)$ *and has density*

$$\delta = \delta^* \prod_{p \text{ meets } U} \{(1 - \frac{1}{\psi^*(p)}) \sum_{\alpha \geq 0} \frac{1}{\psi(p^\alpha)}\}. \tag{12}$$

<u>Proof</u>. For $\nu = 1,2,\ldots$ define multiplicative functions ψ_ν by setting

$$\psi_\nu(p^\alpha) = \begin{cases} \psi(p^\alpha), \text{ if } p^\alpha \notin U \text{ or } p \le \nu \\ \psi^*(p)^\alpha, \text{ if } p^\alpha \in U \text{ and } p > \nu. \end{cases}$$

We first show that each ψ_ν is u.d. in $(0,\infty)$. Then we show that $N(y,\psi_\nu) \longrightarrow N(y,\psi)$ as $\nu \longrightarrow \infty$.

We prove by induction that as $y \longrightarrow \infty$

(13)
$$N(y,\psi_\nu) \sim \delta^* y \prod_{\substack{p \text{ meets } U \\ p \le \nu}} \left(1 - \frac{1}{\psi^*(p)}\right) \sum_{\alpha \ge 0} \frac{1}{\psi(p^\alpha)}.$$

This relation holds for $\nu = 1$, since $\psi_1 = \psi^*$. Suppose that (13) holds for $\mu = \nu - 1$ and suppose that $\nu = q$ is a prime which meets U. Letting $yM(\nu)$ denote the right side of (13) we have

$$\sum_{\substack{\psi_\mu(n) \le y \\ q \nmid n}} 1 = \sum_{\psi_\mu(n) \le y} 1 - \sum_{\psi_\mu(n) \le \frac{y}{\psi^*(q)}} 1 \sim \left(1 - \frac{1}{\psi^*(q)}\right) M(\mu) y.$$

Then

$$N(y,\psi_\nu) = \sum_{\alpha \ge 0} \sum_{\substack{\psi_\nu(n) \le y \\ q^\alpha \| n}} 1 = \sum_{\alpha \ge 0} \sum_{\substack{\psi_\nu(n) \le y/\psi(q^\alpha) \\ q \nmid n}} 1$$

$$\sim \left(1 - \frac{1}{\psi^*(q)}\right) M(\mu) y \sum_{\alpha \ge 0} \psi(q^\alpha)^{-1} = M(\nu) y.$$

Now we show that $\{N(y,\psi_\nu) - N(y,\psi)\}/y$ is small for ν large. The definition of ψ_ν implies that

$$\sum_{\psi(n) \le y}' 1 = \sum_{\psi_\nu(n)\le y}' 1,$$

where the dash indicates restriction of the sums to exclude integers n wi $p^\alpha \| n$, $p^\alpha \in U$, and $p > \nu$.

It remains to estimate

$$\Delta \overset{\text{def}}{=} N(y,\psi) - \sum_{\psi(n)\le y}' 1 \le \sum_{p>\nu} \sum_{\substack{\alpha \\ p^\alpha \in U}} \sum_{\substack{\psi(m)\le y/\psi(p^\alpha) \\ (p,m)=1}} 1$$

and the similarly defined Δ_ν, which has the upper bound

$$\Delta_\nu \le \sum_{p>\nu} \sum_{\substack{\alpha \\ p^\alpha \in U}} \sum_{\substack{\psi_\nu(m)\le y/\psi^*(p^\alpha) \\ (p,m)=1}} 1$$

We drop the $(p,m) = 1$ condition and estimate the resulting expressions simultaneously and uniformly with respect to ν. Let ψ_0 be a multiplicative function defined by

$$\psi_0(p^\alpha) = \min\{\psi(p^\alpha), \psi^*(p^\alpha)\}.$$

We make an O-estimate for the counting function of "U free" numbers:

$$Q(y) \overset{\text{def}}{=} \sum_{\substack{\psi_0(n)\le y \\ p^\alpha \| n \Rightarrow p^\alpha \notin U}} 1 \le \sum_{\psi^*(n)\le y} 1 \le By$$

for some absolute constant B and all y.

Next we make an $\underline{\underline{O}}$-estimate for $N(y,\psi_0)$. We have

$$N(y,\psi_0) = Q(y) + \sum_{\ell\geq 1}\ \sum_{p_1<\dots<p_\ell}\ \sum_{\substack{\alpha_1,\dots,\alpha_\ell \\ p_i^{\alpha_i}\in U}}\ \sum_{\substack{\psi_0(p_1^{\alpha_1}\dots p_\ell^{\alpha_\ell}m) \dots \\ (p_1\dots p_\ell, m) = \dots \\ q^\beta \| m \implies q^\beta \notin U}} \dots$$

$$= Q(y) + \sum_{\ell\geq 1}\ \sum_{p_1<\dots<p_\ell}\ \sum_{\substack{\alpha_1,\dots\alpha_\ell \\ p_i^{\alpha_i}\in U}} Q\left(\frac{y}{\psi_0(p_1^{\alpha_1})\dots\psi_0(\dots}\right.\dots$$

$$\leq B\, y \prod_p \{1 + \sum_{\substack{\alpha\geq 1 \\ p^\alpha\in U}} \frac{1}{\psi_0(p^\alpha)}\}$$

$$\leq B\, y \prod_p \{1 + \sum_{\substack{\alpha\geq 1 \\ p^\alpha\in U}} (\frac{1}{\psi(p^\alpha)} + \frac{1}{\psi^*(p^\alpha)})\} = B'y.$$

Thus we have for all y

$$|\Delta| + |\Delta_\nu| \leq 2 \sum_{\substack{p^\alpha\in U \\ p>\nu}} N(y/\psi_0(p^\alpha),\psi_0)$$

$$\leq 2\, B'y \sum_{\substack{p^\alpha\in U \\ p > \nu}} \psi_0(p^\alpha)^{-1} < \varepsilon y$$

for given $\varepsilon > 0$, provided that ν is sufficiently large.

For such a ν we have

$$|N(y,\psi_\nu) - N(y,\psi)| < \varepsilon y,$$

and since each ψ_ν is u.d. in $(0,\infty)$, so is ψ. The density of values of ψ is $\lim_{\nu\to\infty} M(\nu)$. #

5. Proof of Theorem 1. Let ψ be a positive multiplicative function which is u.d. in $(0,\infty)$. We show first that (3) holds. Suppose that

$$N(x) = \sum_{\psi(n)\leq x} 1 \sim \delta\, x \qquad (x \longrightarrow \infty)$$

for some positive constant δ. Then

$$F(\sigma) = \sum_{n\geq 1} \psi(n)^{-\sigma} = \int_{1-}^{\infty} x^{-s} dN(x)$$

$$= \sigma \int_{1}^{\infty} x^{-\sigma-1} N(x) dx \sim \delta/(\sigma-1) \quad \text{as} \quad \sigma \longrightarrow 1+.$$

This relation and familiar facts about Euler products and the zeta function imply that if ψ is u.d. in $(0,\infty)$, then the density of values equals

$$\lim_{\sigma\to 1+} (\sigma-1)\, F(\sigma) = \lim_{\sigma\to 1+} \prod_{p} \{(1 - p^{-\sigma}) \sum_{\alpha\geq 0} \psi(p^{\alpha})^{-\sigma}\}.$$

Now suppose that ψ is a positive multiplicative function satisfying (1), (2) and (3). We show that ψ is u.d. in $(0,\infty)$. We introduce a c.m. function ψ^* by setting

$$\psi^*(p^\alpha) = \begin{cases} p^\alpha & \text{if } \psi(p) \leq 1 \\ \\ \psi(p)^\alpha & \text{if } \psi(p) > 1. \end{cases}$$

The set U of Lemma 1 consists of

$$\bigcup_{\psi(p)\leq 1} \{p^\alpha : \alpha \geq 1\} \cup \bigcup_{\psi(p)>1} \{p^\alpha : \alpha \geq 2\}.$$

It is easy to check that conditions (9), (10), and (11) of Lemma 1 hold, and that ψ^* satisfies (1) and (2). To show (3) for ψ^*, write

$$F^*(\sigma) = \sum_{n\geq 1} \psi^*(n)^{-\sigma} = F(\sigma)\{F(\sigma)/F^*(\sigma)\}^{-1},$$

and

$$\frac{F(\sigma)}{F^*(\sigma)} = \prod_{\psi(p)\leq 1} \{1 + \psi(p)^{-\sigma} + \psi(p^2)^{-\sigma} + \ldots\}\{1 - p^{-\sigma}\}\cdot$$

$$\prod_{\psi(p)>1} \{1 - \psi(p)^{-2\sigma} + (1-\psi(p)^{-\sigma})(\psi(p^2)^{-\sigma} + \psi(p^3)^{-\sigma} + \ldots)\}.$$

The first product extends over a finite number of primes p (since $\psi(p) \longrightarrow \infty$ as $p \longrightarrow \infty$) and for each prime the series converges at $\sigma = 1$; the second product converges absolutely at $\sigma = 1$ by (1) and (2). Thus $F^*(\sigma) \sim \delta^*/(\sigma - 1)$ as $\sigma \longrightarrow 1+$ with

$$\delta^* = \delta \prod_{\psi(p)\leq 1} \{(1 - p^{-1})(1 + \psi(p)^{-1} + \psi(p^2)^{-1} + \ldots)\}^{-1}.$$

$$\prod_{\psi(p)>1} \{1 - \psi(p)^{-2} + (1 - \psi(p)^{-1})(\psi(p^2)^{-1} + \psi(p^3)^{-1} + \ldots)\}^{-1}.$$

<u>Thus, it suffices to prove Theorem 1 for functions which are completely multiplicative</u>.

Our proof of Theorem 1 is based on an identity akin to Chebyshev's formula of elementary prime number theory.

Lemma 2. *Let* ψ *be completely multiplicative and* $\psi(n) \longrightarrow \infty$ *as* $n \longrightarrow \infty$. *Then*

$$(14) \qquad \sum_{\psi(k)\leq x} \log \psi(k) = \sum_{\substack{m,n\in\mathbb{N} \\ \psi(mn)\leq x}} \Lambda'(m),$$

where $\Lambda'(m) = \log \psi(p)$ *if* $m = p^{\alpha}$ *and* $\Lambda'(m) = 0$ *otherwise*.

Proof of Lemma 2. By the definition of Λ', for $k = \prod p_i^{\alpha_i}$ we have

$$\log \psi(k) = \sum_i \alpha_i \log \psi(p_i) = \sum_{m|k} \Lambda'(m) = \sum_{mn=k} \Lambda'(m).$$

Summing over all k with $\psi(k) \leq x$, we obtain (14). #

Returning to the proof of Theorem 1, we express the right side of (14) as $\sum_{\psi(n)\leq x} S(x/\psi(n))$, where

$$S(z) = \sum_{\psi(p)\leq z} \log \psi(p) + \sum_{\substack{p,\alpha\geq 2 \\ \psi(p)^{\alpha}\leq z}} \log \psi(p).$$

Since $\psi(p) > 1$ for all p and $\psi(p) \longrightarrow \infty$ as $p \longrightarrow \infty$, the ratio of the second sum to the first tends to zero as $z \longrightarrow \infty$. Also, since $\psi(p) \sim p$, we have that $S(z) \sim z$ as $z \longrightarrow \infty$, and hence

$$\sum_{\psi(k)\leq x} \log \psi(k) = (1 + \underline{o}(1))\, x \sum_{\psi(n)\leq x} 1/\psi(n).$$

Condition (3) and the Tauberian theorem of Hardy, Littlewood, and Karamata [7, Theorem 98] imply that the last sum is asymptotic to $\delta \log x$ as $x \longrightarrow \infty$. Now we have

$$\sum_{\psi(k) \leq x} \log \psi(k) \sim \delta x \log x,$$

and summation by parts yields $N(y) \sim \delta y$. #

<u>Proof of Corollary</u> 1. If ψ is u.d. in $(0,\infty)$, then $F(\sigma) \sim \delta/(\sigma-1)$ as $\sigma \longrightarrow 1+$ by Theorem 1, and so $\sum_p \psi(p)^{-\sigma} + \log(\sigma-1)$ converges as $\sigma \longrightarrow 1+$. Then

$$\sum \psi(p)^{-\sigma} - p^{-\sigma} = c + \underline{o}(1), \quad \sigma \longrightarrow 1+.$$

The function

$$f: x \longmapsto \sum_{p \leq \exp x} \psi(p)^{-1} - p^{-1}$$

is slowly oscillating because

$$f(u+\eta u) - f(u) << \sum_{u < \log p \leq u+\eta u} p^{-1} << \eta$$

as $u \longrightarrow \infty$. We have noted above that $\int_0^\infty e^{-\varepsilon x} df(x)$ converges as $\varepsilon \longrightarrow 0+$. A Tauberian theorem of Hardy and Littlewood [7, Theorem 105] asserts that $f(x)$ converges to the same limit as $x \longrightarrow \infty$, i.e. (4) is valid.

Conversely, suppose (4) holds and $\beta^{-1} \leq \psi(p)/p \leq \beta$. Now

$$\left| \sum_{\psi(p) \leq x} \psi(p)^{-1} - \sum_{p \leq x} \psi(p)^{-1} \right| \leq \sum_{x/\beta < p \leq \beta x} \frac{\beta}{p} = o(1)$$

as $x \longrightarrow \infty$. Also, by (4), $\sum_p \psi(p)^{-1} - p^{-1}$ converges. It follows that

$$\sum_{\psi(p) \leq x} \psi(p)^{-1} - \sum_{p \leq x} p^{-1} \text{ converges } (x \longrightarrow \infty).$$

Summation by parts now gives that

$$\sum_p \psi(p)^{-\sigma} - p^{-\sigma} \longrightarrow c \quad (\sigma \longrightarrow 1+).$$

Thus $F(\sigma) \sim \delta/(\sigma-1)$ for some positive δ as $\sigma \longrightarrow 1+$. By the non-trivial implication of Theorem 1, ψ is u.d. in $(0,\infty)$. #

6. Proof of Theorem 2. Suppose first that ψ satisfies (2). We begin by showing that the bad behavior of $\psi(p)/p$ is limited.

Lemma 3. Suppose that ψ satisfies (5). Then for each $\varepsilon > 0$

$$\sum_{|\psi(p)-p|>\varepsilon p} \frac{1}{\psi(p)} + \frac{1}{p} = K_\varepsilon < \infty.$$

Proof. We divide the primes of the sum into two classes. We have

$$\infty > \sum_{\psi(p)<p-\varepsilon p} \frac{p-\psi(p)}{p\psi(p)} > \sum_{\psi(p)<p-\varepsilon p} \frac{\varepsilon}{\psi(p)} > \sum_{\psi(p)<p-\varepsilon p} \frac{\varepsilon/(1-\varepsilon)}{p},$$

$$\infty > \sum_{\psi(p)>p+\varepsilon p} \frac{\psi(p)-p}{p\psi(p)} > \sum_{\psi(p)>p+\varepsilon p} \frac{\varepsilon/(1+\varepsilon)}{p} > \sum_{\psi(p)>p+\varepsilon p} \frac{\varepsilon}{\psi(p)}. \quad \#$$

For each positive integer k define $B(k)$ to be the smallest positive number for which

$$\sum_{\substack{|\psi(p)-p|>p/k \\ p > B(k)}} \frac{1}{\psi(p)} + \frac{1}{p} < 2^{-k}.$$

We define a "bad set" E to consist of those primes p for which $\psi(p) \leq 1$ and those primes p for which for some $k \geq 2$ we have $|\psi(p) - p| > p/k$ and $p > B(k)$.

For n a positive integer, let (n,E) denote the product of all primes in E which divide n. We shall create a new multiplicative function ψ^* by altering ψ on E. Set

$$\psi^*(p^\alpha) = \begin{cases} \psi(p^\alpha), & p \notin E \\ p^\alpha, & p \in E. \end{cases}$$

We shall show that Theorem 1 applies to ψ^* and then use Lemma to conclude that ψ is u.d. in $(0,\infty)$.

Suppose that $|\psi(p_n) - p_n| > p_n/k$ holds for some positive integer k and some sequence $p_n \longrightarrow \infty$. All p_n exceeding $B(k)$ must lie in E. Thus (1) holds for ψ^*. Condition (2) is obvious.

To show (3) for ψ^*, write for $\sigma > 1$,

$$F^*(\sigma) = \sum_{n\geq 1} \psi^*(n)^{-\sigma} = \prod\nolimits_1(\sigma) \cdot \prod\nolimits_2(\sigma),$$

where

$$\prod\nolimits_1(\sigma) = \prod_p (1 - \psi^*(p)^{-\sigma}) \sum_{\alpha=0}^{\infty} \psi^*(p^\alpha)^{-\sigma},$$

$$\prod\nolimits_2(\sigma) = \prod_p (1 - \psi^*(p)^{-\sigma})^{-1}.$$

The first factor converges uniformly for $\sigma \geq 1$ by (2) and the fact that $\sum \psi(p)^{-2}$ converges.

The second factor equals

$$\zeta(\sigma) \prod_{p \notin E} (1 - \psi(p)^{-\sigma})^{-1}(1 - p^{-\sigma}) =$$

$$\zeta(\sigma) \exp \{ \sum_{p \notin E} \psi(p)^{-\sigma} - p^{-\sigma} \} \exp \{ \sum_{\substack{p \notin E \\ \alpha \geq 2}} \alpha^{-1}(\psi(p)^{-\alpha\sigma} - p^{-\alpha\sigma})\}.$$

The last sum converges uniformly for $\sigma \geq 1$ by (2).

To estimate the remaining sum we write

$$\sum_{p \notin E} \frac{1}{\psi(p)^{\sigma}} - \frac{1}{p^{\sigma}} = \sum_{p \notin E} (\frac{1}{\psi(p)} - \frac{1}{p}) \frac{1}{p^{\sigma-1}} + \sum_{p \notin E} \frac{1}{\psi(p)^{\sigma}} \{1 - (\frac{\psi(p)}{p})^{\sigma-1}\}.$$

Since $|\psi(p) - p| < p/2$ for all sufficiently large $p \notin E$, the quantity in brackets is $\underline{O}(\sigma - 1)$ and the last sum above is $\underline{O}\{(\sigma - 1) \log (\sigma - 1)^{-}$ $\longrightarrow 0$ as $\sigma \longrightarrow 1+$. Also

$$\sum_{p \notin E} \frac{1}{\psi(p)} - \frac{1}{p} = \sum_{p} \frac{1}{\psi(p)} - \frac{1}{p} - \sum_{p \in E} \frac{1}{\psi(p)} - \frac{1}{p},$$

and both sums on the right converge, the first by assumption and the second by the fact that

$$\sum_{p \in E} \frac{1}{\psi(p)} + \frac{1}{p} = \underline{O}(1). \tag{15}$$

The Dirichlet continuity theorem guarantees that

$$\sum_{p \notin E} (\frac{1}{\psi(p)} - \frac{1}{p}) p^{1-\sigma} \longrightarrow \sum_{p \notin E} \frac{1}{\psi(p)} - \frac{1}{p} \quad (\sigma \longrightarrow 1+).$$

It follows that ψ^* satisfies (3) with

$$\delta^* = \prod{}_1(1) \prod_{p \notin E} (1 - \frac{1}{\psi(p)})^{-1} (1 - \frac{1}{p})$$

$$= \prod_{p \notin E} \{(1 - \frac{1}{p}) \sum_{\alpha \geq 0} \psi(p^{\alpha})^{-1}\},$$

and so, by Theorem 1, $N(y,\psi^*) \sim \delta^* y \quad (y \longrightarrow \infty)$.

Now take $U = \{p^\alpha : p \in E, \alpha \geq 1\}$. It is clear that ψ^* satisfies (9) and (10). We deduce from (2) and (15) that ψ^* satisfies (11). Now Lemma 1 asserts that ψ is u.d. in $(0,\infty)$ and has density δ.

We conclude by showing the necessity of (2) in Theorem 2. Suppose that (4) and (5) hold but that (2) fails to hold. For ν positive define a multiplicative function ψ_ν by setting

$$\psi_\nu(p^\alpha) = \begin{cases} \psi(p^\alpha), & \alpha = 1 \text{ or } p^\alpha \leq \nu \\ \max(p^\alpha, \psi(p^\alpha)), & \alpha \geq 2 \text{ and } p^\alpha > \nu. \end{cases}$$

Now ψ_ν satisfies (2), is u.d. in $(0,\infty)$, and has density

$$\delta_\nu = \prod_p \{(1 - \tfrac{1}{p}) \sum_{\alpha \geq 0} \psi_\nu(p^\alpha)^{-1}\}$$

$$= \prod_p (1 - \tfrac{1}{p})(1 + \tfrac{1}{\psi(p)}) \prod_p \left\{1 + \frac{\sum_{\alpha \geq 2} \psi_\nu(p^\alpha)^{-1}}{1 + \psi(p)^{-1}}\right\}.$$

The first product converges since $\psi(p)^{-1} - p^{-1}$ and $1/p\psi(p)$ are each summable. The second product tends to ∞ with ν. Since $\psi_\nu(n) \geq \psi(n)$ for each n and ν, we have

$$\varliminf_{y\to\infty} \frac{1}{y} \{n: \psi(n) \leq y\} \geq \delta_\nu \longrightarrow \infty, \; \nu \longrightarrow \infty. \quad \#$$

7. Proof of Theorem 3 for completely multiplicative functions.

Suppose first that ψ is u.d. in $(0,\infty)$, i.e. $N(y) \sim \delta y$ for some $\delta > 0$. Then we have, uniformly for $\mathrm{Re}\, s > 1$,

$$F(s) = \int_{1-}^{\infty} x^{-s}\, dN(x) = s \int_{1}^{\infty} x^{-s-1}\, N(x)\, dx$$

$$= \delta s \int_{1}^{\infty} x^{-s}(1 + o(1))\, dx = \frac{\delta s}{s-1} + \underline{o}\left(\frac{|s|}{\sigma-1}\right) \quad (\sigma \longrightarrow 1+)$$

$$= \frac{\delta}{s-1} + \underline{o}\left(\frac{1}{\sigma-1}\right) \quad (-T \leq t \leq T, \quad \sigma \longrightarrow 1+).$$

Conversely, we shall estimate $N(y,\psi)$ for a c.m. function ψ satisfying $\psi(p) > 1$ for all p. We apply the Mellin inversion formula to

$$-F'(s) = \sum_{n \geq 1} \psi(n)^{-s} \log \psi(n), \quad \mathrm{Re}\, s > 1,$$

as in Halász's mean value theorem for multiplicative functions [5]. We shall sketch rather briefly those arguments which occur in [5].

For $\sigma = \mathrm{Re}\, s > 1$ we have the formula

$$\sum_{\psi(n) \leq y} \log \psi(n) \log y/\psi(n) = \frac{-1}{2\pi i} \int_{(\sigma)} F'(s)\, y^{s}\, s^{-2}\, ds.$$

If we set $\sigma = 1 + 1/\log y$ and take K to be a (large) positive number, then the integral can be broken into part I with $|\mathrm{Im}\, s| < K/\log y$, part II with $K/\log y \leq |\mathrm{Im}\, s| \leq K$, and part III with $|\mathrm{Im}\, s| > K$.

In part I we apply Cauchy's formula to obtain the estimate

$$-F'(s) = \frac{\delta}{(s-1)^2} + \underline{o}\left(\frac{|s|}{(\sigma-1)^2}\right),$$

uniformly for $\sigma > 1$. We integrate this expression to obtain

$$I = \frac{-1}{2\pi i} \int_{\sigma - iK/\log y}^{\sigma + iK/\log y} F'(s)\, y^{s}\, s^{-2}\, ds \sim c\, y \log y.$$

For II we write

$$|II|^2 = \left|\frac{1}{2\pi i} \int_{\substack{(\sigma) \\ K(\sigma-1)\le|t|\le K}} \frac{-F'(s)}{F(s)} \; s^{-3/4} \; \frac{F(3)}{s^{5/4}} \, y^s \, ds\right|^2$$

$$\le \frac{e^2 y^2}{4\pi^2} \int_{(\sigma)} \left|\frac{F'(s)}{F(s)}\right|^2 \frac{|ds|}{|s|^{3/2}} \int_{\substack{(\sigma) \\ K(\sigma-1)\le|t|\le K}} \frac{|F(s)|^2 \, |ds|}{|s|^{5/2}}$$

$$= \frac{e^2 y^2}{4\pi^2} \; IIa \cdot IIb, \quad \text{say},$$

and estimate each of the two new integrals.

For IIa we use our assumption that $\psi(p)$ is not small too often. For λ real we have

(16)
$$S_\lambda(x) \overset{\text{def}}{=} \sum_{\psi(p^\alpha)\le x} \psi(p^\alpha)^{-i\lambda} \log \psi(p) \le S_0(x)$$

$$\le \log x \sum_{\psi(p)^\alpha \le x} 1 \ll \log x \sum_{\psi(p)\le x} 1 \ll x.$$

Now, for $\sigma > 1$ we write

$$\frac{-F'}{F}(s+i\lambda) = \sum_{n\ge 1} \Lambda'(n) \; \psi(n)^{-i\lambda-s} = \int_1^\infty x^{-s} dS_\lambda(x),$$

$$-\frac{1}{s} \; \frac{F'}{F}(s+i\lambda) = \int_0^\infty e^{-\sigma u} \; S_\lambda(e^u) \; e^{-itu} \, du,$$

and apply a weak form of Plancherel's theorem to obtain

$$\int_{t=-1/2}^{1/2} \left|\frac{F'}{F}(\sigma + i\lambda + it)\right|^2 \frac{dt}{\sigma^2+t^2} \le 2\pi \int_0^\infty e^{-2\sigma u} S_\lambda(e^u)^2\, du,$$

(17) $$\int_{t=\lambda-1/2}^{\lambda+1/2} \left|\frac{F'}{F}(\sigma + it)\right|^2 dt \ll \frac{1}{\sigma-1}.$$

Thus

$$IIa = \int_{t=-\infty}^{\infty} \left|\frac{F'}{F}(\sigma + it)\right|^2 \frac{dt}{(\sigma^2 + t^2)^{3/4}}$$

$$\ll \sum_{n=-\infty}^{\infty} (1 + n^2)^{-3/4} \frac{1}{\sigma-1} \ll \log y.$$

We estimate IIb by writing

$$\int_{\substack{(\sigma) \\ K(\sigma-1)\le|t|\le K}} \frac{|F(s)|^2}{|s|^{5/2}} |ds| \le \max_{K(\sigma-1)\le|t|\le K} \left|\frac{F(\sigma+it)}{\sigma+it}\right|^{1/2} \int_{(\sigma)} \frac{|F(s)^{3/2}|ds|}{|s|^2}$$

$$= IIc \cdot IId, \text{ say.}$$

The hypothesis upon F implies that

$$IIc \le \left\{\frac{\delta}{K(\sigma-1)} + \underline{o}\left(\frac{1}{\sigma-1}\right)\right\}^{1/2} \ll \sqrt{\frac{\log y}{K}}.$$

We estimate IId with the aid of

Lemma 4. Let ψ be a c.m. function which satisfies (3) and (6), and let $F(s) = \sum_{n\ge1} \psi(n)^{-s}$ $(\sigma > 1)$. For $0 < \mu \le 1$ set $G(s) = G(s,\mu) = F(s)^\mu$.

Then G has the Dirichlet series

$$G(s,\mu) = \sum_{n\geq 1} g(n,\mu)\ \psi(n)^{-s} \tag{18}$$

where $g(n) = g(n,\mu)$ $(n = 1,2,\ldots)$ are non negative coefficients and

$$\gamma(x,\mu) \overset{\text{def}}{=} \sum_{\psi(n)\leq x} g(n,\mu) \ll x\ (\log x)^{\mu-1}.$$

(The O-constant may depend upon μ.)

Proof of the lemma. We have

$$F(s,\mu) = \exp\ \{\mu \sum_{n\geq 1} \lambda(n)\ \psi(n)^{-s}\},$$

where $\lambda(n) = 1/\alpha$ for $n = p^{\alpha}$ with α a positive integer and $\lambda(n) = 0$ otherwise. If we develop the exponential in its Maclaurin series we obtain the Dirichlet series (18) with each $g(n,\mu) \geq 0$. (Indeed the $g(n,\mu)$ are coefficients of the Dirichlet series of $\zeta(s)^{\mu}$ as can be seen by taking $\psi(n) = n$.)

Now

$$-\frac{G'}{G}(s) = -\mu\ \frac{F'}{F}(s) = \mu \sum_{m\geq 1} \Lambda'(m)\ \psi(m)^{-s},$$

$$-G'(s) = \sum_{n\geq 1} (g(n)\ \log\ \psi(n))\ \psi(n)^{-s}.$$

The Dirichlet series identity theorem implies that

$$\sum_{\psi(n)\le x} g(n)\log\psi(n) = \mu \sum_{\psi(mn)\le x} \Lambda'(m)\, g(n)$$

$$= \mu \sum_{\psi(n)\le x} g(n) \sum_{\psi(m)\le x/\psi(n)} \Lambda'(m).$$

By (6) the inner sum is $\ll x/\psi(n)$, so

$$\sum_{\psi(n)\le x} g(n)\log\psi(n) \ll x \sum_{\psi(n)\le x} g(n)/\psi(n).$$

Now

$$\sum_{\psi(n)\le x} g(n)/\psi(n) \le e\, G\left(1+\frac{1}{\log x}\right) \ll (\log x)^{\mu},$$

and so

$$\sum_{\psi(n)\le x} g(n)\log\psi(n) \ll x\,(\log x)^{\mu}.$$

The claimed estimate follows by summation by parts. #

Returning to the estimate of IId, we have

$$\frac{1}{s} F(s)^{3/4} = \int_0^{\infty} e^{-\sigma u}\, \gamma(e^u, 3/4) e^{-itu}\, du.$$

Plancherel's theorem and the preceding estimate yield

$$\int_{t=-\infty}^{\infty} \frac{|F(\sigma+it)|^{3/2}}{\sigma^2+t^2}\, dt = 2\pi \int_0^{\infty} e^{-2\sigma u}\, \gamma(e^u,3/4)^2\, du$$

$$\ll (\sigma-1)^{-1/2} = \sqrt{\log y}.$$

Combining all the estimates of II we find that

$$II \ll y(\log y)/K^{1/4}.$$

For part III write

$$|III|^2 = \left|\frac{1}{2\pi i}\int_{\substack{(\sigma)\\ |t|>K}} -\frac{F'}{F}(s)\cdot\frac{1}{s}\cdot\frac{F(s)}{s}\,y^s\,ds\right|^2$$

$$\le \frac{e^2y^2}{4\pi^2}\int_{\substack{(\sigma)\\ |t|>K}}\left|\frac{F'}{F}(s)\right|^2\frac{|ds|}{|s|^2}\cdot\int_{(\sigma)}|F(s)|^2\frac{|ds|}{|s|^2}$$

$$= \frac{e^2y^2}{4\pi^2}\ \text{IIIa}\cdot\text{IIIb},\ \text{say}.$$

We estimate IIIa by use of (17) and obtain

$$\text{IIIa} \ll \sum_{n\ge K}\frac{\log y}{n^2} \ll K^{-1}\log y.$$

We estimate IIIb by using Lemma 4 again, this time with $\mu = 1$, and applying Plancherel's theorem. We have

$$\text{IIIb} = 2\pi\int_0^\infty e^{-2\sigma u}\,\gamma(e^u,1)^2\,du \ll \frac{1}{\sigma-1} = \log y.$$

Thus $|III| \ll (y\log y)/\sqrt{K}.$

If we take K large and combine the estimates I, II, and III, we obtain

$$(19)\qquad \sum_{\psi(n)\le y}\log\psi(n)\,\log(y/\psi(n)) \sim \delta y\log y.$$

Setting $L(x) = \sum_{\psi(n) \leq x} \log \psi(n)$, we see that the left side of (19) is expressible as

$$\int_{t=1}^{y} \log(y/t) dL(t) = \int_{1}^{y} L(t)\, t^{-1}\, dt.$$

Since L is increasing, an easy tauberian argument based on differencing the last integral yields $L(y) \sim \delta\, y \log y$ and summation by parts gives $N(y) \sim \delta y$. #

8. Vanishing of Euler product factors. In the preceding secti we proved Theorem 3 for c.m. functions. In order to extend the theorem to multiplicative functions, we shall show that if ψ is a multiplicative function which satisfies (7) and ψ^* is an associated c.m function, then ψ^* also satisfies (7).

Under the hypotheses of Theorem 3 at most a finite number of factors

$$F(p,s) \overset{def}{=} 1 + \psi(p)^{-s} + \psi(p^2)^{-s} + \ldots$$

of the Euler product of $F(s)$ can have zeros in the closed half plane $\{s\colon \mathrm{Re}\, s \geq 1\}$. In this section we show that if we remove from $F(s)$ those factors $F(p,s)$ which vanish or have small values somewhere in the half plane, then the resulting function still satisfies (7).

Lemma 5. Let $s = \sigma + it$. Suppose that ψ satisfies (2) and that (7) holds uniformly as $\sigma \longrightarrow 1+$ on any fixed interval $-T \leq t \leq$ T Let V be the (finite) set of primes p for which $|F(p,s)| \leq 1/2$ for some s with $\sigma \geq 1$. Let

$$P(s) = \prod_{p\in V} F(p,s) \quad \underline{\text{and}} \quad G(s) = F(s)/P(s).$$

Then G satisfies (7) with constant $\delta' = \delta/P(1))$ uniformly as $\sigma \longrightarrow 1+$ on any fixed interval $-T \leq t \leq T$.

Proof. We have $P(1) > 0$, and each factor $F(p,s)$ is analytic at $s = 1$. Thus $1/P(s)$ is analytic near 1 and

$$G(s) = \frac{F(s)}{P(s)} = \left\{\frac{\delta}{s-1} + \underline{o}\left(\frac{1}{\sigma-1}\right)\right\}\left\{\frac{1}{P(1)} + \underline{\underline{O}}(|s-1|)\right\}$$

$$= \frac{\delta/P(1)}{s-1} + \underline{o}\left(\frac{1}{\sigma-1}\right)$$

holds uniformly as $\sigma \longrightarrow 1+$ for some interval $-\Delta < t < \Delta$.

If $F(q,s_\nu) = 0$ for some $q \in V$ and some $s_0 = \sigma_0 + it_0$ with $\sigma_0 \geq 1$, then $\sigma_0 = 1$ for otherwise G would have a pole with real part exceeding 1. This is impossible, since for $\sigma > 1$

$$|G(s)| \leq G(\sigma) \leq F(\sigma) < \infty.$$

Now suppose that the lemma is false and there exists a sequence $\sigma_n + it_n$, $\Delta \leq |t_n| \leq T$, $\sigma_n \downarrow 1$ and a positive constant b such that

$$|G(\sigma_n + it_n)| > b/(\sigma_n - 1). \tag{20}$$

By compactness there is a convergent subsequence, which we also call t_n, having limit t_0, $\Delta \leq |t_0| \leq T$. We have

$$\frac{b}{\sigma_n - 1} < \frac{|F(\sigma_n+it_n)|}{|P(\sigma_n+it_n)|} = \frac{\underline{o}(1/(\sigma_n-1))}{|P(\sigma_n+it_n)|}$$

so $P(1 + it_0) = 0$ and at least one factor $F(p,s)$ vanishes at $1 + it_0$.

In the remainder of the proof we show that the blow up of G near $1 + it_0$ is repeated near $1 + 2it_0$, $1 + 3\,it_0, \ldots$, but that the exists some positive integer m for which $P(1 + imt_0) \neq 0$. Thus F wil be large near $1 + imt_0$ in violation of (7).

For $\mathrm{Re}\, s > 1$ we have

$$\log G(s) = \sum_{p \notin V} \log \{ \sum_{\alpha \geq 0} \psi(p^{\alpha})^{-s} \}$$

$$= \sum_{p \notin V} \psi(p)^{-s} + \sum_{p \notin V} (\log \{ \sum_{\alpha \geq 0} \psi(p^{\alpha})^{-s} \} - \psi(p)^{-s})$$

$$= \sum_{p} \psi(p)^{-s} + \underline{\underline{O}}(1)$$

by (2), the finiteness of V, and the fact that $\sum_p \psi(p)^{-2}$ converges. Let $g(s) = \sum_p \psi(p)^{-s}$. We have

$$g(\sigma) = \log G(\sigma) + \underline{\underline{O}}(1) = \log \frac{1}{\sigma - 1} + \underline{\underline{O}}(1),$$

and for $n = 1, 2, \ldots$

$$\mathrm{Re}\; g(\sigma_n + it_n) \geq \log \frac{1}{\sigma_n - 1} + \underline{\underline{O}}(1).$$

Since $g(\sigma) \geq \mathrm{Re}\; g(\sigma + it)$ for any $\sigma > 1$ and any real t, it follows tha

$$\mathrm{Re}\; g(\sigma_n + it_n) = \log \frac{1}{\sigma_n - 1} + \underline{\underline{O}}(1).$$

We claim that

(21) $$\mathrm{Re}\; g(\sigma_n + ir\,t_n) = \log \frac{1}{\sigma_n - 1} + \underline{\underline{O}}_r(1)$$

holds for $r = 2, 3, 4, \ldots$, where $\underline{\underline{O}}_r$ may depend on r (but not on n).

If we expand the relation

$$0 \leq (N - \cos\theta - \cos 2\theta - \ldots - \cos N\theta)^2,$$

we obtain the inequality

$$(22) \qquad \sum_{r=N+1}^{2N} (2N+1-r)\cos r\theta \geq \sum_{r=1}^{N} (2N+r+1)\cos r\theta - 2N^2 - N.$$

We use this inequality inductively for $N = 1,2,4,8,\ldots$

For $N = 1$ we have $\cos 2\theta \geq 4\cos\theta - 3$, so

$$\begin{aligned}\operatorname{Re} g(\sigma_n+2it_n) &\geq 4 \operatorname{Re} g(\sigma_n+it_n) - 3g(\sigma) \\ &\geq \log \frac{1}{\sigma_n - 1} + \underline{\underline{O}}(1),\end{aligned}$$

and (21) holds for $r = 2$. Now suppose that (21) holds for all positive integers $r \leq 2^k = N$. Then (22) gives

$$\sum_{r=N+1}^{2N} (2N+1-r) \operatorname{Re} g(\sigma_n+irt_n) \geq \{\sum_{r=1}^{N} (2N+r+1) - 2N^2 - N\} \log \frac{1}{\sigma_n - 1} + \underline{\underline{O}}(1)$$

Now

$$\sum_{r=N+1}^{2N} (2N+1-r) = \sum_{r=1}^{N} (2N+r+1) - 2N^2 - N$$

and for $r = N+1,\ldots,2N$

$$\operatorname{Re} g(\sigma + irt) \leq g(\sigma) = \log \frac{1}{\sigma - 1} + \underline{\underline{O}}(1).$$

Thus (21) holds for $N + 1 \leq r \leq 2N$.

Now we show that $P(1 + imt_0) \neq 0$ for some positive integer m

Choose ν sufficiently large that for each $p \in V$,

$$\sum_{i>\nu} \psi(p^i)^{-1} \leq 1/2.$$

Then each factor in $P(s)$ has the form

$$F(p,s) = 1 + \psi(p)^{-s} + \ldots + \psi(p^{\nu})^{-s} + \theta/2, \quad |\theta| \leq 1.$$

Dirichlet's theorem on simultaneous approximation insures that there is some positive integer m such that for each $p \in V$ and $1 \leq \alpha \leq \nu$ we have

$$\cos(m\, t_0 \log \psi(p^{\alpha})) \geq 0.$$

Thus each factor $F(p,s)$ of $P(s)$ satisfies

$$|F(p,1+imt_0)| \geq \operatorname{Re} F(p,1+imt_0) \geq 1/2,$$

and so $|P(1+imt_0)| \geq 2^{-|V|} > 0$.

Now we have

$$|F(\sigma_n + imt_n)| = |G(\sigma_n + imt_n)|\ |P(\sigma_n + imt_n)|$$

$$\geq \frac{A}{\sigma_n - 1} \{|P(1 + imt_0)| - \underline{o}(1)\} \qquad (n \longrightarrow \infty)$$

for some positive A, in violation of (7). Thus G satisfies (7) uniformly on any interval $[-T,T]$. #

9. Proof of Theorem 3 for multiplicative functions.

Given ψ a multiplicative function, the deduction of (7) from the assumption that ψ is u.d. in $(0,\infty)$ is made exactly as in §6. To prove the converse we introduce the c.m. function ψ^* defined by

$$\psi^*(p) = \begin{cases} \psi(p) & \text{if } \psi(p) > 1 \\ p & \text{if } \psi(p) \leq 1 \end{cases}$$

and let

$$F^*(s) = \sum_{n\geq 1} \psi^*(n)^{-s} = \prod_p \{1 - \psi^*(p)^{-s}\}^{-1}.$$

We show first that ψ^* inherits property (7) from ψ. Then ψ^* is u.d. by the special case of Theorem 3, and finally we conclude that ψ is u.d. in $(0,\infty)$ exactly as we did in proving Theorem 1.

Let K be a (large) positive number. We show that (7) holds for ψ^* in the wedge

$$\{s: \sigma > 1, |t| < K(\sigma-1)\}.$$

Arguing exactly as we did in showing $F^*(\sigma) \sim \delta^*/(\sigma-1)$ for Theorem 1, we obtain

$$F^*(s) = F(s)\,\frac{F^*(s)}{F(s)} = \{\frac{\delta}{s-1} + \underline{o}(\frac{1}{\sigma-1})\}\ (c + \underline{o}(1))$$

$$= \frac{\delta^*}{s-1} + \underline{o}(\frac{1}{\sigma-1})$$

as $s \longrightarrow 1$ in the wedge.

For the region $\{s: \sigma > 1, K(\sigma-1) \leq |t| \leq T\}$ we give an $\underline{o}$-estimate for $F^*(s)$ as $\sigma \longrightarrow 1+$. Let

$$F(s,p) = 1 + \psi(p)^{-s} + \psi(p^2)^{-s} + \ldots$$

and

$$V = \{p: |F(s,p)| \leq 1/2 \text{ for some } s \text{ with } \sigma \geq 1\}.$$

We have

$$F^*(s) = \prod_{p\in V} (1 - \psi^*(p)^{-s})^{-1} \prod_{p\notin V} F(s,p) \Big/ \prod_{p\notin V} (1 - \psi^*(p)^{-s})F(s,p)$$

$$= \prod_1(s) \prod_2(s) \Big/ \prod_3(s), \quad \text{say.}$$

There are only a finite number of factors in $\prod_1$, and each is bounded for $\sigma \geq 1$. Thus $| \prod_1(s) | \leq B_1$ for $\sigma \geq 1$. By Lemma 5

$$\prod_2(s) = \delta'/(s-1) + \underline{o}(1/(\sigma-1))$$

uniformly for $-T \leq t \leq T$ as $\sigma \longrightarrow 1+$. Thus

$$| \prod_2(s) | \leq \delta' K^{-1}/(\sigma-1) + \underline{o}(1/(\sigma-1))$$

for $K(\sigma-1) \leq |t| \leq T$ as $\sigma \longrightarrow 1+$.

Each factor of $\prod_3$ is bounded below in modulus for $\sigma \geq 1$. We remove the finite number of factors for which $\psi(p) \leq 1$ and note that

$$\prod_3{}'(s) = \prod_{\substack{p\notin V \\ \psi(p)>1}} \{1 - \psi(p)^{-2s} + (1-\psi(p)^{-s})(\psi(p^2)^{-s} + \psi(p^3)^{-s} + \ldots\}$$

is uniformly bounded below for $\sigma \geq 1$. Thus $| \prod_3(s) | \geq B_3 > 0$ for $\sigma \geq 1$.

For $K(\sigma-1) \leq |t| \leq T$ and any $\varepsilon > 0$ we now have

$$|F^*(s)| \leq B_1 \{\frac{\delta'}{K(\sigma-1)} + \underline{o}(\frac{1}{\sigma-1})\}/B_3 < \frac{\varepsilon}{\sigma-1}$$

provided that K is sufficiently large and $\sigma-1$ sufficiently small.

Thus ψ^* satisfies (7) uniformly on $[-T,T]$ as $\sigma \longrightarrow 1+$. Now ψ^* is u.d. in $(0,\infty)$ and so ψ is also u.d. in $(0,\infty)$. #

10. Proof of Theorem 4. Suppose first that ψ is u.d. in $(0,\infty)$. The trivial implication of Theorem 3 implies that

$$\sum_p \psi(p)^{-\sigma} = \log \frac{1}{\sigma-1} + c' + \underline{o}(1) \quad (\sigma \longrightarrow 1).$$

The condition $\log \psi(p)/p = \underline{\underline{O}}(1)$ gives

$$\sum_p \frac{p^{1-\sigma}}{\psi(p)} - \frac{1}{\psi(p)^{\sigma}} = \sum_p \psi(p)^{-\sigma} \{(\frac{\psi(p)}{p})^{\sigma-1} - 1\}$$

$$<< (\sigma-1) \sum_p p^{-\sigma} \sim (\sigma-1) \log \frac{1}{\sigma-1} \longrightarrow 0.$$

These estimates and an elementary zeta function relation yield

$$\sum_p (\frac{1}{\psi(p)} - \frac{1}{p}) p^{1-\sigma} = c + \underline{o}(1) \quad (\sigma \longrightarrow 1+).$$

As in the proof of Corollary 1, it follows that

(23) $$\sum_p 1/\psi(p) - 1/p = c.$$

For $t \neq 0$ we have

$$\sum_p \frac{1}{p} - \mathrm{Re} \frac{1}{\psi(p)^{1+it}} = \sum_p \frac{1}{p} - \frac{1}{\psi(p)} + \sum_p \frac{1}{\psi(p)} (1 - \mathrm{Re}\, \psi(p)^{-it}).$$

The first series on the right converges as we have seen. The terms of the second series are non negative and thus equal

(24) $$\lim_{\sigma\to 1+} \sum_p \psi(p)^{-\sigma} \{1 - \mathrm{Re}\, \psi(p)^{-it}\}.$$

By the easy implication of Theorem 3, $(\sigma-1)|F(\sigma+it)| \longrightarrow 0$

as $\sigma \longrightarrow 1+$ for each $t \neq 0$. It follows that

$$\sum_p p^{-\sigma} - \mathrm{Re}\, \psi(p)^{-\sigma-it} \longrightarrow +\infty \qquad (\sigma \longrightarrow 1+).$$

Also, $\sum_p \psi(p)^{-\sigma} - p^{-\sigma}$ converges to a finite limit as $\sigma \longrightarrow 1+$. It follows that the limit in (24) is $+\infty$ and thus

$$\text{(25)} \qquad \sum_p \frac{1}{p} - \mathrm{Re}\, \frac{1}{\psi(p)^{1+it}} = \begin{cases} +\infty, & t \neq 0 \\ c \in \mathbb{R}, & t = 0. \end{cases}$$

Now suppose that (25) holds. We shall show that condition (7) of Theorem 3 holds uniformly for each compact interval $-T \leq t \leq T$ as $\sigma \longrightarrow 1+$ and thus deduce that ψ is u.d. in $(0,\infty)$. Suppose that $\eta \geq |\log \psi(p)/p|$ for all p. Given $\varepsilon > 0$ we divide the rectangle $\{(\sigma>1) \times (0\leq t\leq T)\}$ into three ranges:

$$0 < t < \frac{\sigma-1}{\varepsilon}, \quad \frac{\sigma-1}{\varepsilon} \leq t < 1/\eta, \text{ and } 1/\eta \leq t \leq T.$$

(The estimates will be valid for $[-T,0]$ by symmetry.)

Suppose that $0 < t < (\sigma-1)/\varepsilon$. If we can show that

$$\text{(26)} \qquad \sum_p \psi(p)^{-s} - p^{-s} \longrightarrow -c$$

uniformly as $s \longrightarrow 1$ in the wedge, then we will have

$$\log F(s) = \log \frac{\delta}{s-1} + \underline{o}(1)$$

and hence (7) holds uniformly as $s \longrightarrow 1$ within the wedge.

By assumption

$$\sum_{p\leq x} \psi(p)^{-1} - p^{-1} \longrightarrow -c \quad (x \longrightarrow \infty),$$

and arguing as in the proof of Corollary 1 we show that

$$\sum_{\psi(p)\leq x} \frac{1}{\psi(p)} - \sum_{p\leq x} \frac{1}{p} \longrightarrow -c \quad (x \longrightarrow \infty).$$

Summation by parts implies that (26) and hence (7) holds for $|t| < (\sigma-1)/\varepsilon$.

Now suppose that $1/\eta \leq t \leq T$. The two conditions of (25) together imply that

$$\sum_{p} \psi(p)^{-1} \{1 - \text{Re}\ \psi(p)^{-it}\} = +\infty$$

holds for each $t \neq 0$. Since the terms of this series are non negative continuous functions of t, Dini's theorem insures that the series sums uniformly to $+\infty$ for $1/\eta \leq t \leq T$. We then have by partial summation that

$$\log\{F(\sigma)/|F(\sigma + it)|\} \longrightarrow +\infty$$

uniformly as $\sigma \longrightarrow 1+$. Since $\log F(\sigma) - \log \frac{\delta}{\sigma-1} \longrightarrow 0$ as $\sigma \longrightarrow 1+$ by the estimate for $|t| < (\sigma-1)/\varepsilon$, we have $F(\sigma + it) = o\{1/(\sigma-1)\}$ uniformly for $1/\eta \leq t \leq T$.

For $(\sigma-1)/\varepsilon \leq t < 1/\eta$ we shall show that $(\sigma-1)|F(s)| \longrightarrow 0$ uniformly as $\sigma \longrightarrow 1+$. For $\sigma > 1$ we have

$$\log F(\sigma) - \log|F(s)| = \sum_{p} \psi(p)^{-\sigma} \{1 - \text{Re}\ \psi(p)^{-it}\} + O(1),$$

and we show the right side to be uniformly large as $\sigma \longrightarrow 1+$ in the region.

Let $\sigma = 1 + 1/\log x$. Then

$$\sum_p \psi(p)^{-\sigma}\{1 - \operatorname{Re}\psi(p)^{-it}\} \geq c \sum_{\substack{p \leq x \\ \cos(t \log \psi(p)) \leq 0}} 1/p$$

since $\psi(p)^{\sigma-1} < (xe^{\eta})^{\sigma-1} << 1$. Now $\cos(t \log \psi(p)) \leq 0$ for

$$a_n = t^{-1}(2\pi n + \pi/2) + \eta < \log p < t^{-1}(2\pi n + 3\pi/2) - \eta = b_n.$$

For σ near 1 we have

$$\sum_{\substack{p \leq x \\ \cos(t \log \psi(p)) \leq 0}} 1/p \geq \sum_{0 \leq n \leq .1t \log x} \sum_{a_n < \log p \leq b_n} 1/p$$

$$\geq \sum_{0 \leq n \leq .1t \log x} \log b_n - \log a_n - c/a_n^2.$$

Now

$$\log b_n - \log a_n \geq \frac{b_n - a_n}{b_n} = \frac{\pi - 2\eta t}{2\pi n + 3\pi/2 - \eta t} > \frac{1}{2\pi(n+1)},$$

and so

$$\sum_{\substack{p \leq x \\ \cos(t \log \psi(p)) \leq 0}} 1/p \geq \frac{1}{2\pi} \sum_{1 \leq n \leq .1t \log x} \frac{1}{n} - c't^2 \sum_{n=0}^{\infty} (n + 1/4)^{-2}$$

$$\geq \frac{1}{2\pi} \log \{t/(\sigma-1)\} - c''$$

$$\geq \frac{1}{2\pi} \log 1/\varepsilon - c''.$$

The inequalities imply that

$$|F(s)| \leq B\varepsilon^{1/2\pi}/(\sigma-1)$$

holds for $(\sigma-1)/\varepsilon \leq t < 1/\eta$ and all σ sufficiently near 1+. Here B is a number depending on ψ but not on σ or ε.

Thus (7) holds uniformly for $0 \leq t \leq T$, and by symmetry for $-T \leq t < 0$ as well. Now Theorem 3 insures that ψ is u.d. in $(0,\infty)$. #

11. Connections with the mean value of $n/\psi(n)$.

Let $h(n) \overset{\text{def}}{=} n/\psi(n)$. In this section we shall prove Theorems 5 and 6. The first of these asserts that if ψ is a "reasonable" multiplicative function whose values are u.d. in $(0,\infty)$, then h has a mean value. We shall show by example (§13) that this theorem is false if condition (8) $(1 \ll \psi(p)/p \ll 1)$ is omitted. Another example (§13) shows that the converse Theorem 6 is not valid if condition (1) $(\psi(p) \sim p)$ is replaced by (8).

The proof of Theorem 5 depends upon the following two lemmas which together gives estimates of $H(s) = \sum_{n\geq 1} h(n)n^{-s}$ in a neighborhood of the line $\sigma = 1$.

Lemma 6. Suppose that ψ is multiplicative and satisfies (2) and (8). Given any positive K and ε there exists $\eta > 0$ such that $|H(s)/F(s) - 1| < \varepsilon$ holds uniformly on the wedge shaped region $\{s: 1 < \sigma < 1 + \eta, \ |t| \leq K(\sigma-1)\}$.

Proof of the lemma. For $\sigma > 1$, write

$$\frac{H(s)}{F(s)} = \prod_p \{1 + \frac{H(p) - F(p)}{F(p)}\},$$

where

$$H(p) = H(p,s) = \sum_{\alpha\geq 0} p^{(1-s)\alpha}/\psi(p^\alpha),$$

$$F(p) = F(p,s) = \sum_{\alpha\geq 0} \psi(p^\alpha)^{-s}.$$

For all sufficiently large primes p we have

$$|F(p,s)| \geq 1 - \sum_{\alpha\geq 1} \psi(p^\alpha)^{-1} > 1/2$$

uniformly on $\{s: \sigma > 1\}$. For small primes p we note that $F(p,1) > 1$ and so $|F(p,s)| > 1/2$ for all s sufficiently near 1 by continuity. If η is sufficiently small then $|F(p,s)| > 1/2$ for all p and all s in the wedge.

To complete the proof of the lemma it suffices to show that $\sum_p |H(p,s) - F(p,s)|$ is uniformly small throughout the wedge. We have

$$\sum_p |H(p,s) - F(p,s)| \leq \sum_{p,\alpha} \left|\frac{p^{\alpha(1-s)}}{\psi(p^\alpha)} - \psi(p^\alpha)^{-s}\right| =$$

$$\sum_p \frac{1}{\psi(p)^\sigma} \left| \left(\frac{\psi(p)}{p}\right)^{s-1} - 1\right| + \sum_{p,\alpha\geq 2} \frac{1}{\psi(p^\alpha)} \left|p^{\alpha(1-s)} - \psi(p^\alpha)^{1-s}\right|$$

$$= \sum\nolimits_1 + \sum\nolimits_2, \text{ say.}$$

Now $\log \psi(p)/p$ is bounded and we have

$$|(\psi(p)/p)^{s-1} - 1| \leq B|s-1| \leq B(K+1)(\sigma-1)$$

for all s in the wedge. It follows that

$$\sum\nolimits_1 \leq B(K+1)(\sigma-1) \sum_p \frac{1}{(ap)^\sigma} < B'K(\sigma-1) \log \frac{1}{\sigma-1} \longrightarrow 0$$

uniformly as $s \longrightarrow 1$ in the wedge.

The absolute value expression in $\sum_2$ is bounded for $1 < \sigma < 2$ for all pairs p,α. (An upper bound for the expression is 2 for all pairs p,α except possibly for at most a finite number of cases where $\psi(p^\alpha) < 1$.) Since we have assumed that $\sum_{p,\alpha\geq 2} \psi(p^\alpha)^{-1}$ converges, it follows that $\sum_2 \longrightarrow 0$ uniformly as $s \longrightarrow 1$ in the wedge.

Thus, for η sufficiently small, the ratio of the generating functions is within ε of 1 for all points s within the wedge. #

Lemma 7. Suppose that ψ is multiplicative and satisfies (2) and (8). Let K and T be any positive numbers. Then there exist positive numbers γ, M and η such that for $1 < \sigma < 1 + \eta$ and $K(\sigma-1) < |t| < T$ we have

(27) $$|H(s)| \leq M\, H(\sigma) K^{-\gamma}.$$

Proof. For at most a finite number of primes p will we have $|H(p,s) - 1| \geq 1/2$ at any point s with $\sigma \geq 1$. Since all factors $H(p,s)$ of $H(s)$ are bounded for $1 \leq \sigma \leq 2$ and $H(p,\sigma) \geq 1$ for each p, it suffices to prove the lemma under the condition that $|H(p,s) - 1| < 1/2$ holds for all primes p and all s with $\sigma > 1$.

For $\sigma > 1$ write

$$\log H(s) = \sum_p \log \{ \sum_{\alpha\geq 0} p^{\alpha(1-s)}/\psi(p^\alpha)\}$$

$$= \sum_1 + \sum_2 + \sum_3,$$

where

$$\sum_1 = \sum_p \log \{ \sum_{\alpha \geq 0} p^{\alpha(1-s)}/\psi(p^\alpha)\} - p^{1-s}/\psi(p),$$

$$\sum_2 = \sum_p \psi(p)^{-\sigma} \{(\frac{\psi(p)}{p})^{\sigma-1} - 1\}p^{-it}$$

$$\sum_3 = \sum_p p^{-it} \psi(p)^{-\sigma}.$$

The first sum converges absolutely for $\sigma \geq 1$, as we see by considering the Taylor series of $\log (1 + z)$, and is bounded on this half plane. The second sum is small for σ near 1 by the estimate of $\sum_1$ of the preceding lemma.

We estimate the third sum by exploiting the fact that for $t \neq 0$ the values of p^{-it} are well distributed mod 2π. By calculations like these used in the end of §10, we find that

$$\text{Re} \sum_3 \leq \log M + \log H(\sigma) - \gamma \log K,$$

and thus (27) holds for $1 < \sigma < 1 + \eta$ and $K(\sigma-1) < |t| < T$. #

Proof of Theorem 5. By [Halász, Satz 3], it suffices to show that

$$H(s) - \frac{\delta}{s-1} = \underline{o}(\frac{|s|}{\sigma-1}) \tag{28}$$

uniformly as $\sigma \longrightarrow 1+$. Let $\varepsilon > 0$ be given. We show that the right side of (28) is less than $\varepsilon|s|/(\sigma-1)$ uniformly as $\sigma \longrightarrow 1+$ on each of three regions.

First, take $T = 3\delta/\varepsilon$, where δ is the density of values of ψ. We have, trivially,

$$\left|H(s) - \frac{\delta}{s-1}\right| \leq H(\sigma) + \frac{\delta}{\sigma-1} < \frac{3\delta}{\sigma-1} < \frac{\varepsilon|s|}{\sigma-1}$$

provided σ is near enough to 1 that

$$H(\sigma) \sim F(\sigma) \sim \frac{\delta}{\sigma-1} < \frac{2\delta}{\sigma-1}$$

and $|t| > T = 3\delta/\varepsilon$.

Next, choose K so large that $K^{\gamma} > 3\,\delta M/\varepsilon$ and $K > 2\,\delta/\varepsilon$, where M and γ are constants from Lemma 7.

Lemmas 7 and 6 imply that

$$|H(s)| < \frac{\varepsilon}{3\delta} H(\sigma) \sim \frac{\varepsilon}{3(\sigma-1)} < \frac{\varepsilon}{2(\sigma-1)}$$

for $1 < \sigma \leq$ some σ_1 and $K(\sigma-1) \leq |t| \leq T$. Thus

$$\left|H(s) - \frac{\delta}{s-1}\right| \leq |H(s)| + \frac{\delta}{|t|} < \frac{\varepsilon}{2(\sigma-1)} + \frac{\delta}{K(\sigma-1)} < \frac{\varepsilon}{\sigma-1}$$

holds uniformly for $K(\sigma-1) \leq |t| \leq T$, $1 < \sigma \leq \sigma_1$.

Finally, for $|t| < K(\sigma-1)$ we apply Lemma 6. Write

$$|H(s) - F(s)| = \left|\frac{H(s)}{F(s)} - 1\right| \, |F(s)| < \frac{\varepsilon}{3\delta} F(\sigma)$$

$$\sim \frac{\varepsilon}{3\,(\sigma-1)} < \frac{\varepsilon}{2(\sigma-1)}$$

for $1 < \sigma \leq$ some σ_2 and $|t| < K(\sigma-1)$. Also, by the easy assertion of Theorem 3 we have that

$$|F(s) - \delta/(s-1)| < \varepsilon|s|/(2\sigma-2)$$

for $1 < \sigma \leq$ some σ_3 and all t. Thus (28) holds also for $|t| < K(\sigma-1)$.

The other hypotheses of Halász's theorem on mean values are easily seen to be satisfied. Thus h has mean value δ. #

Proof of Corollary 3. The c.m. function ψ assumes each positive integer value exactly once. Thus ψ is u.d. in $(0,\infty)$ with density 1. Now Theorem 5 applies and we conclude that $n \longmapsto n/\psi(n)$ has a mean value, which also equals 1. #

Proof of Theorem 6. Since h has a mean value we have

$$\sum_{n\geq 1} h(n)n^{-\sigma} \sim \delta/(\sigma-1), \quad (\sigma \longrightarrow 1+).$$

Lemma 6 guarantees that

$$\sum_{n\geq 1} \psi(n)^{-\sigma} \sim \sum h(n)n^{-\sigma} \quad (\sigma \longrightarrow 1+).$$

Together these relations imply that ψ satisfies condition (3) of Theorem 1, and by that result ψ is u.d. in $(0,\infty)$ with density δ. #

12. Zero and infinite densities. Here we prove Theorems 7 and 8.

Proof of Theorem 7. If the values of ψ have zero density in $(0,\infty)$, then summation by parts shows that $(\sigma-1)F(\sigma) \longrightarrow 0$. (The bound on S is not needed here.)

For the converse we proceed as in the proof of Theorem 1. We first introduce a c.m. function ψ^* and verify that it inherits the assumed properties of ψ and satisfies the conditions of Lemma 1 (with the obvious interpretation of $\sim$). Then we prove the theorem for a c.m. function. In place of the tauberian theorem of Hardy, Littlewood, and Karamata we use the following simple inequality. Let $\sigma = 1 + 1/\log y$. Then

$$\sum_{\psi(n)\le y} \frac{1}{\psi(n)} \le e\, F(\sigma) = \underline{o}(\sigma-1)^{-1} = \underline{o}\ (\log y). \qquad \#$$

Proof of Theorem 8. If the values of ψ have infinite density in $(0,\infty)$, then summation by parts shows that $(\sigma-1)F(\sigma) \longrightarrow \infty$. (The bound on S is not used here.)

For the converse we use an argument occurring in an unpublished manuscript of Halász on large deviations of additive arithmetic functions, generalizing results in [6].

Lemma 2 and the lower bound assumptions on

$$S(z) \overset{\text{def}}{=} \sum_{\psi(p)^{\alpha}\le z} \log \psi(p)$$

give

$$\sum_{\psi(n)\le x} \log \psi(k) = \sum_{\psi(n)\le x} S(x/\psi(n)) \ge ax \sum_{\psi(n)\le x} \psi(n)^{-1}$$

$$\ge\ ax\{F(\sigma) - \sum_{\psi(n)>x} \psi(n)^{-\sigma}\} \qquad (\sigma > 1).$$

We now find an upper bound for the last sum. Let $\sigma^* = (\sigma+1)/2$. Then

$$\sum_{\psi(n)>x} \psi(n)^{-\sigma} \le x^{-(\sigma-1)/2} \sum_{\psi(n)>x} \psi(n)^{-\sigma^*} \le x^{-(\sigma-1)/2} F(\sigma^*).$$

The hypothesis $\sum_{\psi(x)\le x} \log \psi(p)/\psi(p) \ll \log x$ implies that

$$-\frac{F'}{F}(u) = \sum_{n\ge 1} \Lambda'(n)\psi(n)^{-u} \le A/(u-1)$$

holds for all $u > 1$ and so

$$\log\{F(\sigma^*)/F(\sigma)\} = \int_{\sigma^*}^{\sigma} -\frac{F'}{F}(u)\, du \le \frac{A}{\sigma^*-1}(\sigma - \sigma^*) = A.$$

We find that

$$\sum_{\psi(n)>x} \psi(n)^{-\sigma} \le e^{A} x^{-(\sigma-1)/2} F(\sigma).$$

If we choose $\sigma = 1 + (2A + \log 4)/\log x$, then $e^{A} x^{-(\sigma-1)1/2} = 1/2$ and we have

$$\sum_{\psi(n)\le x} \log \psi(n) \ge ax\, F(\sigma)/2 = \frac{ax \log x}{2(2A+\log 4)} (\sigma-1)\, F(\sigma).$$

Now

$$x^{-1} \sum_{\psi(k)\le x} 1 \ge \frac{1}{x \log x} \sum_{\psi(k)\le x} \log \psi(n) \longrightarrow \infty$$

as $x \longrightarrow \infty$, since $(\sigma-1)\, F(\sigma) \longrightarrow \infty$ as $\sigma \longrightarrow 1+$. #

13. Examples.

Example 1. ψ not u.d. in $(0,\infty)$. Let $b > 1$, $\beta = (\log b)/(b-1)$, ψ c.m. and for $k = 0,1,2,..$ set

$$\psi(p) = b^k, \quad \beta b^k \leq p < \beta b^{k+1}.$$

For $0 < u \leq 1$ we have

$$\sum_{\beta b^k \leq p < \beta b^{k+u}} \frac{1}{\psi(p)} - \frac{1}{p} = \frac{\beta(b^u-1)}{k \log b} - \frac{u}{k} + \underline{\underline{O}}\left(\frac{1}{k^2}\right).$$

For $u = 1$, this sum $= \underline{\underline{O}}(k^{-2})$, which is summable on k. For $0 < u < 1$, the above sum tends to zero as $k \longrightarrow \infty$. It follows that

$$\sum_p \psi(p)^{-1} - p^{-1} \text{ converges.}$$

Since $\psi(p)^{-s} = \psi(p)^{-s-2\pi i/\log b}$ holds for each prime p, we have

$$F(s) \overset{\text{def}}{=} \sum_{n \geq 1} \psi(n)^{-s} = F(s + 2\pi i/\log b).$$

In particular

$$F(\sigma + 2\pi i/\log b) \neq \underline{o}(F(\sigma)),$$

and by Theorem 3, ψ is not u.d. in $(0,\infty)$.

Remarks. If b is a number rather near 1, then $\psi(p)/p$ is quite close to 1. Thus Theorem 1 is false if condition (1) is weakened. Also, in this case, condition (5) of Theorem 2 holds (vacuuously) for any $\varepsilon \geq (b-1 - \log b)/\log b$, but not hold for smaller ε. Thus Theorem 2 is false if (5) is required only for a fixed positive ε.

With ψ as above, the function $h\colon n \longmapsto n/\psi(n)$ <u>has a mean value</u>. Let $H(s) = \sum_{n\geq 1} h(n)n^{-s}$ $(\sigma > 1)$ and

$$\log H(s) = \sum_{p} p^{1-\sigma-it}/\psi(p) + \underline{O}(1) \quad (\sigma > 1).$$

The arguments used in the proof of Theorem 4 to show that (7) holds for F for $|t| < (\sigma-1)/\varepsilon$ and for $(\sigma-1)/\varepsilon \leq |t| \leq 1/\eta$ can be adapted (simplified!) to show that H satisfies (7) as $\sigma \longrightarrow 1+$, uniformly for $-T \leq t \leq T$, any fixed T. Thus h has a mean value even though ψ is not u.d. in $(0,\infty)$. This shows that Theorem 6 is false without condition (1).

<u>Example</u> 2. ψ <u>u.d. in</u> $(0,\infty)$, h <u>has no mean value</u>. Let ψ be the c.m. function which rearranges the primes in each interval $[2^{2^n}, 2^{2^{n+1}})$ in decreasing order, i.e. if $p_e < \ldots < p_m$ are all the primes in such an interval, then $\psi(p_e) = p_m, \ldots, \psi(p_m) = p_e$. (This rearrangement is not of bounded ratio in the sense of Corollary 2.)

Since ψ assumes each positive integer value exactly once, ψ is u.d. in $(0,\infty)$. We suppose that $h\colon n \longmapsto n/\psi(n)$ has a mean value and show a contradiction.

Chebyshev's identity of elementary prime number theory asserts that

$$\log n = \sum_{jk=n} \Lambda(k).$$

where

$$\Lambda(k) = \begin{cases} \log p & \text{if } k = p^{\alpha},\ \alpha \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

If we multiply by the c.m. function h and sum we get

$$\sum_{n\le x} h(n)\log n = \sum_{jk\le x} \frac{j}{\psi(j)}\frac{k}{\psi(k)}\Lambda(k)$$

$$= \sum_{k\le x} \frac{k\Lambda(k)}{\psi(k)} \sum_{j\le x/k} \frac{j}{\psi(j)}$$

$$\sim \sum_{p\le x} \frac{p\log p}{\psi(p)} \frac{cx}{p}$$

for some positive number c.

Consider an interval $[a,a^2)$, where $a = 2^{2^{N-1}} \ge 4$. Recalling the definition of ψ and summing in reverse order, we obtain

$$\sum_{a\le p<a^2} \frac{\log p}{\psi(p)} = \sum_{a\le p<a^2} \frac{\log \psi(p)}{p}.$$

The identity

$$\pi(a^2) - \pi(\psi(p)) = \pi(p) - \pi(a) - 1 \qquad (a < p < a^2)$$

implies that

$$\log \psi(p) \sim \log(a^2 + a - p) \qquad (a < p < a^2).$$

Now

$$\sum_{a\le p<a^2} \frac{\log(a^2+a-p)}{p} \ge \sum_{a\le p<a^{3/2}} \frac{\log(a^2+a-a^{3/2})}{p} + \sum_{a^{3/2}\le p<a^2} \frac{\log a}{p}$$

$$\sim (\log 3)\log a.$$

Thus, for $x_\nu = 2^{2^\nu}$ we have

$$\sum_{n\le x_N} h(n)\log n \sim c\, x_N \sum_{m=1}^{N} \sum_{x_{m-1}\le p<x_m} \frac{\log\psi(p)}{p}$$

$$\ge (\log 3 + \underline{o}(1))c\, x_N \sum_{m=1}^{N} 2^{m-1}\log 2$$

$$\sim (\log 3)c\, x_N \log x_N.$$

However, if h had a mean value, then summation by parts would give

$$\sum_{h\le x_N} h(n)\log n \sim c\, x_N \log x_N,$$

in contradiction to the preceding estimate. Thus h does not have a mean value.

Example 3. $F(\sigma) = \underline{o}\{(\sigma-1)^{-1}\}$, $\overline{\lim}\, N(y)/y > 0$. Let ψ be c.m. with $\psi(2) = 2$ and for $p \ge 3$, define ψ on the $\ell^{\underline{th}}$ prime by

$$\psi(p_\ell) = e^{k^k} \text{ for } \sum_{1\le i<k} e^{i^i} < \ell < \sum_{1\le i\le k} e^{i^i}.$$

Now

$$N(e^{k^k}) \ge \sum_{1\le i\le k} e^{i^i} > e^{k^k},$$

while for $\sigma > 1$,

$$\log F(\sigma) \ll \sum_p \psi(p)^{-\sigma} \le 2^{-\sigma} + \sum_{k=1}^{\infty} (e^{k^k}+1)e^{-\sigma k^k}$$

$$\ll (\log\frac{1}{\sigma-1})/(\log\log\frac{1}{\sigma-1}).$$

Example 4. $(\sigma-1)F(\sigma) \longrightarrow \infty$, $\underline{\lim}\, N(y)/y < \infty$. We create a c. m. function ψ with $\psi(p) = \nu_n$, a constant, on blocks of the form $x_n \leq p < x_{n+1}$. For $n = 1, 2, \ldots$ define

$$N_n(y) = \# \{k : p|k \Rightarrow p < x_n \text{ and } \psi(k) < y\}.$$

Let $x_1 = 2$ and suppose $\psi(p)$ is determined for all $p < x_n$. Let $y = y_n$ be a number which is so large that $N_n(y) < y$. Such a number y exists since each N_n grows like some power of log. Choose $\nu_n = y + 1$. By construction $N(y)/y = N_n(y)/y < 1$ for a sequence $y = y_n \longrightarrow \infty$.

Next choose an integer $z = z_n$ so large that

$$1 - \nu_n^{-1-2^{-n}} < e^{-1/z},$$

and take x_{n+1} such that $\pi(x_{n+1}) = \pi(x_n) + z_n$. Now for $\sigma_n = 1 + 2^{-n}$ we have

$$F(\sigma_n) > \prod_{j \leq n} (1-\nu_j^{-\sigma_n})^{-z_j} > \prod_{j \leq n} (1-\nu_j^{-1-2^{-j}})^{-z_j} \geq e^n.$$

It follows that for $\sigma_{n+1} < \sigma < \sigma_n$

$$(\sigma-1)F(\sigma) > (\sigma_{n+1}-1)F(\sigma_n) = \frac{1}{2}(\sigma_n-1)F(\sigma_n) \longrightarrow \infty.$$

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