GENERALIZED RAMSEY NUMBERS INVOLVING SUBDIVISION GRAPHS, AND RELATED PROBLEMS IN GRAPH THEORY

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Let G_1 and G_2 be (simple) graphs. The <u>Ramsey number</u> $r(G_1,G_2)$ is the smallest integer n such that if one colors the complete graph K_n in two colors I and II, then either color I contains G_1 as a subgraph or color II contains G_2. The systematic study of $r(G_1,G_2)$ was initiated by F. Harary, although there were a few previous scattered results of Generisér, Gyárfás, Lehel, Erdős, and others. For general information on the subject, see the surveys [1], [7], [8]. We also note here that notation not defined follows Harary [6].

Chvátal [3] proved that if T_n is any tree on n vertices, then

$$r(T_n,K_\ell) = (\ell-1)(n-1) + 1$$

Trivially, then, if G_n is a connected graph on n points, we have $r(G_n,K_\ell) \geq (\ell-1)(n-1) + 1$. It appears to be a general principle that if such a graph is sufficiently "sparse", equality holds. With this in mind, call a connected graph G_n on n points <u>ℓ-good</u> if

$$r(G_n, K_\ell) = (\ell-1)(n-1) + 1.$$

We are preparing a systematic study of ℓ-good graphs [2]. We will not discuss the results of [2], but we will mention the following interesting unsolved problem: Let Q_m be the graph determined by the edges of the m-dimensional cube, so that Q_m has 2^m vertices, and $m2^{m-1}$ edges. Is Q_m ℓ-good if m is large enough?

One type of sparse graph not dealt with in [2] is that of subdivision graphs. If G is a graph, its <u>subdivision graph</u> S(G) is formed by putting a vertex on every edge of G. We will show that $S(K_n)$, $n \geq 8$, is 3-good. In fact, we will treat a denser graph than this. Denote by K"(n) the subdivision graph of K_n, together with all the edges of the original K_n. In other words, each edge of the K_n is replaced by a triangle. This graph has $n + \binom{n}{2} = \binom{n+1}{2}$ vertices and $3\binom{n}{2}$ edges. (for consistency, we denote $S(K_n)$ by K'(n).) We will prove the following result.

<u>Theorem 1</u>: If $n \geq 8$, then K"(n) is 3-good, that is

$$r(K''(n), K_3) = n^2 + n - 1.$$

The proof of this theorem is somewhat long and we defer it. It appears likely that the method can be extended to show that if ℓ is fixed, $K'(n)$ is ℓ-good when n is large enough, but we have not carried out the details. Other possible extensions are discussed at the end of this paper.

We now turn our attention in another direction. Following Erdős and Hajnal [4], denote by $K_{top}(n)$ any graph homeomorphic to K_n, that is a graph formed from K_n by putting various numbers of extra vertices on its edges. The paper [4] is reproduced in [9], pages 167-173. Thus K_n and $K'(n)$ are both examples of a $K_{top}(n)$. Note that a $K_{top}(n)$ has n vertices of degree n-1 and any number of degree 2. Let $\mathcal{K}_{top}(n)$ be the class of all $K_{top}(n)$. In [4] Erdős and Hajnal investigate the Ramsey numbers $r(\mathcal{K}_{top}(n),\mathcal{K}_{top}(n))$ and $r(\mathcal{K}_{top}(m),K_\ell)$. (Here we have slightly extended the definition of r: If G_1 or G_2 are classes of graphs, we are satisfied if any number of a class appears in its appropriate color.) They prove (in our notation):

$$r(\mathcal{K}_{top}(n),K_3) > cn^{4/3}(\log n)^{-2/3}.$$

Our method will give, without much difficulty,

$$r(\mathcal{K}_{top}(n),K_3) < c_1 n^{3/2}.$$

Before we prove this, we need another result. Denote by $f(n)$ the largest integer for which there is a graph G on $f(n)$ vertices which has no triangle, and moreover every induced subgraph of G has at least $f(n)$ edges. We prove the following result.

Theorem 2:

$$cn^{4/3}(\log n)^{-2/3} < f(n) < 2^{-1/2}n^{3/2}.$$

Proof: The proof of the lower bound is implicitly contained in [4-see pg. 147] (and also in the proof of Theorem 3 which follows), so we only have to prove the upper bound. Let G be a graph with $f(n)$ vertices, all of whose n-vertex induced subgraphs have at least $f(n)$ edges. Let q be the number of edges of G. Then, by a simple averaging argument, we obtain

$$q \geq f(n)\binom{f(n)}{2}\binom{f(n)-2}{n-2}^{-1} = \frac{f^2(n)(f(n)-1)}{n(n-1)} > \frac{f^3(n)}{n^2} \geq \frac{nf(n)}{2},$$

if $f(n) \geq 2^{-1/2}n^{3/2}$. Since G has $f(n)$ vertices, it has a vertex x of valency at least n. Since G has no triangle, all the vertices adjacent to x are mutually nonadjacent. But this contradicts (strongly) the assumption that any n vertices induce at least $f(n)$ edges, so necessarily $f(n) < 2^{-1/2}n^{3/2}$, completing the proof.

Clearly, the constant $2^{-1/2}$ could be replaced by a smaller one. However, we will not pursue this farther since we believe that $f(n) = O(n^{3/2})$, although we don't know how to prove it. We can now prove our result on $r(K_{top}(n),K_3)$.

Theorem 3: For some constants c and c_1,

$$cn^{4/3}(\log n)^{-2/3} < r(K_{top}(n),K_3) < c_1n^{3/2}.$$

Proof: We have already said that the lower bound was proved in [4]. We prove the upper bound by showing that

$$r(K_{top}(n),K_3) \leq f(n) + 3n - 5.$$

Consider a graph G on $f(n) + 3n - 5$ vertices such that $\overline{G}$ has no triangle. Observe that if any vertex has degree at least n in $\overline{G}$, we are done, since otherwise we have even a K_n in G. (In fact, this also is immediate from Chvátal's result.)

From the definition of $f(n)$, we see that $\overline{G}$ has a set of vertices $A = \{a_1,\ldots,a_n\}$ which induces fewer than $f(n)$ edges. We will develop a $K_{top}(n)$ in G for which A is the set of vertices of degree n. These vertices already span at least $\binom{n}{2} - f(n) + 1$ edges, so that at most $f(n) - 1$ must be joined by other paths. We will in fact do so with paths of length two, with the midpoints being distinct, of course.

Suppose, on the contrary, that we have joined k pairs of a's, $k < f(n) - 1$, but that we cannot join a_i to a_j by a path of length two in G which avoids all vertices already used. We have used $n + k \leq n + f(n) - 2$, leaving a set B of at least $2n - 3$ vertices. Since, by our assumption, none of these are adjacent to both a_i and a_j in G, either a_i or a_j is joined in $\overline{G}$ to at least $n - 1$ vertices in B. Since we also have that a_i and a_j are adjacent in $\overline{G}$, we have a point of degree at least n in $\overline{G}$. But this has been shown to be impossible, which completes the proof.

It would be of great interest to estimate f(n), or $r(K_{top}(n),K_3)$, as accurately as possible. At the moment we cannot prove $f(n) > n^{4/3+\varepsilon}$ or $f(n) = 0(n^{3/2})$. It might not be out of the question to determine the existence and value of

$$\lim_{n\to\infty} f(n)/\log n.$$

To determine the exact value of f(n) or $r(K_{top}(n),K_3)$ is probably hopeless.

Now we return to the proof of Theorem 1. It is very likely that this theorem actually holds for $n \geq 3$. Once or twice (for instance in Fact 4) we prove a trifle more than necessary in what follows in the hope that this will help eventually to fill in the missing cases.

Proof of Theorem 1: Of course, $K''(n)$ has $n + \binom{n}{2} = N$ vertices, so we wish to show that $r(K''(n),K_3) \leq 2N - 1$. (That $r(K''(n),K_3) \geq 2N-1$ follows immediately from the fact that $K''(n)$ is connected.) Let G be a graph on $2N - 1$ points and assume, contrary to the theorem, that $K''(n) \not\subset G$ and $K_3 \not\subset \overline{G}$.

It will be convenient to make the following definition of a partial $K''(n)$. Let A and B be disjoint sets of vertices with $|A| = n$ and with $|B| \leq \binom{n}{2}$. Then a $K''(A,B)$ is any graph consisting of a complete graph on A, together with a pair of edges connecting each point of B with a different pair of points of A. Such graphs are not unique in general, but of course if $|B| = \binom{n}{2}$, a $K''(A,B)$ is a $K''(n)$. Furthermore, if F is a $K''(A,B)$, define H_F to be the graph with A as its vertices, with a pair of vertices joined in H_F if they are joined in F through a point of B. Moreover, call a $K''(A,B)$ in G <u>maximal</u> in a given graph if there exists no $K''(A,B_1)$ in the graph with $|B_1| > |B|$.

We will now prove a series of facts about G, leading finally to a contradiction.

Fact 1: If F is a maximal $K''(A,B)$ then $\overline{H}_F$ contains no triangle.

To see this, assume to the contrary that $a_1a_2a_3$ is a triangle in $\overline{H}_F$ and let v be any vertex not contained in F. Since no two a_i can be joined through v in G, v is connected to at least two a_i in $\overline{G}$. Let v_1 be any other vertex not contained in F; it, too, is connected to at least two a_i in $\overline{G}$. Hence, for some a_i, the edges a_iv and a_iv_1 are both in $\overline{G}$. Since $\overline{G}$ contains no triangle, the edge vv_1 must be in G. But v and v_1 were arbitrary vertices not in F, so these vertices span a complete graph in G. If F had as many as N vertices, F would be a K"(n); so G contains a K_N, which is again a contradiction.

Fact 2: $\overline{G}$ has no vertex of degree as large as L, where $L = \left[\frac{n^2}{4}\right] + n$.

Suppose that this is false; since $\overline{G}$ has no triangle, G must have a K_L. Let A be a set of n vertices from the K_L. Omit for the moment the other $\left[\frac{n^2}{4}\right]$ vertices of the K_L, and let F be a maximal K"(A,B) using the remaining part of G. By Fact 1, $\overline{H}_F$ contains no triangle, so by Turán's Theorem, $\overline{H}_F$ has no more than $\left[\frac{n^2}{4}\right]$ edges, and so H_F has at least $\binom{n}{2} - \left[\frac{n^2}{4}\right]$ edges. Therefore, $|B| \geq \binom{n}{2} - \left[\frac{n^2}{4}\right]$. Furthermore, there are L-n unused vertices in the K_L, where we have $L - n = \left[\frac{n^2}{4}\right]$. Therefore, we can form a $K(A,B_1)$, where $|B_1| = \binom{n}{2}$, using $\left[\frac{n^2}{4}\right]$ of these unused vertices, and $\binom{n}{2} - \left[\frac{n^2}{4}\right]$ vertices from B. This contradiction establishes Fact 2.

Fact 3: Any two points of G are joined by at least $2N - 2L - 1$ different paths of length 2.

This fact follows immediately from Fact 2.

Fact 4: Let $n \geq 7$ and let $F = K''(A,B)$ be maximal. Suppose that a_1, a_2, a_3 be distinct vertices in A, and suppose that a_i and a_j are connected through $b_{ij} \in B$. Let u_1u_2 be any edge in $\overline{H}_F$. Then G does not contain all six edges of the form u_ib_{jk}.

(Note that $u_i = a_j$ is permitted.) Assume this fact is false, so that G does contain s such edges. Let C be the set of vertices not in A or B, so $|C| \geq N$. Let $c \in C$. Suppose G had two edges ca_i and ca_j. Then G would contain the two paths a_ica_j and $u_1b_{ij}u_2$. In F, adjoin these two and delete the path $a_ib_{ij}a_j$. This new graph is a $K''(A, B \cup \{c\})$, contradicting the maximality of F. Thus for any $c \in C$, there is at most one edge from c to a_1, a_2, a_3. Therefore, at least 2N edges join the a_i to C in $\overline{G}$, and hence some a_i has degree at least $2N/3$. It is easy to see that this contradicts Fact 2 if $n \geq 7$.

Fact 5: $K_n \subset G$.

This fact follows easily from the well-known result that $r(K_m, K_n) \leq \binom{m+n-2}{m-1}$, already proved in effect in [5]. (The paper [5] is reproduced on pages 5-12 of [9].)

We are now ready to complete the proof of Theorem 1. By Fact 5, G contains a $K''(A,\{\emptyset\})$ for some A. Let $K''(A,B) = F$ be maximal. By hypothesis, $|B| < N$; this will lead to a contradiction. Let u_1u_2 be an edge of $\overline{H}_F$. By Fact 3, u_1 and u_2 are joined by at least $2N - 2L - 1$ different paths of length 2, the midpoints of which all must lie in $A \cup B$, by the maximality of F. Of these midpoints, n-2 lie in A. Thus 2N - 2L - 1 - (n-2) of these are in B, and therefore correspond to edges in H_F. It is easy to check that $2N - 2L - 1 - (n-2) > \left[\frac{n^2}{4}\right]$ if $n \geq 8$. Because of this, some three of these midpoints correspond to a triangle $a_1a_2a_3$ in H_F, the midpoints being of course b_{12}, b_{23}, b_{31}. But this is just the configuration prohibited by Fact 4. This contradiction completes the proof of Theorem 1.

Now we prove one final result which is very simple, but interesting. Let G be a graph with 2n-1 vertices such that $K_3 \not\subset \overline{G}$ and $K_n \not\subset G$. Then G has diameter 2. To see this, note, as we have before, that $\overline{G}$ cannot have a vertex of degree as large as n. Hence every vertex of G has degree at least n - 1. From this it is immediate that any two vertices are either adjacent or joined by a path of length 2.

We close with some remarks about improvements and generalizations of Theorem 1. We have already conjectured that Theorem 1 actually holds for $n \geq 3$, and we have indeed proved it for n = 3. The cases $4 \leq n \leq 7$ remain open. Although the methods of this paper would certainly help, dealing with these cases is likely to be tedious without at least one new idea. A more important direction is replacing K_3 by K_ℓ. Standard estimates of $r(K_n, K_\ell)$ show that $K''(n)$ cannot be ℓ-good if $\ell > 3$, but there is every reason to believe that for each ℓ, $K'(n)$ is ℓ-good when n is large enough. In fact, as we have said, it should be possible to extend the proof to this case fairly directly, but we have not carried this out.

Another interesting generalization would be to consider the subdivision graphs, or the modification we have treated here, of arbitrary graphs, rather than just K'(n) or K"(n). This may be easy, but it would not be surprising if new difficulties arise. One might also consider higher-order subdivision graphs $S_2(K_n), S_3(K_n), \ldots$; this is probably straightforward. It may be more difficult to deal with arbitrary, but fixed, members of $\mathcal{K}_{top}(n)$, even with the requirement that all the paths joining the n special points have lengths at least two. (Of course, some such requirement is necessary, since $K_n \in \mathcal{K}_{top}(n)$, and K_n is certainly not even 3-good.)

One further generalization of K'(n) is of interest. Let $\{a_1, \ldots, a_n\}$ be a set of vertices, and for each triple $\{a_i, a_j, a_k\}$ of them, join each to a new vertex b_{ijk}. It seems certain that if ℓ is fixed, all large graphs of this form are ℓ-good, and similarly for the obvious generalizations. Parts of our proof of Theorem 1 generalize easily; some may not, especially those using Turan's Theorem, since these seem to need hypergraph versions of that theorem, and such versions are not nearly as precise as for graphs.

REFERENCES

[1] S. A. Burr, Generalized Ramsey Theory for Graphs - A Survey, Graphs and Combinatorics, Springer, Berlin 1974, 52-75.

[2] S. A. Burr and P. Erdős, Generalizations of a Ramsey-Theory Result of Chvátal, to appear.

[3] V. Chvátal, Tree-Complete Graph Ramsey Numbers, J. Graph Theory 1 (1977), 93.

[4] P. Erdős and A. Hajnal, On Complete Topological Subgraphs of Certain Graphs, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 7 (1964), 143-149.

[5] P. Erdős and Gy. Szekeres, A Combinatorial Problem in Geometry, Compositio Math. 2 (1935), 463-470.

[6] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass., 1969.

[7] F. Harary, A Survey of Generalized Ramsey Theory, Graphs and Combinatorics, Springer, Berlin 1974, 10-17.

[8] F. Harary, The Foremost Open Problems in Generalized Ramsey Theory, Proc. Fifth British Combinatorial Conference, Aberdeen, 1975, Utilitas Math. Pub. Inc., Winnipeg, 269-282.

[9] J. Spencer (ed.), Paul Erdős: The Art of Counting, MIT Press, Cambridge, Mass., 1973.